

# Factor Investing: A Welfare-Improving New Investment Paradigm or Yet Another Marketing Fad?

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# Foreword

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The present publication was produced as part of the "Risk Allocation Solutions" research chair at EDHEC-Risk Institute, in partnership with Lyxor Asset Management. This chair is examining performance portfolios with improved hedging benefits, hedging portfolios with improved performance benefits, and inflation risk and asset allocation solutions.

The current paper, "Factor Investing: A Welfare-Improving New Investment Paradigm or Yet Another Marketing Fad?", examines the relative efficiency of standard forms of practical implementation of the factor investing paradigm based on commonly-used factors in the equity, fixed-income and commodity universes.

Investment practice has recently witnessed the emergence of a new approach known as factor investing, which recommends that allocation decisions be expressed in terms of risk factors, as opposed to standard asset class decompositions. To answer the question of whether factor investing is truly a welfare-improving new investment paradigm or whether it is merely yet another marketing fad, the paper identifies mathematical conditions under which it is expected to generate welfare gains for asset owners and provides an empirical measure of such gains.

The study suggests that the most meaningful way to group individual securities is not to form arbitrary asset class indices, but rather to form replicating portfolios for asset pricing factors that can collectively explain cross-sectional differences in security returns.

In this paper, the authors found that the out-of-sample benefits of factor investing are potentially extremely large when short-selling is allowed, and remain substantial in the presence of long-only constraints, especially when improved weighting schemes are used in implementation.

I would like to thank Professor Lionel Martellini and Dr Vincent Milhau for their work on this research, and Laurent Ringelstein and Dami Coker for their efforts in producing the final publication.

I would also like to extend particular thanks to Lyxor Asset Management for their support of this research chair.

We wish you a useful and informative read.



Noël Amenc  
Professor of Finance  
Director of EDHEC-Risk Institute

# About the Authors

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**Lionel Martellini** is Professor of Finance at EDHEC Business School and Scientific Director of EDHEC-Risk Institute. He has graduate degrees in economics, statistics, and mathematics, as well as a PhD in finance from the University of California at Berkeley. Lionel is a member of the editorial board of the *Journal of Portfolio Management* and the *Journal of Alternative Investments*. An expert in quantitative asset management and derivatives valuation, his work has been widely published in academic and practitioner journals and he has co-authored textbooks on alternative investment strategies and fixed-income securities.



**Vincent Milhau** is Deputy Scientific Director of EDHEC-Risk Institute. He holds master's degrees in statistics (ENSAE) and financial mathematics (Université Paris VII), as well as a PhD in finance (Université de Nice-Sophia Antipolis). His research focus is on portfolio selection problems and continuous-time asset-pricing models.

# Executive Summary



## Executive Summary

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A new approach known as *factor investing* has recently emerged in investment practice, which recommends that allocation decisions be expressed in terms of risk factors, as opposed to standard asset class decompositions. While intuitively appealing, this approach poses a major challenge, namely the choice of the meaningful factors and the corresponding investable proxies.

Simply put, factor investing proposes regarding each constituent in an investor's portfolio, and therefore the whole portfolio, as a bundle of factor exposures. Obviously, factor models, such as those of Sharpe (1963) and Fama and French (1993), have long been used for performance measurement purposes, and several factors correspond to classical investment styles, such as value-growth investing, trend following or short volatility, that were in use in the industry before they were formally identified as asset pricing factors. In this context, the question arises whether factor investing is truly a welfare-improving new investment paradigm or whether it is merely yet another marketing fad.

A first remark is that if there are as many factors as individual securities and the factors are themselves portfolios of such securities, then thinking in terms of factors is strictly equivalent to thinking in terms of asset classes, and would therefore not add any value. More relevant is the situation where a parsimonious factor mode is used, with a number of factors smaller than the number of constituents. The first challenge posed to investors who decide to express their decisions in terms of factor exposures is then the identification of meaningful factors. In this perspective, the theoretical section of this paper reviews the academic literature on asset pricing and makes a list of conditions that such factors should satisfy.

We then survey a (vast) empirical literature in order to identify the most consensual factors in three major asset classes, namely stocks, bonds and commodities. The second challenge in factor investing is the implementation of decisions in a cost-efficient way with investable proxies. On the empirical front, our paper provides an analysis of the welfare gains that can be expected from the use of proxies for factors, and discusses the choice of long-only versus long-short factors, which is relevant for many investors given that they are not allowed to take short positions.

In asset pricing theory, the relevant and important factors are the "pricing factors", the exposures to which explain all differences in expected returns across assets.

Asset pricing theory makes a distinction between "pricing factors", which explain differences in expected returns across assets, and "priced factors", which earn a premium over the long run. The theory, exposed in the textbooks of Duffie (2001) and Cochrane (2005), expresses the risk premium on an asset, i.e. the expected return on this asset in excess of the risk-free rate, as a function of the covariance between the pay-off and an abstract quantity known as the *stochastic discount factor*. The goal of theoretical and empirical asset pricing models is to find a representation for this random variable in terms of economically interpretable variables. For instance, in consumption-based models, the stochastic discount factor is proportional to the marginal utility of consumption. This conveys an important economic intuition: risk premia exist as rewards required by investors in exchange for holding assets that have a low pay-off in "bad times", defined as times where investors' wealth is low and marginal utility is high.

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Pricing factors arise when one attempts to find observable proxies for the aggregate investor's marginal utility. In a factor model, the risk premium on an asset is a linear combination of the factor risk premia, weighted by the betas of the asset with respect to the factor. As a consequence, all alphas are zero and the cross-sectional differences between expected returns are entirely explained by the differences in factor exposures. As for an asset, the premium of a factor is determined by its covariance with the stochastic discount factor, so that a factor deserves a positive premium if, and only if, it is high in "bad times" and low in "good times". A factor is said to be "priced" if it has a non-zero premium. It can be shown that there is no loss of generality from searching for pricing factors among returns, but further assumptions are needed to identify their economic nature. Two main classes of theoretical models have been developed to this end. A first category uses economic equilibrium arguments. In the static Capital Asset Pricing Model (CAPM) of Sharpe (1964), the only factor, or "market factor", is the return on aggregate wealth. The intertemporal version (ICAPM) of Merton (1973) adds as new factors the variables that predict changes in expected returns and volatilities. A second class of models refers to the Arbitrage Pricing Theory (APT) of Ross (1976) and characterises factors as variables that explain returns from a statistical standpoint. One of the questions studied in the recent asset pricing literature is whether the factors proposed in empirical asset pricing models do meet these theoretical criteria.

The property of assets having zero alphas with respect to the factors has an interesting implication. Indeed, it can be shown that a theoretical single-step solution to a mean-variance optimisation

problem coincides with an optimal linear combination of mean-variance efficient benchmark portfolios invested in individual securities if each individual security has a zero alpha when regressed on the benchmark portfolios. This result therefore suggests that the most meaningful way for grouping individual securities is not by forming arbitrary *asset class indices*, but instead by forming *factor indices*, that is replicating portfolios for a set of *asset pricing factors* that can collectively be regarded as linear proxies for the unobservable stochastic discount factor, thus providing a theoretical justification for factor investing.

**Empirically, the search for pricing factors in asset classes such as stocks, bonds and commodities begins with the identification of persistent and economically interpretable patterns in average returns. Recent research has subsequently started to look for multi-class factors.**

While the CAPM is relatively explicit about the nature of the underlying pricing factor (the return on aggregate wealth), multi-factor models derived from the ICAPM or the APT do not provide an explicit definition of their factors. Thus, the traditional approach in empirical asset pricing has been to examine the determinants of cross-sectional differences in expected returns and to find sound economic interpretations for regular patterns (presence of a risk factor, market frictions or behavioural biases).

Most of the empirical asset pricing literature has focused on factors explaining equity returns. This literature starts in the early 1970s with empirical verifications of the CAPM and concludes that the model's central prediction, namely the positive

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and linear relationship between expected excess return and the covariance with aggregate wealth, is not well validated by the data for two reasons. First, there is the existence of patterns, or "anomalies", that are not explained by the market exposure, and second, the relation between expected returns and the market betas is at best flat, or even negative. The most consensual patterns are those that have shown to be robust to various statistical tests, to exist in almost all international equity markets, to persist over time (in particular after their discovery) and to admit plausible economic explanations. They include the size and the value effects, which are historically among the first reported anomalies: small cap stocks tend to outperform their large cap counterparts, and there is a positive relationship between the book-to-market ratio and future average returns. The size and value factors are used together with the market factor in the model of Fama and French (1993). Another remarkably robust pattern is the momentum effect: the winners (resp., losers) of the past 3 to 12 months tend to outperform (resp., underperform) over the next 3 to 12 months. The number of reported empirical regularities has grown rapidly in the recent literature, and a survey by Harvey et al. (2013) lists at least 315 of them. Among them is the controversial "low volatility puzzle", namely the documented outperformance of low volatility stocks over high volatility stocks, the existence and persistence of which remains somewhat debated in the academic literature. Among the other noticeable patterns are the investment and profitability effects.

In fixed-income, the two main traditional factors are term and credit. As put by Fama and French (1993), unexpected changes in interest rates and in the probability of default are "common risk factors" for

bonds, so it is expected that they should be rewarded. Given that long-term bonds are more exposed to interest rate risk than short-term bills through their longer duration, one expects them to earn higher returns on average. Similarly, defaultable bonds are expected to earn a premium over default-free ones because default events are more likely to happen in bad economic conditions. However, a mathematical decomposition of the term premium, such as those performed in the studies of bond return predictability, suggests that it varies in sign and magnitude with changes in the slope of the term structure and changes in expectations about future interest rates. Historical evidence confirms this variation. In addition to these two standard factors, recent literature has found patterns similar to those encountered in the equity class, such as momentum, value (which refers to a long-term reversal effect) and low risk.

For commodities, one can obtain a first important factor by examining the determinants of the performance of passive strategies that roll over futures contracts. Research has shown that the long-term returns to such strategies are mainly driven by the roll returns, which are the positive or negative returns earned by replacing the nearest contract with the second nearest when the former matures. Hence, the prevailing shape of the term structure of futures prices is an essential determinant of long-term returns. Specifically, backwardated futures markets (for which the term structure is decreasing) outperform contangoed futures markets (for which it is increasing). This calls for the introduction of a term structure factor defined as the excess return of backwardated contracts over contangoed contracts. A related factor is the hedging pressure, which is suggested by the eponymous theory and indirectly captures the shape of the term structure.

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Beyond these "fundamental" factors, a momentum factor has also been empirically reported for commodities.

A class-by-class study reveals that some patterns exist repeatedly in various classes. Asness et al. (2013) show that this is the case for short-term momentum and long-term reversal in equities, bonds, commodities and currencies. Furthermore, the single-class momentum factors are positively correlated, and the same goes for value factors. Taken together, these findings justify a new approach, which is the construction of multi-class value and momentum factors, obtained by aggregating the corresponding single-class components.

Empirical tests show that investable proxies for factors add value in single-class or multi-class portfolios when they are used as complements or substitutes for broad asset class indices. Moreover, in the equity class, a portfolio of factor indices dominates a portfolio of sector indices.

Our empirical study focuses on the following factors, which have been selected because they have a well-documented historical performance, are theoretically grounded and are widely accepted by practitioners: size, value, momentum and volatility for equities; term and credit for bonds; and term structure and momentum for commodities. In addition, we test multi-class value and momentum factors computed after the methodology of Asness et al. (2013). A first analysis of the descriptive statistics for these factors highlights a few simple but important facts. Each long-only factor outperforms its opposite tilt, in line with the theoretical and empirical literature, and outperforms the corresponding broad asset class index. Correlations within a class

are high (above 75%), although they are lower across classes, and they are much lower for long-short versions of the factors.

The benchmark universes that we consider contain the broad indices of one or more asset class(es), which represent the market factors. For equities, we also test the benefits of using a standard sector classification, as an alternative to grouping securities according to their factor exposures. A first method for assessing the usefulness of factors is to compare the efficient frontiers in the benchmark universe and in an extended universe that also contains the factors. Formal mean-variance spanning tests (see Kan and Zhou (2012)) reject the null hypothesis that the efficient frontier of the benchmark universe is included in that of the extended universe for all long-short factors and for most long-only factors. This is first evidence that the introduction of factors improves the efficient frontier, even though these tests rely on in-sample long-short efficient frontiers, so that they may give an overly optimistic picture.

For this reason, we also conduct a series of out-of-sample tests, where we compare portfolios of traditional indices (asset class indices or equity sectors) and portfolios of factors, by imposing long-only constraints and by estimating parameters without a look-ahead bias. Since a test of the relevance of factor investing is a joint test of the relevance of the chosen factors and the chosen allocation methodology, we run the comparison for various diversification schemes that avoid the estimation of expected returns: equally-weighted, minimum variance, risk parity and "factor risk parity", where the implicit factors are extracted from the covariance matrix. The "relative" counterparts of these schemes, which focus on the tracking error as opposed to the absolute volatility,

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are also tested. Exhibit 1 shows a sample of results for the equity class: for most weighting schemes, the four-factor portfolios have higher average return, higher Sharpe ratios and higher information ratios compared to their ten-sector equivalents. In addition, they have a lower turnover. Exhibit 2 extends the analysis to a multi-class context by comparing "policy-neutral" portfolios of equity, bond and commodity factors to a fixed-mix policy portfolio of 60% equities, 30% bonds and 10% commodities. Again, both the average return and the Sharpe ratio are improved.

As a conclusion, theoretical arguments in favour of factor investing exist, that is in favour of grouping individual securities into factor indices as opposed to arbitrary

forms of indices. An extensive empirical literature has documented a number of recurring patterns in the returns of equities, bonds and commodity futures, and provides investors with a rich list of insights regarding the choice of meaningful factors in each of these classes. The identification of a parsimonious set of factors capturing the largest possible number of sources of risk is an ongoing task on the academic side. On the practical side, a challenge is to develop factor indices that aim to capture factor risk premia at reasonable implementation costs. It is being addressed in the equity class with a new generation of "smart beta" indices, but similar products are not as widely developed in other classes and no multiple-class products are available to date.

Exhibit 1: Cap-weighted equity factor indices as substitutes for equity sector indices (1975-2013).

Universe	Avg. return (%)	Volatility (%)	Sharpe ratio	Tracking error (%)	Information ratio	Ann. 1-way Turnover (%)
Equally-Weighted						
Equ 10 Sectors	12.56	16.37	0.45	2.82	0.34	8.9
Equ 4 Factors	13.23	16.73 ***	0.48	2.59	0.63	3.5
Minimum Variance						
Equ 10 Sectors	12.67	14.26	0.52	6.18	0.17	29.1
Equ 4 Factors	12.06	15.87 ***	0.43	3.48	0.13	27.2
Risk Parity						
Equ 10 Sectors	12.76	15.60	0.48	3.64	0.32	11.9
Equ 4 Factors	13.17	16.60 ***	0.48	2.64	0.60	4.6
Factor Risk Parity						
Equ 10 Sectors	12.76	14.18	0.53	5.58	0.21	34.6
Equ 4 Factors	12.55	15.68 ***	0.47	3.59	0.27	26.2
Minimum Tracking Error						
Equ 10 Sectors	11.98	16.97	0.40	1.47	0.27	14.0
Equ 4 Factors	12.59	16.60 ***	0.44	1.99	0.50	15.3
Relative Risk Parity						
Equ 10 Sectors	11.98	16.96	0.40	1.47	0.26	13.2
Equ 4 Factors	12.62	16.65 ***	0.44	1.97	0.52	11.2
Relative Factor Risk Parity						
Equ 10 Sectors	12.45	16.52	0.44	2.64	0.33	38.1
Equ 4 Factors	12.63	16.72 ***	0.44	2.10	0.50	17.5

Note: The tracking error and the information ratio are computed with respect to the broad cap-weighted index. \*\*\*denotes significance of the difference with respect to the benchmark at the 1% level.

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Exhibit 2: Substitution of asset class indices by factor indices in policy portfolio (1988-2011).

Universe	Avg. return (%)	Volatility (%)	Sharpe ratio	Tracking error (%)	Information ratio
Policy portfolio	8.82	9.44	0.52	0.00	–
Absolute weighting schemes					
MDC 8 Factors	9.90 *	9.46	0.63 *	2.57	0.42
GMV 8 Factors	9.54	8.78 ***	0.64	3.22	0.22
MENCB 8 Factors	9.37	9.56	0.57	2.68	0.21
MENUB 8 Factors	9.76	8.82 ***	0.66	3.55	0.26
Relative weighting schemes					
MTE 8 Factors	9.74 *	9.24	0.63 *	2.23	0.41
MENRCB 8 Factors	9.56	9.57	0.59	2.90	0.25
MENRUB 8 Factors	9.67 *	9.36	0.62 *	2.25	0.38

Note: GMV and MTE refer respectively to global minimum variance and minimum tracking error. MDC, MENCB and MENUB are the respective equivalents of equally-weighted, risk parity and factor risk parity with policy constraints. Similarly, MENRCB and MENRUB are the policy-neutral equivalents of relative risk parity and relative factor risk parity. \*\*\* (resp., \*) denotes significance of the difference with respect to the benchmark at the 1% (resp., 10%) level.

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# 1. Introduction



# 1. Introduction

A new approach known as factor investing has recently emerged in investment practice, which recommends that allocation decisions be expressed in terms of risk factors, as opposed to standard asset class decompositions (see Ang (2014) for a comprehensive overview of the subject). Given that risk factors are already commonly used for risk and performance evaluation of actively managed portfolios, the question arises whether factor investing is truly a welfare-improving new investment paradigm or whether it is merely yet another marketing fad. The main objective of this paper is to answer this question by identifying mathematical conditions under which factor investing is expected to generate welfare gains for asset owners, as well as providing an empirical measure of such gains, if any, with respect to the standard approach based on asset classes, and an assessment of their robustness with respect to various implementation choices.

While the standard investment approach in delegated portfolio management consists in first grouping individual securities in (somewhat arbitrary) asset class categories and then allocating wealth to these different asset classes, it is well-known that this two-step process involves a potentially large efficiency loss (Sharpe (1981), Elton et al. (2004) or van Binsbergen et al. (2008)), and one might therefore wonder whether a better articulation could be achieved between the construction of benchmark portfolios from individual securities, and allocation decisions to such benchmarks.<sup>1</sup> Given that security and asset class returns can be explained by their exposure to pervasive systematic risk factors, looking through the asset class decomposition

level to focus on the underlying factor decomposition level appears to be a perfectly legitimate approach, which is supported by standard equilibrium or arbitrage arguments (see in particular the intertemporal CAPM of Merton (1973) or the arbitrage pricing theory of Ross (1976)). To provide a more formal analysis of the question, we first show that a necessary and sufficient condition for the single-step solution to a mean-variance optimisation problem to coincide with the optimal linear combination of mean-variance efficient benchmark portfolios invested in individual securities, is that each individual security has a zero alpha when regressed on the benchmark portfolios (Proposition 4). This condition is trivially satisfied when the benchmark portfolio returns are defined as a rotation of the returns on the individual securities. In realistic situations, where there are fewer benchmarks than securities, the two-step process involves a loss of efficiency, unless all alphas are zero. A sufficient condition for this to happen is to take the indices as perfect replicating portfolios for asset pricing factors (Proposition 2). This result suggests that the most meaningful way for grouping individual securities, that is the way leading to the smallest opportunity cost, is not by forming arbitrary *asset class indices*, but instead by forming factor indices, that is replicating portfolios for a set of *asset pricing factors* that can collectively be regarded as linear proxies for the unobservable stochastic discount factor. Building on this theoretical insight and a number of associated formal statistical tests, we then provide a detailed empirical analysis of the relative efficiency of various forms of practical implementation of the factor investing paradigm, including the

1 - In addition to the induced lack of diversification, van Binsbergen et al. (2008) analyse other sources of efficiency loss caused by the two-step process, including the presence of liabilities, of different appetites for risk, or of different investment horizons.

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approaches using long-only versus long-short factor indices, cap-weighted versus smart-weighted factor indices, or multi-asset factor indices versus asset class factor indices. Overall, we find that the out-of-sample benefits of factor investing are extremely substantial when short sales are permitted, but remain strong in a long-only context, especially when improved weighting schemes are used in implementation. We also find that factor diversification dominates standard forms of diversification based on asset classes or sector allocation decisions. While a cost-efficient implementation of this approach is relatively straightforward in the equity space, extending the approach to a multi-asset context involves, however, a number of conceptual and practical challenges that are discussed in this paper.

Our paper is related to the vast literature on the identification of pricing factors and priced factors, where a pricing factor is defined as a factor which is useful to price available securities, while a priced factor is one that earns a non-zero reward (see Section 2 for more detail). This literature can be broadly divided into theoretical and empirical papers. The first strand of the literature, which we review in Section 2, relates to the asset pricing theory, and develops a number of models that lead to a series of economic and/or statistical requirements that the factors should satisfy. The second strand of the literature, which we review in Section 4, takes an empirical approach, and tests the robustness of some empirical regularities that might exist in expected returns and their compatibility with various explanations, including notably behavioural biases or the presence of genuine factors. The main focus of our

paper is not to propose new methodologies for extracting or testing new individual factors, but instead to assess whether commonly used factor indices in their investable versions are sufficiently good proxies for true factor replicating portfolios for their use to generate substantial welfare gains compared to the use of standard asset classes indices to perform the allocation step. It turns out that most, if not all, of the standard factors are explicit factors: this means that they are defined as portfolio returns with explicitly defined constituents and weights. In contrast, implicit factors are extracted from observed returns by statistical techniques such as principal component analysis. While the implicit approach has the advantage of "letting the data talk" without imposing a view about the economic origin of systematic risk, it raises stability and interpretability issues: more often than not, the loadings of principal factors on the constituents are unstable from one date to the other and the weight profile does not admit an intuitive economic interpretation. For instance, in the equity class, only the first principal component corresponds to a well-identified factor (the market factor), but the next ones do not reflect traditional factors such as value and size. On the other hand, in the bond class, principal component analysis yields more stable results, with the level, slope and possibly credit (in the case of non-homogeneous issuers) factors. We restrict this paper to explicit factors selected because they are standard in the academic literature and the investment practice.

The rest of the paper is organised as follows. In Section 2, we present a detailed discussion of the role of factors in asset

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pricing theory. In Section 3, we present various allocation methods to factor indices and introduce new results regarding the relation between maximum Sharpe ratio portfolios constructed from factor indices versus maximum Sharpe ratio portfolios constructed from individual securities. In Section 4, we provide a review of the vast body of literature that has focused on the identification of empirical proxies for asset pricing factors, and propose a list of standard factors used in investment practice. In Section 5, we perform a thorough empirical investigation of the benefits of factor investing versus other forms of investment strategies that differ through the criteria employed to group individual securities, and the methodologies used to form benchmark portfolios from the set of selected securities. A conclusion and suggestions for further research can be found in Section 6, while technical details are relegated to an Appendix.

## 2. Risk Premia and the Stochastic Discount Factor in Asset Pricing Theory



## 2. Risk Premia and the Stochastic Discount Factor in Asset Pricing Theory

In a nutshell, asset pricing theory relates expected returns to the comovements between future realised returns and a *stochastic discount factor*. This section is intended to provide a synthetic overview of some important results regarding the discount factor and how they relate to *factor models*. It is organised as follows. We introduce in Section 2.1 the stochastic discount factor (SDF), which is the fundamental tool in asset pricing theory. In Section 2.2, we turn to a presentation of generic factor models, which allow one to replace the (unobservable) SDF by a linear combination of "pricing factors". Section 2.3 contains a description of the theoretical factor models built from equilibrium or arbitrage considerations (CAPM, ICAPM, CCAPM, APT), to which empirical work on factors typically refers. Section 2.4 presents a list of conditions that empirical factors should satisfy in order to be admissible as pricing factors in the various models, and discusses the identification of "priced factors", i.e. factors that have a non-zero risk premium. A more detailed presentation of the concepts introduced in these sections can be found in the reference textbooks on the subject, e.g. Duffie (2001) and Cochrane (2005). Finally, Section 3.2 contains new theoretical results regarding the benefits of investing in factors with respect to investing in other groups of securities such as asset classes or country indices. These theoretical results will be used in Section 5, where we conduct an empirical comparison between factor investing and other more standard approaches to grouping securities.

### 2.1 The Stochastic Discount Factor

The *stochastic discount factor* (in short, SDF), also known as *pricing kernel*, allows one to relate the current price of an asset to its future pay-off. In this section,

we write the relationship between the expected excess return on an asset and the SDF, and we sketch different methods for constructing an SDF.

#### 2.1.1 Fundamental Asset Pricing Equation

In what follows, we assume that uncertainty is represented by a standard probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , where  $\Omega$  is a set of outcomes,  $\mathcal{A}$  is a sigma-algebra that represents the set of all measurable events and  $\mathbb{P}$  is a probability measure. Depending on the model considered, the time period can be either a discrete set of dates or a continuous interval. For each multi-period model, we implicitly assume the existence of a filtration  $(\mathcal{F}_t)_t$ , which is a collection of sigma-algebras indexed by time: the element  $\mathcal{F}_t$  is the information set available to the investor at date  $t$ , and expectations, variances and covariances computed conditional to this information are denoted by  $\mathbb{E}_t$ ,  $\mathbb{V}_t$  and  $\text{Cov}_t$ . We will often use column vectors of ones, which we denote by  $\mathbf{1}_N$ ,  $N$  being the vector size. Similarly, a column vector of  $N$  zeros is denoted as  $\mathbf{0}_N$ . The identity matrix of size  $N$  is denoted by  $\mathbf{I}_N$ .

The fundamental asset pricing equation is the one relating the price of a pay-off to its pay-off through the SDF. Using the same notations as Cochrane (2005), we let  $x_{t+1}$  denote a pay-off at date  $t + 1$ ,  $p_t$  its price at date  $t$  and  $m_{t+1}$  the SDF at date  $t + 1$ . The fundamental equality states that:

$$p_t = \mathbb{E}_t [m_{t+1} x_{t+1}]. \quad (2.1)$$

In words, Equation (2.1) means that the price of the pay-off equals the conditional expectation of the pay-off multiplied by the stochastic discount factor. Heuristically, if  $m_{t+1}$  is small in a state of the world, then the investor attaches little value to receiving

## 2. Risk Premia and the Stochastic Discount Factor in Asset Pricing Theory

the pay-off in this state, which contributes to decreasing the price for the pay-off that is given by an average of such outcomes of the product of the pay-off times the SDF. This mechanism is perhaps clearer if the equation is rewritten in the following equivalent form, where we have introduced the return,  $r_{t+1} = \frac{x_{t+1}-p_t}{p_t} = \frac{x_{t+1}}{p_t} - 1$ , and have used the decomposition of covariance:

$$\mathbb{E}_t[r_{t+1}] = \frac{1}{\mathbb{E}_t[m_{t+1}]} - 1 - \frac{1}{\mathbb{E}_t[m_{t+1}]} \text{Cov}_t[m_{t+1}, r_{t+1}]. \quad (2.2)$$

To the extent that  $m_{t+1}$  has positive expectation, it follows that an asset whose return covaries positively with the SDF will have a lower expected return than an asset whose return covaries negatively. That is, investors are ready to accept lower returns for assets that covary positively with the SDF.

If a risk-free asset exists, that is, an asset whose pay-off at date  $t + 1$  is known with certainty as of date  $t$ , Equation (2.2) can be rewritten in an equivalent form that involves the expected excess return. First, let us apply (2.2) to the risk-free asset. The certain rate of return,  $r_{dt}$  satisfies:

$$r_{dt} = \frac{1}{\mathbb{E}_t[m_{t+1}]} - 1.$$

This equation says that the price of a pure discount bond that will pay \$1 with certainty in the next period equals the expectation of the pricing kernel. The expected excess return, or *risk premium*, of an asset is defined as the expected return minus the risk-free rate. Subtracting  $r_{dt}$  from both sides of (2.2), we obtain that the expected excess return of the asset,  $\tilde{\mu}_t$ , is:

$$\begin{aligned} \tilde{\mu}_t &= \mathbb{E}_t[r_{t+1}] - r_{dt} \\ &= -\frac{1}{\mathbb{E}_t[m_{t+1}]} \text{Cov}_t[m_{t+1}, r_{t+1}]. \end{aligned} \quad (2.3)$$

This decomposition is general and can be applied to any asset. In words, it means that an asset earns a positive risk premium if, and only if, it covaries negatively with the SDF. We now provide more intuitive content to these results by providing a natural interpretation for the stochastic discount factor in a consumption-based model.

### 2.1.2 The Stochastic Discount Factor in a Consumption-Based Model

There are different routes to arrive at (2.1) or the equivalent expression (2.2). A consumption-based asset pricing model shows that this equation holds for any investor who maximises a time-separable utility function. Formally, we let  $u$  denote the utility function satisfying standard conditions (see for example Cochrane (2005) for details) and  $\delta$  the elasticity of intertemporal substitution. The investor chooses his consumption plan in order to achieve the maximum expected utility subject to the budget constraint, and the optimality condition derived from the program states that at the optimum, (2.2) holds with the SDF being given by:

$$m_{t+1} = \delta \frac{u'(c_{t+1})}{u'(c_t)}. \quad (2.4)$$

This relation is interesting because it relates the SDF to the marginal utility of consumption and gives a very intuitive content to (2.2). The "bad" states of the world are those where consumption is low, that is, marginal utility of consumption is high. Hence, an investor is ready to accept a lower expected return, i.e. pay a higher price, to hold an asset that pays off well in those bad states. Conversely, an asset that tends to pay off well when marginal utility of consumption is low is less appealing to investors, who thus require a higher expected return, i.e. a lower price, to purchase it.

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The same mechanics is at work in Equation (2.3): the risk premium of an asset is positive if, and only if, the asset covaries negatively with the marginal utility of consumption. This is a logic of insurance. Investors seek assets that help them smooth consumption across states of the world. The risk-free asset represents a neutral reference point because its pay-off is the same in all states. A risky asset that performs well when marginal utility is low and poorly when marginal utility is high is not particularly desirable for smoothing purposes, so investors demand a higher expected return than for the risk-free asset. On the other hand, if the risky asset performs well when marginal utility is high, investors find it attractive and are ready to accept a lower return than the risk-free rate.

2 - This equivalence holds in a discrete-time model, as here, but it no longer holds in a continuous-time model, where the existence of the SDF is not sufficient to exclude the possibility of an arbitrage (see Duffie (2001) Proposition 6.C and Dybvig and Huang (1988) for examples of conditions to be imposed in order to recover an equivalence).

While (2.2) is easiest to derive under the assumption of time-additive utility, it extends to much more general contexts. For instance, it obtains also if the investor has "recursive preferences" as defined in Epstein and Zin (1989): in this framework, Campbell (1993) shows that (2.2) holds, with  $m_{t+1}$  being a function of the consumption growth rate and the market portfolio return.

### 2.1.3 Other Conditions for the Existence of a Stochastic Discount Factor

It is well-known that one may derive the existence of an SDF without referring to individual optimisation, from assumptions on the formation of prices or the absence of arbitrage. A first assumption is the "law of one price", which states that the price to pay to acquire a portfolio of assets equals the sum of the constituent prices. In more formal terms, the pricing rule is linear. That is, if  $x_1$  and  $x_2$  are two pay-offs and  $a$  and  $b$  are two portfolio weights, the price of the pay-off  $ax_1 + bx_2$  is:

$$p(ax_1 + bx_2) = ap(x_1) + bp(x_2).$$

Under this assumption, Campbell et al. (1997) Chap. 8.1 provide an explicit construction of an SDF in a model with a finite number of states of the world. Cochrane (2005) Chap. 4.1 extends this result by relaxing the assumption of a finite state space, and he shows that under the law of one price, a unique pay-off  $x^*$  exists which is an SDF. This theorem neither says that the pay-off  $x^*$  is positive, nor that there exists a positive SDF. In order to obtain the existence of a positive SDF, Cochrane (2005) Chap. 4.2 imposes the additional restriction of no arbitrage opportunities: the law of one price and the condition of no arbitrage opportunities hold if, and only if, a positive SDF exists. A similar result can be found in Duffie (2001) (Theorem 2.C), who proves that the absence of arbitrage is equivalent to the existence of a positive SDF.<sup>2</sup> These existence theorems are very general, but they do not provide an explicit construction of the SDF, unlike a consumption-based model. In such a model, the SDF is given by (2.4): since marginal utility is negative, the SDF is positive, and there is no arbitrage.

### 2.1.4 Risk Premia and Systematic Risk

Equation (2.3) has an important implication regarding the pricing of "systematic" versus "idiosyncratic" risks. In this general model, the "systematic part" of the return is defined as the orthogonal projection of the return on the SDF and the "idiosyncratic part" as the residual of this projection, so that we have the following decomposition for the realised return:

$$r_{t+1} = \text{proj}_m[r_{t+1}] + \varepsilon_{t+1},$$

with  $\varepsilon_{t+1}$  being uncorrelated from  $m_{t+1}$ , and the orthogonal projection being given by:

$$\text{proj}_m[r_{t+1}] = \frac{\text{Cov}_t[m_{t+1}, r_{t+1}]}{\text{V}_t[m_{t+1}]} m_{t+1}.$$

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By (2.3), we thus have that:

$$\tilde{\mu}_t = -\text{Cov}_t[m_{t+1}, \text{proj}_m[r_{t+1}]].$$

As a result, the idiosyncratic component carries no risk premium, and the risk premium arises only from the part of the return that is perfectly correlated with the SDF. In other words, what matters for the risk premium is the systematic risk of the asset as defined with respect to the SDF, not the total risk. In particular, an asset whose return is uncorrelated from the SDF bears no risk premium, no matter how large its total risk might be.

### 2.2 Risk Premia in Factor Models

The expression of a risk premium given in (2.3) is of limited practical use because it involves the unobservable SDF. The consumption-based model allows us to relate the SDF to consumption, which is observable. Consumption, however, is not straightforward to measure since it is a macroeconomic aggregate only available at a relatively low frequency, and which is released with a lag and is subject to revisions. These limitations motivate the search for a *factor model*, that is a collection of factors which can be used as proxies for the SDF while having the advantage of being more easily observable than consumption. In this section, we give two general alternative (and equivalent) formulations of a factor model, and we define the notion of "factor risk premia", which determine the expected excess return on assets.

For notational simplicity, we now drop the subscripts  $t$  and  $t + 1$  from the notations of processes and expectation operators (except in Section 2.2.4 where this notation is needed to emphasise the difference between conditional and unconditional risk premia).

#### 2.2.1 Definition of a Factor Model

Given a  $K \times 1$  vector of factor values  $\mathbf{f}$ , a factor model postulates the existence of a scalar  $a$  and a  $K \times 1$  vector  $\mathbf{b}$  such the quantity defined by

$$m = a + \mathbf{b}'\mathbf{f} \quad (2.5)$$

is an SDF. A factor  $f_k$  that has a non-zero loading  $b_k$  is said to be a "pricing factor".

#### Beta Representation of Expected Returns

Cochrane (2005) shows that the formulation of the factor model in terms of the SDF is equivalent to a "beta representation" of expected returns. This important result is stated in the following proposition.

#### Proposition 1 (Beta Representation of Expected Returns)

Let  $\mathbf{f}$  be a  $K \times 1$  random vector. A scalar  $a$  and a  $K \times 1$  vector  $\mathbf{b}$  exist such that  $m = a + \mathbf{b}'\mathbf{f}$  prices all assets if, and only if, a scalar  $\kappa$  and a  $K \times 1$  vector  $\mathbf{\Lambda}$  exist such that for each asset  $i$ , the expected return is:

$$\mathbb{E}[r_i] = \kappa + \mathbf{\Lambda}'\boldsymbol{\beta}_i, \quad i = 1, \dots, N, \quad (2.6)$$

where the  $K \times 1$  vector  $\boldsymbol{\beta}_i$  is the vector of multivariate regression coefficients of  $r_i$  on  $\mathbf{f}$  with a constant.

**Proof.** The proof can be found in Chap. 6.3 of Cochrane (2005). For the sake of completeness, it is also written in Appendix A.1.

It follows from the proof of the proposition that the equations that allow one to switch between representations (2.5) and (2.6) are:

$$\kappa = \frac{1}{\mathbb{E}[m]} - 1,$$

$$\mathbf{\Lambda} = -\frac{1}{\mathbb{E}[m]}\text{Cov}[m, \mathbf{f}],$$

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$$a = \frac{1}{1 + \kappa} (1 + \mathbf{\Lambda}' \mathbf{\Sigma}_f^{-1} \boldsymbol{\mu}_f),$$

$$\mathbf{b} = -\frac{1}{1 + \kappa} \mathbf{\Sigma}_f^{-1} \mathbf{\Lambda},$$

where  $\mathbf{\Sigma}_f$  and  $\boldsymbol{\mu}_f$  are the covariance matrix and the vector of expected values of the factors. Note that if a risk-free asset exists, then  $\kappa$  equals  $r_d$ . More generally,  $\kappa$  can be regarded as the return on a zero-beta portfolio, as in the CAPM with no risk-free asset of Black (1972).

Proposition (1) is a very general result, in which factors can be any set of random variables, and are not required to be portfolio returns. It says that since the constant  $\kappa$  is the same for all assets, the differences in expected returns in the cross section result only from the differences in factor exposures. is called the zero-beta expected return. If a risk-free asset exists, then  $\kappa$  is equal to  $r_d$ , so that expected excess returns satisfy:

$$\mathbb{E}[r_i] - r_d = \mathbf{\Lambda}' \boldsymbol{\beta}_i, \quad i = 1, \dots, N, \quad (2.7)$$

In words, this means that the expected excess returns are linear functions of the betas.

### Factor-Mimicking Portfolios

In the general definition of a factor model, factors are not given as pay-offs or returns. But Cochrane (2005) Chap. 6.3 shows that it is possible to replace a given set of pricing factors by a set of pay-offs that carries the same pricing information. To this end, one searches for the minimum distance between a pay-off and the SDF:

$$\min_{P \in \mathcal{X}} \mathbb{E} [(P - m)^2] \quad (2.8)$$

where  $\mathcal{X}$  is the set of feasible pay-offs. The existence and the uniqueness of the

minimum can be established under abstract conditions on  $\mathcal{X}$  (e.g.  $\mathcal{X}$  is convex and closed), but in order to have an explicit expression for  $P$ , we make the assumption that  $\mathcal{X}$  is the vector space generated by a finite set of pay-offs  $\mathbf{X} = (x_1, \dots, x_N)'$ . The vector space assumption means in particular that short positions in the pay-offs are allowed, a condition that ensures that the minimisation problem has an explicit expression. A straightforward quadratic minimisation exercise shows that the pay-off that achieves the minimum distance with respect to the SDF is  $m^* = \mathbb{E}[m \mathbf{X}'] \mathbb{E}[\mathbf{X} \mathbf{X}']^{-1} \mathbf{X}$ . Substituting the equality  $m = a + \mathbf{b}' \mathbf{f}$ , we obtain the expressions for the constant-mimicking and the factor-mimicking pay-offs:

$$c^* = \mathbb{E}[\mathbf{X}]' \mathbb{E}[\mathbf{X} \mathbf{X}']^{-1} \mathbf{X},$$

$$f_k^* = \mathbb{E}[f_k \mathbf{X}]' \mathbb{E}[\mathbf{X} \mathbf{X}']^{-1} \mathbf{X},$$

$$k = 1, \dots, K. \quad (2.9)$$

$c^*$  and  $f_k^*$  are the orthogonal projections of the constant 1 and the random vector  $\mathbf{f}$  on the vector space of pay-offs. The following proposition then shows that  $m^*$  is an SDF.

### Proposition 2 (Factor-Mimicking Portfolios)

Assume that the set of pay-offs is the vector space generated by  $\mathbf{X} = (x_1, \dots, x_N)'$  and that  $m = a + \mathbf{b}' \mathbf{f}$  is an SDF,  $\mathbf{f}$  being a  $K \times 1$  vector of factors. Define the constant-mimicking and the factor-mimicking pay-offs as in (2.9). Then,  $m^* = ac^* + \mathbf{b}' \mathbf{f}^*$  is an SDF.

**Proof.** See Appendix A.2.

This proposition implies that there is no loss of generality from searching for pricing factors among pay-offs. Replacing the factor-mimicking pay-offs by the factor-mimicking gross returns

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$$R_c^* = \frac{c^*}{p(c^*)}, \quad R_k^* = \frac{f_k^*}{p(f_k^*)},$$

$$k = 1, \dots, K,$$

we see that it is equally general to search for factors among gross returns.<sup>3</sup>

### Test Assets

In the model, the factors price all  $N$  assets existing in the universe. In empirical work, not all assets are assumed to be observable, if only because their number is potentially infinite. Thus, the pricing equation (2.6) is tested on a selected set of "test assets". These test assets are often defined as portfolios of stocks or bonds sorted on an attribute (e.g. size, book-to-market, industry, rating, etc.).

### 2.2.2 Definition of Factor Risk Premia

As shown previously, the vector  $\Lambda$  is given by:

$$\Lambda = -\frac{1}{\mathbb{E}[m]} \text{Cov}[m, \mathbf{f}]. \quad (2.10)$$

This expression is formally similar to the expression (2.3) for an asset risk premium, with the asset return being replaced by the vector of factors. Thus, it is natural to call  $\Lambda$  the vector of *factor risk premia*.

### Definition of a Rewarded (or Priced) Factor

A factor  $f_k$  is said to be rewarded (or priced) if it carries a non-zero premium, that is if  $\Lambda_k \neq 0$  in (2.10). By the beta representation of expected returns, it is equivalent to say that a factor is rewarded or that other things (i.e. other betas) being equal, differences in exposure to this factor imply differences in expected returns. It should be noted that the definition of a rewarded factor does not assume that the factor premium is positive. If the premium is negative, then it is the assets with the lowest betas that have the highest expected returns.

Moreover, the parallel between the definitions of asset risk premia in (2.3) and factor risk premia in (2.10) shows that the intuitions behind the sign of risk premia are the same for factors and assets. A factor commands a positive premium if, and only if, it covaries negatively with the SDF. In a consumption-based model, it is equivalent to say that the factor premium is positive if, and only if, the factor is high in "good times" (when marginal utility is low) and low in "bad times" (when marginal utility is high). If a factor has zero correlation with marginal utility, it carries a zero premium. Thus, if marginal utility is not observable but the factor is and the sign of the premium is known, it is possible to design a rule of thumb to guess whether the economy is in a good or a bad state. For a positively rewarded factor, a high factor value is likely to signal good times, while observing a low factor value is suggestive of bad times. For a negatively rewarded factor, the rule is reversed.

Another way to interpret (2.10) is that if investors see a rise in the factor as an unfavourable event, i.e. if the factor tends to rise at the same time as marginal utility, then they are ready to accept a negative premium for this factor. Indeed, the insurance motive makes the case for a long exposure to the factor. On the other hand, if a rise in the factor signals a favourable event, i.e. if the factor tends to decrease when marginal utility decreases, then a positive premium is required to justify a long exposure to the factor.

### Pricing Factors versus Priced Factors

As Cochrane (2005) Chap. 13.4 points out, it is important to make a distinction between the fact that the coefficient  $b_k$  of factor  $k$  in  $m = \mathbf{a} + \mathbf{b}'\mathbf{f}$  is non-zero and the fact that  $\Lambda_k$  is not zero.  $b_k$  is the coefficient of factor  $k$  in the *multivariate* regression of

3 - The gross return to a portfolio with value  $p_t$  at date  $t$  and pay-off  $x_{t+1}$  at  $t+1$  is defined as

$$R = \frac{x_{t+1}}{p_{t+1}}.$$

It is related to the return through  $R = r + 1$ .

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$m$  on the factors, and the fact that it is non-zero means that the factor helps to price assets given the presence of the other factors. On the other hand, as appears from (2.10),  $\lambda_k$  is proportional to the *univariate* beta of  $m$  with respect to the factor, and a non-zero  $\lambda_k$  means that the factor is rewarded. When factors are uncorrelated from each other, it is equivalent to test whether a factor is relevant for pricing or to test whether it is rewarded, but in the presence of correlation, the two concepts are not equivalent. For instance, any random variable that has non-zero covariance with the SDF may appear as a "priced factor". But this factor is spurious if it has zero loading in the SDF. In what follows, we define a "priced factor" as a factor whose risk premium is non zero, while a "pricing factor" is a factor that has a non-zero coefficient  $b_k$  in  $m = a + \mathbf{b}'\mathbf{f}$ .

### 2.2.3 Factor Premia when Factors are Returns or Excess Returns

We now show that when factors are themselves portfolio returns or excess returns, factor risk premia can be expressed as expected excess returns. Campbell et al. (1997) Chap. 6.3 discuss the implications of these relations for the estimation of factor premia.

#### *Factors as Returns*

Suppose first that a factor  $f_k$  is the return on a portfolio of assets with positive initial value. Then, we have  $\mathbb{E}[mf_k] = 1 - \mathbb{E}[m]$ , and the factor premium is:

$$\Lambda_k = -\frac{1}{\mathbb{E}[m]} \text{Cov}[m, f_k] = \mathbb{E}[f_k] - \left( \frac{1}{\mathbb{E}[m]} - 1 \right).$$

If a risk-free asset exists, it must be the case that the risk-free rate is  $r_d = 1/\mathbb{E}[m] - 1$ , so that  $\Lambda_k$  equals  $\mathbb{E}[f_k] - R_f$ . This is the expected excess return on the portfolio. Hence, denominations are consistent: if a

factor is a portfolio return, its premium equals the risk premium as defined in Section 2.1.1. If there is no risk-free asset, the factor premium is the difference between the expected factor return and the zero-beta return.

#### *Factors as Dollar-Neutral Returns*

A dollar-neutral factor is the pay-off to a portfolio with a zero price. Hence, it satisfies by definition  $\mathbb{E}[mf_k] = 0$ . The factor premium is thus given by:

$$\Lambda_k = -\frac{1}{\mathbb{E}[m]} \times (-\mathbb{E}[m]\mathbb{E}[f_k]) = \mathbb{E}[f_k],$$

and is then simply equal to the expected value of the factor.

This situation is common in empirical work. Following the work of Fama and French (1993), many empirical factors are defined as the excess return of some portfolio – known as the long leg – over another portfolio – known as the short leg. The premia on such a factor equals the expected excess return of the long leg over the short leg.

If all factors are returns or dollar-neutral returns, then define the  $K \times 1$  vector  $\tilde{\mathbf{f}}$  as:

$$\tilde{f}_k = \begin{cases} f_k - r_d & \text{if } f_k \text{ is a return,} \\ f_k & \text{if } f_k \text{ is a dollar-neutral return} \end{cases}$$

We call  $\tilde{\mathbf{f}}$  the vector of factor excess returns. By what precedes, we have that  $\mathbb{E}[\tilde{\mathbf{f}}] = \Lambda$ , hence (2.7) implies that:

$$\mathbb{E}[r_i - r_d] = \mathbb{E}[\beta_i' \tilde{\mathbf{f}}]$$

Hence, in the regression of the security excess returns over the factor excess returns:

$$r_i - r_d = \alpha_i + \beta_i' \tilde{\mathbf{f}} + \varepsilon_i,$$

the intercept  $\alpha_i$  is zero.

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### 2.2.4 Time-Varying Premia

The definition of factor premia in Section 2.2.2 can be adapted to both single-period and multi-period settings. In a multi-period framework, it reads:

$$\Lambda_t = -\frac{1}{\mathbb{E}_t[m_{t+1}]} \text{Cov}_t[m_{t+1}, \mathbf{f}_{t+1}],$$

where  $m_{t+1}$  is the SDF, which by definition allows us to express prices at date  $t$  as functions of pay-offs at date  $t + 1$  (see (2.1)). This equation defines the "conditional factor premia".

One can also define "unconditional premia" by replacing conditional expectation and covariance by unconditional moments:

$$\bar{\Lambda}_t = -\frac{1}{\mathbb{E}[m_{t+1}]} \text{Cov}[m_{t+1}, \mathbf{f}_{t+1}].$$

The index  $t$  in the notation of the unconditional premium can be dropped if the SDF and the factors are stationary in the sense that their first two moments are constant over time. It should be noted that the unconditional premium does not always equal the unconditional expectation of the conditional premium. This equality holds when factors are dollar-neutral because the factor premia are then simply the expected factor values (see Section 2.2.3). Indeed, we have then  $\Lambda_t = \mathbb{E}_t[\mathbf{f}_{t+1}]$  and  $\bar{\Lambda}_t = \mathbb{E}[\mathbf{f}_{t+1}]$ , so that  $\bar{\Lambda}_t = \mathbb{E}[\Lambda_t]$ .

### 2.3 Examples of Theoretical Factor Models

The fundamental question raised in a factor model is: what is (are) the true pricing factor(s)? We start by giving examples that do not require any economic assumption, and we then review the two main classes of theoretical factor models, which provide two types of answers to the previous question: equilibrium models (CAPM, ICAPM, CCAPM) derive a beta representation

of expected returns of the form (2.6) from assumptions on investors' preferences and a market clearing condition, while the Arbitrage Pricing Theory (APT) emphasises the importance for factors to explain returns from a statistical standpoint.

#### 2.3.1 "Model-Free" Examples

We first note the existence of sets of variables that always act as pricing factors, regardless of assumptions made on individual behaviour, market equilibrium or statistical properties of asset returns. We refer to these pricing factors as "model-free" factors. All of the models enumerated below have either one factor only – and are thus extremely parsimonious – or as many factors as risky assets to price. The theoretical multi-factor models developed in the literature are located between these two extremes.

##### *SDF as Single Factor*

This example is trivial: it is always possible to construct a one-factor model by taking  $m$  as the single pricing factor, in which case  $\mathbf{a} = 0$  and  $b = 1$ .

##### *Asset Returns as Factors*

If there are  $N - 1$  risky (i.e. with a non-zero variance) securities to price and a risk-free asset exists, it is possible to construct a  $(N - 1)$ -factor model. To see this point, it suffices to use Proposition 1. With  $\mathbf{f} = \mathbf{r}$ , the factor premia are the asset excess returns, that is  $\Lambda = \tilde{\boldsymbol{\mu}}$ , and the vector of multivariate betas of asset  $i$  is the  $i^{\text{th}}$  canonical vector  $\mathbf{e}_i$ . We thus have:

$$\begin{aligned} \mathbb{E}[r_i] &= r_d + \mathbf{e}_i' \tilde{\boldsymbol{\mu}}, \\ \mathbb{E}[r_d] &= r_d + \mathbf{0}'_N \tilde{\boldsymbol{\mu}}. \end{aligned}$$

By Proposition 1, these equalities imply that  $\mathbf{r}$  is a vector of pricing factors. Specifically, the SDF is given by:

$$m = \frac{1}{1 + r_d} [1 - \tilde{\boldsymbol{\mu}}' \boldsymbol{\Sigma}^{-1} (\mathbf{r} - \boldsymbol{\mu})], \quad (2.11)$$

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where  $\Sigma$  and  $\mu$  are the covariance matrix and the vector of expected returns of the risky assets, and  $\tilde{\mu}$  is the vector of expected excess returns.

### Return of MSR Portfolio of Assets as Single Factor

Assume again that there are  $N - 1$  risky assets and a risk-free asset. It is well known that the weights of the long-short maximum Sharpe ratio (MSR) portfolio are:

$$\mathbf{w}_{MSR} = \frac{\Sigma^{-1}\tilde{\mu}}{\mathbf{1}'_N \Sigma^{-1}\tilde{\mu}}.$$

Substituting this equality in (2.11), we obtain the existence of an SDF given by:

$$m = \frac{1}{1 + r_d} [1 - \nu(r_{MSR} - \mu_{MSR})],$$

where  $\nu = \mathbf{1}'_N \Sigma^{-1}\tilde{\mu}$ ,  $r_{MSR} = \mathbf{w}'_{MSR}\mathbf{r}$  is the realised return of the MSR and  $\mu_{MSR}$  is its expected return. Hence, the return of the MSR portfolio of the risky assets can be taken as the single pricing factor. Note that any combination (possibly leveraged) of the MSR portfolio and the risk-free asset is a pricing factor too.

There is also a converse implication, which states that if a portfolio return prices all assets, then it maximises the Sharpe ratio. The following proposition summarises the equivalence between Sharpe ratio maximisation and the search for a portfolio return pricing assets.

### Proposition 3 (MSR Portfolio as the Unique Pricing Portfolio)

Assume that there are  $N - 1$  risky assets and a risk-free one, and consider a portfolio  $\mathbf{w}_0$  with return  $r_0$  and expected return  $\mu_0$  strictly greater than the risk-free rate. Then:

- If an SDF with  $r_0$  exists as the single pricing factor, then  $\mathbf{w}_0$  has the maximum Sharpe ratio over

all feasible portfolios;

- If  $\mathbf{w}_0$  is the long-short MSR portfolio, then an SDF exists with  $r_0$  as the single pricing factor.

**Proof.** See Appendix A.3.

### MSR Portfolio of Factor Indices as Single Factor

Assume now the existence of a  $K \times 1$  vector of pricing factors  $\mathbf{f}$  whose elements are portfolio returns or dollar-neutral returns. As shown in Section 2.2.2, the vector of factor risk premia is the vector of expected excess factor returns:

$$\Lambda = \tilde{\mu}_f,$$

and Proposition 1 and the formulas given in Section 2.2.1 imply the existence of an SDF given by:

$$m = \frac{1}{1 + r_d} \left[ 1 - \tilde{\mu}'_f \Sigma_f^{-1} (\mathbf{f} - \mu_f) \right].$$

The vector  $\Sigma_f^{-1}\tilde{\mu}_f$  is proportional to the weights of the long-short MSR portfolio of the factors, so the return to this portfolio is a pricing factor.

### 2.3.2 Static CAPM

The Capital Asset Pricing Model of Sharpe (1964) and Lintner (1965) is an equilibrium model in which the only pricing factor is the excess return on the "market portfolio". Hence, the model predicts that the expected excess return on an asset is proportional to its market beta.

#### Model Assumptions

The CAPM builds on the seminal work by Markowitz (1952) on the construction of mean-variance efficient portfolios. It essentially makes five classes of assumptions:

1. Frictionless markets: in particular, leverage is allowed without limit;
2. Existence of a risk-free asset;

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3. Homogeneity of investors' horizon, expectations and preferences: they have the same horizon, the same beliefs regarding the distribution of asset returns, and maximise a quadratic utility function, though they do not necessarily have identical risk aversion levels;
4. Static portfolios: the agents invest for one period only and cannot rebalance over this period;
5. Market is in equilibrium: the aggregate holdings for the shares of an asset by economic agents equal the supply for this asset.

### *Expressions of Asset Risk Premia and Stochastic Discount Factor*

Under these assumptions, the expected return on an asset is given by:

$$\mathbb{E}[r_i] = r_d + \beta_{im}(\mathbb{E}[r_m] - r_d), \quad (2.12)$$

$R_f$  being the risk-free rate,  $R_m$  the return on the market portfolio and  $\beta_{im}$  the beta of the asset with respect to the market. By Proposition 1, this implies the existence of an SDF given by:

$$m = a + bf,$$

where the factor is the centred excess market return,  $f = r_m - r_d - \tilde{\mu}_m$ , and the coefficients  $a$  and  $b$  are given by:

$$a = \frac{1}{1 + r_d}, \quad b = -\frac{1}{1 + r_d} \frac{\tilde{\mu}_m}{\tilde{\sigma}_m^2}.$$

In these equations,  $\tilde{\mu}_m$  and  $\tilde{\sigma}_m$  denote respectively the expectation and the volatility of the excess market return.

Black (1972) develops a more general version of the model, by relaxing the assumption that a risk-free asset exists. In this model, the risk-free rate is replaced by the expected return on the "zero-beta portfolio", and the pricing factor is the excess return on the market portfolio over the zero-beta portfolio.

### 2.3.3 Merton (1973)'s Intertemporal CAPM

The ICAPM of Merton (1973) relaxes several assumptions of the static CAPM, by assuming notably that investors can rebalance their portfolios and that investment opportunities vary over time, and by allowing for more heterogeneous horizons and preferences across investors. Its pricing implication is that asset risk premia are linear functions of the market beta and betas with respect to the factors that drive changes in the opportunity set. Hence, the market factor is no longer the only pricing factor.

### *Model Assumptions*

Samuelson (1969), Merton (1969) and Merton (1971) laid the grounds for the theory of optimal portfolio choice in a multi-period setting. The ICAPM is cast in a continuous-time framework and assumes that each investor maximises an expected utility from future consumption and terminal wealth at horizon  $T$ :

$$\max_{c, w} \mathbb{E} \left[ \int_0^T e^{-\delta t} u(c_t) dt + U(W_T) \right], \quad (2.13)$$

subject to the intertemporal budget constraint

$$dW_t = W_t \sum_{i=0}^N w_{it} \frac{dS_{it}}{S_{it}} - c_t dt,$$

with  $S_{it}$  being the price of asset  $i$ ,  $w_{it}$  its weight in the portfolio and  $c_t$  consumption rate per unit of time. By convention, asset 0 is the locally risk-free asset, i.e. the asset that earns the continuously compounded risk-free rate  $r_{dt}$ . The other  $N$  assets are said to be "locally risky".

Each risky asset is assumed to follow a diffusion process of the form:

$$\frac{dS_{it}}{S_{it}} = \mu_{it} dt + \sigma_{it} dz_{it},$$

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where  $z_i$  is a standard Wiener process and  $\mu_{it}$  and  $\sigma_{it}$  are the expected return and the volatility over the next infinitesimal period,  $[t, t + dt]$ , conditional on the information available at date  $t$ . These risk and return parameters can vary stochastically over time, but the key assumption in the derivation of the beta representation of expected returns is that they depend on a finite number of state variables  $X_1, \dots, X_{K-1}$ .<sup>4</sup> This means that all the uncertainty in the evolution of expected returns and volatilities is captured by a finite number of factors. This implies in particular that the risk-free rate itself is a function of the state variables. The main assumptions of the ICAPM can then be summarised as follows:

1. Frictionless markets: in particular, short sales and leverage are allowed;
2. Existence of a locally risk-free asset;
3. Homogeneity of investors' expectations: investors have the same beliefs regarding the distribution of asset returns. However, they do not necessarily have the same horizon and utility function when they maximise an objective of the form (2.13);
4. Dynamic portfolios: portfolios can be rebalanced continuously (this assumption is linked to that of frictionless markets);
5. Market is in equilibrium.

### Expression of Asset Risk Premia and Stochastic Discount Factor

Under the previous assumptions, Merton (1992) Chap. 15 and Breeden (1979) show that the expected excess return on asset  $i$  can be written as:

$$\mu_{it} - r_{dt} = \beta_{imt}(\mu_{mt} - r_{dt}) + \sum_{k=1}^{K-1} \beta_{ikt}(\mu_{Xkt} - r_{dt}). \tag{2.14}$$

where  $\mu_{mt}$  is the expected return on the market portfolio,  $\mu_{Xkt}$  is the expected return on the portfolio that maximises the squared correlation with the  $k^{th}$  state variable,

and where the expression for the portfolio weights is:

$$w_{kt} = \frac{1}{\mathbf{1}'\Sigma_t^{-1}\Sigma_{kt}}\Sigma_t^{-1}\Sigma_{kt},$$

with  $\Sigma_t$  being the (instantaneous) covariance matrix of the assets and  $\Sigma_{kt}$  being the vector of covariances between assets and the  $k^{th}$  variable. This portfolio is called a "hedging portfolio". Finally, the betas that appear in (2.14) are the multivariate betas obtained by regressing asset returns on the excess returns to the market portfolio and the  $K - 1$  hedging portfolios, with a constant.

By (2.14), the ICAPM implies a factor model where the  $K$  factors are the excess returns on the market portfolio and  $K - 1$  "mutual funds", dedicated to hedging against unfavourable changes in investment opportunities. The factor risk premia are the expected excess returns on these hedging portfolios.

We now write the expression of an SDF as a function of the factors. The formalism of the continuous-time framework requires we slightly modify the definition of an SDF with respect to the discrete-time setting. Here, an SDF is defined as any positive process  $(M_t)_{0 \leq t \leq T}$  such that for all  $i = 0, \dots, N$ , the discounted price process  $M_t S_{it}$  follows a martingale. In other words, for any asset  $i$  and for any two dates  $s < t$ , we have  $\mathbb{E}_s[M_t S_{it}] = M_s S_{is}$ . In the context of the model, we can show the existence of an SDF given by:

$$\frac{dM_t}{M_t} = -r_{dt} dt + b_{mt}[dr_{mt} - \mu_{mt} dt] + \mathbf{b}'_{Xt}[d\mathbf{r}_{Xt} - \boldsymbol{\mu}_{Xt} dt], \tag{2.15}$$

where  $dr_{mt}$  is the return on the market portfolio,  $\mu_{mt}$  is its expected return,  $\mathbf{r}_{Xt}$

4 - A technical but important condition is that the state vector should follow a Markov process.

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is the  $(K - 1) \times 1$  vector of returns on the hedging portfolios,  $\boldsymbol{\mu}_{Xt}$  is the vector of their expected returns and the coefficients  $(b_{mt}, \mathbf{b}_{Xt})$  are given by:

$$\begin{pmatrix} b_{mt} \\ \mathbf{b}_{Xt} \end{pmatrix} = -\boldsymbol{\Sigma}_{mXt}^{-1} \begin{pmatrix} \mu_{mt} - r_{dt} \\ \boldsymbol{\mu}_{Xt} - r_{dt}\mathbf{1}_{K-1} \end{pmatrix},$$

where  $\boldsymbol{\Sigma}_{mXt}$  is the covariance matrix of the  $K$  factors (excess market return and excess returns on  $K - 1$  hedging portfolios).

Another expression for asset risk premia can be written, under the assumption that a representative investor exists. Let  $J(t, W, \mathbf{X})$  denote the indirect utility of this investor, i.e. the maximum of the objective function in (2.13). Then, it can be shown that:

$$\mu_{it} - r_{dt} = \gamma_t \sigma_{imt} + \sum_{k=1}^{K-1} \gamma_{Xk,t} \sigma_{iXk,t}, \quad (2.16)$$

where  $\sigma_{imt}$  and  $\sigma_{iXk,t}$  are the covariances of the asset with the market and the  $k^{\text{th}}$  hedging portfolio, and the coefficients  $\gamma$  and  $\gamma_{Xk}$  are linked to the partial derivatives of the indirect utility function through:

$$\gamma_t = -\frac{J_{WW}W}{J_W}, \quad \gamma_{Xk,t} = \frac{J_{WXk}}{J_W}.$$

By definition,  $\gamma_t$  is the relative risk aversion of the investor at the optimum. Thus, the ICAPM implies two expressions for expected excess returns on assets. The difference between (2.14) and (2.16) is that the former uses the *multivariate* betas of the asset with respect to the  $K$  factors, while the latter involves the *univariate* betas, through the covariances between the asset and the factors.

For each factor  $f_k$ , where  $k = 1, \dots, K$ , the relationship between the factor risk

premium and the SDF is:

$$\begin{aligned} \Lambda_{kt} &= \mu_{kt} - r_{dt} \\ &= -\frac{1}{dt} \text{Cov}_t \left[ df_{kt}, \frac{dM_t}{M_t} \right]. \end{aligned} \quad (2.17)$$

This relation is similar to the definition of factor risk premia in discrete time (see (2.10)). As a result, the same rule applies: a factor is positively rewarded if, and only if, it covaries negatively with the SDF.

### 2.3.4 Campbell (1993)'s Intertemporal CAPM

Campbell (1993) (hereafter C93) develops an alternative version of the ICAPM, which differs from Merton's model in several assumptions enumerated below. Instead of relating the risk premium on an asset to its covariance with factors that drive the opportunity set, the pricing equation states that the risk premium depends on the covariances with the market and the news about future market returns.

#### Model Assumptions

The main assumptions of C93's model can be summarised as follows:

1. Frictionless markets;
2. Existence of a locally risk-free asset;
3. Existence of a representative investor, who maximises a recursive utility function of consumption of the form

$$\max_{c, w} U_t,$$

where

$$U_t = \left[ (1 - \beta) c_t^{\frac{1-\gamma}{\theta}} + \beta (\mathbb{E}_t[U_{t+1}])^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}}$$

and  $\gamma$  is the relative risk aversion;

4. The portfolio is rebalanced in discrete time;
5. The ratio of aggregate wealth to aggregate consumption has small variance.

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Hence, the differences with respect to Merton's ICAPM are as follows. First, the model is written in discrete, as opposed to continuous, time. In discrete time, the budget constraint is no longer linear in consumption and wealth (see Equation (1) in C93), but the last assumption is used by C93 to perform an approximate linearisation of the constraint, and to derive an approximate analytical solution to the optimal consumption and portfolio choice problem. C93 also provides a study of the accuracy of this "log-linear" approximation. The second difference with respect to Merton's model is that investors have recursive, as opposed to time-separable, preferences over consumption (see Epstein and Zin (1989)). Third, the model does not require the existence of a finite set of state variables, which capture all the uncertainty in risk and return parameters.

### Expression of Asset Risk Premia

As explained by C93, the pricing equation is more easily obtained if one assumes that the variances and covariances of asset returns and consumption are constant. The equation is written in terms of logarithmic returns. For simplicity, let us use here the same symbols as before, namely  $\mu_{it}$  and  $r_{dt}$ , but with a slightly different meaning:  $\mu_{it}$  is the expected logarithmic return on asset  $i$  and  $r_{dt}$  is the logarithmic return on the risk-free asset. Then, the first form of the pricing equation is:

$$\mu_{it} - r_{dt} \approx -\frac{\sigma_{it}^2}{2} + \theta \frac{\sigma_{ic,t}}{\varphi} + (1 - \theta)\sigma_{im,t},$$

where  $\varphi$  is the elasticity of intertemporal substitution and, which is related to  $\theta$  through:

$$\theta = \frac{1 - \gamma}{1 - \frac{1}{\varphi}},$$

and  $\sigma_{ic,t}$  and  $\sigma_{im,t}$  are the conditional covariances of asset  $i$  with respect to consumption and aggregate wealth. The

quantity  $\mu_{it} + \frac{\sigma_{it}^2}{2}$  is an approximation to the expected arithmetic return. Thus, the cross section of expected returns is explained by the covariances with aggregate consumption and wealth, that is, consumption and wealth are the pricing factors.

The second form of the pricing equation eliminates consumption as a pricing factor and makes use of the changes in the expectations of future market returns (note that the return to the market portfolio equals the return to aggregate wealth in this model). In detail, the new factor is defined as:

$$h_{t+1} = (\mathbb{E}_{t+1} - \mathbb{E}_t) \left[ \sum_{j=1}^{\infty} \rho^j r_{m,t+1+j} \right],$$

where  $r_{m,t+1+j}$  denotes the (logarithmic) market return and  $\rho$  is defined as the expectation of the logarithmic ratio of consumption to wealth:

$$\rho = \mathbb{E}[\log c_t - \log W_t].$$

Then, an asset risk premium can be expressed as:

$$\mu_{it} - r_{dt} \approx -\frac{\sigma_{it}^2}{2} + \gamma\sigma_{im,t} + (\gamma - 1)\sigma_{ih,t}, \tag{2.18}$$

where  $\sigma_{ih,t}$  is the conditional covariance with  $h_{t+1}$ . Thus, the market return and the revisions in expectations of future market returns,  $h_{t+1}$ , work as pricing factors.

An asset that covaries positively with  $h_{t+1}$  may be desirable for investment purposes because it pays off well when expectations about future market returns are revised upwards, that is when investment prospects are good. On the other hand, it may be undesirable for hedging purposes because it does not provide a hedge against the risk of worsening investment opportunities.

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C93 notes that the case  $\gamma = 1$  is the frontier between the two cases. When  $\gamma < 1$ , it is the former effect that dominates, and investors accept lower expected returns for assets that have higher covariances with the factor  $h_{t+1}$ . When  $\gamma > 1$ , the opposite holds, and a higher asset risk premium is required.

Finally, C93 extends the pricing equation to the case of time-varying second-order moments and reviews special cases in which consumption is not needed as a pricing factor. For instance, if the conditional variances and covariances of the market return and the random variable  $h_{t+1}$  are affine functions of the expected market return, then:

$$\begin{aligned} \mu_{it} - r_{dt} \approx & -\frac{\sigma_{it}^2}{2} + \gamma\sigma_{im,t} \\ & + \left[ \gamma - 1 - \frac{\theta\psi}{\varphi} \right] \sigma_{ih,t}, \end{aligned}$$

$\psi$  being a coefficient that depends on the model parameters. This equation is similar to (2.18), with a modified coefficient before the covariance  $\sigma_{ih,t}$ .

### 2.3.5 Breeden (1979)'s Consumption-Based CAPM

The original ICAPM requires the identification of the variables that drive changes in the opportunity set. Breeden (1979) introduces a "Consumption-Based CAPM" (CCAPM) that replaces the multiple betas in the decomposition of expected returns (2.14) by a single beta, which is the beta with respect to changes in aggregate consumption. This model can be regarded as an equilibrium version with multiple investors of the consumption-based model of Section 2.1.2, where the SDF is derived from the optimality conditions for a single agent.

### Model Assumptions

The assumptions are those of Merton's ICAPM.

### Expression of Asset Risk Premia and Stochastic Discount Factor

Breeden (1979) shows that expected excess returns are given by:

$$\begin{aligned} \mu_{it} - r_{dt} = & \frac{\mu_{Ct} - r_{dt}}{\beta_{CCt}} \beta_{iCt}, \\ & i = 1, \dots, N, \end{aligned}$$

where  $\beta_{iCt}$  is the asset's beta with respect to aggregate consumption  $C$ ,  $\beta_{CCt}$  is the beta of the portfolio that maximises the squared correlation with aggregate consumption changes, and  $\mu_{Ct}$  is the expected return on this portfolio. The relation further simplifies if one assumes the existence of a portfolio perfectly correlated with consumption changes, in which case we obtain:

$$\begin{aligned} \mu_{it} - r_{dt} = & (\mu_{Ct} - r_{dt}) \beta_{iCt}, \\ & i = 1, \dots, N, \end{aligned}$$

where the beta is measured directly with respect to consumption changes. Then, an SDF exists given by:

$$\begin{aligned} \frac{dM_t}{M_t} = & -r_{dt} dt \\ & + \frac{\mu_{Ct} - r_{dt}}{\sigma_{Ct}^2} [dr_{Ct} - \mu_{Ct} dt], \end{aligned}$$

where  $dr_{Ct}$  is the return on the perfect consumption-hedging portfolio.

As emphasised by Breeden and by Merton (1992) Chap. 15, the sign of a risk premium is determined by the covariance of the asset with aggregate consumption, not aggregate wealth. This is because when investment opportunities are stochastic, the marginal utility from wealth is not only a function of wealth, but also of current

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opportunities. Unlike in the constant opportunity case, a low wealth does not always imply a high marginal utility from an additional dollar, because investment prospects may be poor. But the "envelope theorem", which is a consequence of the optimality conditions in the program (2.13), states that marginal utilities of wealth and consumption are equal at the optimum, that is  $J_W = u'(c)$ , where  $J(W, X, t)$  is the indirect utility. Since the function  $u$ , unlike  $J$ , is independent from the state variables, the level of consumption alone is sufficient to determine the marginal utility from wealth. This explains why it is consumption, not wealth, that matters for the determination of risk premia. As in the model of Section 2.1.1, investors seeking to smooth their consumption seek assets that pay off well when the marginal utility from consumption is high, that is, assets that covary negatively with consumption. An asset that covaries positively with consumption does not provide a hedge against the risk of a decrease in consumption, and investors therefore require a positive premium to hold it as a reward for the risk involved.

### 2.3.6 Arbitrage Pricing Theory

The APT of Ross (1976) attempts to derive expected returns without the economic restrictions imposed by the CAPM or the ICAPM (e.g. market equilibrium and a finite number of risk factors driving the opportunity set). Instead, the model postulates the existence of a factor structure in returns, as well as (when returns are not entirely explained by factors) the existence of a large number of assets that allow investors to diversify away specific risk.

#### Model Assumptions

The CAPM and the ICAPM imply a representation of the SDF as an affine combination of factors. In contrast, the

APT focuses on a factor decomposition for returns. Its starting point is a statistical description of returns or prices as linear combinations of  $K$  common factors and an idiosyncratic term.

If  $x_i$  denotes the pay-off of an asset, then we have:

$$x_i = \mathbb{E}[x_i] + \beta'_i \mathbf{f} + \varepsilon_i, \quad i = 1, \dots, N, \quad (2.19)$$

where the factors are centred and the idiosyncratic return is uncorrelated from the factors. At this level of generality, such a decomposition always holds since it is always possible to regress a pay-off on a given set of factors. The restriction imposed is that the residuals "i should be uncorrelated across assets. As a result, the covariance matrix of pay-offs has the following decomposition:

$$\Sigma = \beta' \Sigma_f \beta + \Sigma_\varepsilon, \quad (2.20)$$

where  $\Sigma_\varepsilon$  is a diagonal matrix,  $\Sigma_f$  is the factor covariance matrix and  $\beta$  is the  $K \times N$  matrix of betas.

#### Expressions for Asset Risk Premia and Stochastic Discount Factor

Suppose first that there is no idiosyncratic risk, that is  $\varepsilon_i = 0$  in (2.19). Then, any pay-off  $x_i$  can be replicated as a portfolio of factors and a constant pay-off. Define  $\hat{\mathbf{f}}$  to be the vector

$$\begin{pmatrix} 1 \\ \mathbf{f} \end{pmatrix}$$

and assume that well-defined prices exist for the factors and the constant,  $p(\mathbf{f})$  and  $p(1)$ . Cochrane (2005) Chap. 9.4 shows that under the law of one price, the quantity

$$m = p(\hat{\mathbf{f}})' \mathbb{E}[\hat{\mathbf{f}} \hat{\mathbf{f}}']^{-1} \hat{\mathbf{f}}$$

is an SDF, that is, the price of each pay-off  $x_i$  satisfies  $p(x_i) = \mathbb{E}[m x_i]$ .

Given the existence of an SDF which is an affine combination of the factors, the

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connection between expected returns and factor premia is the same as in any factor model. In detail, we have:

$$\mathbb{E}[r_i] = \kappa + \mathbf{\Lambda}'\beta_i,$$

where  $\mathbf{\Lambda} = -\frac{1}{\mathbb{E}[m]}\mathbb{E}[m\mathbf{f}]$  is the vector of factor risk premia and  $\kappa = \frac{1}{\mathbb{E}[m]} - 1$ . If a risk-free rate exists, then we have  $\kappa = r_d$ .

But an exact factor structure is a very restrictive assumption. It is surely satisfied if one takes as many factors as test assets and one defines the factors as linear combinations of the returns, but as pointed by Connor and Korajczyk (1993), this approach is not parsimonious because it assumes that any factor that impacts the return of an asset potentially matters for all other assets. More economically plausible is the assumption of an approximate factor structure. If there are non-zero residuals in (2.19), then the pay-off does not exactly coincide with a portfolio of the factors  $\hat{\mathbf{f}}$ . But if the residuals are "not too large", the price of the pay-off is close to the price of the factor portfolio. Formally, we have  $x_i = Fx_i + \varepsilon_i$  where  $F(x_i) = \mathbb{E}[x_i] + \beta_i'\hat{\mathbf{f}}$  if is the pay-off of the factor portfolio. Cochrane (2005) Chap. 9.4 shows that given an SDF  $m$  that prices the factors (i.e. such that  $\mathbb{E}[m\hat{\mathbf{f}}] = p(\hat{\mathbf{f}})$ , the price  $p(x_i)$  converges to the price  $p(F(x_i))$  if the variance of the residual shrinks to zero. Hence, pay-offs that are close to factor portfolios (in the sense that the  $R^2$  of the regression on the factors is high) can be approximately priced as factor portfolios. Cochrane (2005) proves that this property is verified in particular for equally-weighted portfolios that contain a large number of assets.

### 2.3.7 Affine Term Structure Models

In the bond universe, a special class of models has been developed for the computation of no-arbitrage prices. These

"term structure models" exploit the existence of no-arbitrage conditions between bonds of various maturities. They work as relative pricing models in the sense that they take the SDF as given. Hence, unlike in the CAPM, ICAPM or APT, the specification of the SDF is part of the model assumptions, rather than a consequence of these assumptions. Among them, the class of affine models has been the focus of a large theoretical and empirical work. Although the early models of the term structure of Vasicek (1977) and Cox et al. (1985) fit into this category, a systematic study of the affine class was not available until the work of Duffie and Kan (1996) and Duffee (2002).

#### Model Assumptions

The affine models of Duffie and Kan (1996) and Duffee (2002) are continuous-time models, which postulate that the nominal short-term interest rate can be written as a combination of  $K$  factors  $X_t$ :

$$r_{dt} = \delta_0 + \delta_1'X_t,$$

and that an SDF exists in the form:

$$M_t = \exp \left[ - \int_0^t \left( r_s + \frac{\|\lambda_s\|^2}{2} \right) ds - \int_0^t \lambda_s' dz_s \right],$$

where  $\mathbf{z}$  is a  $d$ -dimensional Brownian motion and the "price of risk vector"  $\lambda_t$  is itself an affine function of the factors:

$$\sigma_X(\mathbf{X}_t)$$

The factors are assumed to follow a mean-reverting process:

$$d\mathbf{X}_t = \mathbf{K}(\bar{\mathbf{X}} - \mathbf{X}_t)dt + \sigma_X(\mathbf{X}_t)'dz_t,$$

where  $\mathbf{K}$  is a feedback matrix and  $\bar{\mathbf{X}}$  is the vector of long-term means. The form of the volatility matrix  $\sigma_X(\mathbf{X}_t)$  is what distinguishes the "completely affine" models (see Dai and Singleton (2000)) from the "essentially affine" ones (see Duffee (2002)).

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The affine framework can also accommodate defaultable bonds under convenient assumptions on the default date. In the reduced-form approach to default risk (Duffie and Singleton (1999)), the default of an issuer occurs at a random date  $\tau$ .<sup>5</sup> To this default time are associated two "intensity processes",  $\xi$  and  $\tilde{\xi}$ , respectively under the physical and the risk-neutral default probability measures. Heuristically, saying that  $\tilde{\xi}$  is a default intensity means that the probability for default to occur in the small time interval  $[t, t + dt]$  conditional on the absence of default prior to date  $t$  is  $\tilde{\xi}_t dt$ . As shown by Duffie and Singleton (1999), the price of a defaultable zero-coupon that promises the payment of \$1 on date  $T$  is:

$$p_{t,T} = \mathbb{E}_t^{\mathbb{Q}} \left[ e^{-\int_t^T (r_{fu} + \tilde{s}_u) du} \right], \quad (2.21)$$

where  $\mathbb{Q}$  is the risk-neutral probability measure,  $r_d$  is the short-term risk-free rate and  $\tilde{s}$  is the credit spread. This quantity can be expressed as  $\tilde{s}_t = (1 - \delta_t)\tilde{\xi}_t$ , where  $\delta_t$  is the recovery rate of the zero-coupon in the event of default, that is the ratio of the bond price just after default to the price just before. Equation (2.21) shows that the pricing of a defaultable bond is formally equivalent to the pricing of a default-free bond whose principal is discounted at a greater rate than the risk-free rate. By making the assumption that the spread  $\tilde{s}_t$  is an affine function of the factors, as is the risk-free rate, one obtains that the total discount rate,  $r_t + \tilde{s}_t$ , is also affine in the factors. This allows us to use the affine term structure framework to price risky bonds.

### Expression of Bond Risk Premia

The common feature of all affine models is that the prices of zero-coupon bonds can be written as exponential affine functions of the factors. Formally, if  $b_{t,T}$  is the price on date  $T$  of a bond that pays \$1 on date  $T$

in all states of the world, we have:

$$b_{t,T} = e^{A(T-t) - \mathbf{D}(T-t)' \mathbf{X}_t}, \quad (2.22)$$

where  $A$  and  $\mathbf{D}$  are two deterministic functions of the bond maturity. The function  $\mathbf{D}$  is given by:

$$\mathbf{D}(T-t) = (\tilde{\mathbf{K}}')^{-1} \left[ \mathbf{I}_K - e^{-\tilde{\mathbf{K}}'} \right] \boldsymbol{\delta}_1,$$

where  $\tilde{\mathbf{K}}$  is the feedback matrix under the equivalent martingale measure, equal to  $\mathbf{K} + \boldsymbol{\sigma}_X(\mathbf{X}_t)' \boldsymbol{\ell}_1$ . The function  $A$  can be obtained by solving a differential equation (see the aforementioned references for details). The vector  $\mathbf{D}(T-t)$  can be thought of as a vector of durations, where durations are defined as the negative of the sensitivities of the bond price with respect to the factor values.

By the absence of arbitrage opportunities, the instantaneous expected return on the bond must be equal to the risk-free rate plus the inner product of the bond volatility vector and the price of risk vector. Applying Ito's lemma to (2.22), we obtain that the volatility vector is  $-\boldsymbol{\sigma}_X(\mathbf{X}_t) \mathbf{D}(T-t)$ , so that the expected excess return over the risk-free rate is:

$$\mathbb{E}_t \left[ \frac{db_{t,T}}{b_{t,T}} \right] - r_{dt} dt = -\mathbf{D}(T-t)' \boldsymbol{\sigma}_X(\mathbf{X}_t) \boldsymbol{\lambda}_t dt.$$

The bond risk premium, which is precisely defined as this expected excess return, has thus a deterministic and a stochastic component. The deterministic component comes from the vector of durations, which depends on the maturity, while the stochastic part comes from the factor values, which appear both in the vector  $\boldsymbol{\lambda}_t$  and the volatility matrix  $\boldsymbol{\sigma}_X(\mathbf{X}_t)$ . The dependence with respect to the maturity contradicts the expectation hypothesis, which states that all bonds have the same expected return, regardless their maturity.

5 - See Duffie (2001) Chap. 11 for the technical conditions that the default time must satisfy.

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The previous equation can be rewritten in a way more similar to the beta representation (2.6), by introducing the factor risk premia, defined as:

$$\Lambda_t = -\frac{1}{dt} \text{Cov}_t \left[ \frac{dM_t}{M_t}, dX_t \right].$$

We recall that this equality is the definition of factor risk premia in a continuous-time setting: it is the transposition of the discrete-time definition (2.10) with an adaptation to the continuous-time formalism. With this notation, we have:

$$\mathbb{E}_t \left[ \frac{db_{t,T}}{b_{t,T}} \right] - r_t dt = -\mathbf{D}(T-t)' \Lambda_t dt, \quad (2.23)$$

which is the familiar expression of the risk premium as a linear combination of the factor risk premia and the factor exposures. When the matrix  $\widetilde{\mathbf{K}}$  is diagonal, this expression simplifies and becomes:

$$\mathbb{E}_t \left[ \frac{db_{t,T}}{b_{t,T}} \right] - r_t dt = -\sum_{k=1}^K \frac{1 - e^{-a_k(T-t)}}{a_k} \delta_{1k} \Lambda_{kt}, \quad (2.24)$$

where  $a_k$  is the  $k^{\text{th}}$  diagonal coefficient of  $\widetilde{\mathbf{K}}$  and  $\Lambda_{kt}$  is the premia of the  $k^{\text{th}}$  factor. This equation does not predict alone the sign of the expected excess return. Indeed, this quantity depends on the signs of the products  $\delta_{1k} \Lambda_{kt}$ , about which the model is silent.

As explained above, the expressions (2.22) for the zero-coupon price and (2.23) for the bond risk premium remain valid for a defaultable zero coupon, provided the vector of factors includes both the factors that drive the risk-free rate and those that drive the spread.

### 2.4 Theoretical Recommendations for the Choice of Factors

Following the distinction introduced in Section 2.2.2, we discuss here the identification of "pricing factors" and "priced factors" from a theoretical standpoint. We first make a list of implications from the academic literature regarding the choice of pricing factors. Second, we turn to the problem of measuring the factor risk premia.

#### 2.4.1 Choice of Pricing Factors

The prescriptions below can be classified into two categories, namely those that rely solely on the general definition of a factor model, that is the existence of a combination of factors that prices assets, and those that arise from more specific models (consumption-based models, ICAPM or APT). This list does not aim to be exhaustive: it reviews conditions that the literature has identified as necessary for an economic or a financial series to be a pricing factor. Some of these conditions are standard (e.g. factors should explain common variation in returns or should proxy for the time-varying opportunity set); some others have only been stated in recent papers.

#### *General Definition: Factors Should Explain Common Time Variation in Returns*

By definition of the SDF, we have, for any asset:  $\text{Cov}[m, r_i] = 1 - \mathbb{E}[m]\mathbb{E}[r_i]$ . If the true factor model is  $m = a + \mathbf{b}'\mathbf{f}$  and there exists a risk-free rate (so that  $r_d = \frac{1}{\mathbb{E}[m]} - 1$ ) then we obtain:

$$\mathbf{b}'\text{Cov}[r_i, \mathbf{f}] = 1 - \frac{\mathbb{E}[r_i]}{1 + r_d} \text{ for all asset } i.$$

Hence, for all assets that earn an expected return different from the risk-free rate, the vector of covariances between  $\mathbf{f}$  and  $r_i$  must be non-zero. In other words, such an asset

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cannot have all its multivariate betas equal to zero. Hence, in a multivariate regression of returns on the candidate pricing factors  $\mathbf{f}$ , all assets should have a statistically significant loading on at least one factor.

This condition is verified by Fama and French (1993) for the five factors that they introduce to explain stock and bond returns, namely the stock market index and the size factor (small minus big, or SMB), value factor (high minus low book-to-market, or HML), term factor (TERM) and default factor (DEF). Specifically, the loadings of stock and bond returns on the three stock factors or the two bond factors are statistically significant, although in five-factor stock return regressions, stock factors drive out bond factors, and conversely, stock factors are no longer significant when bond returns are regressed on all five factors.

It is important to point that the prescription of having at least one significant beta per risky asset does not imply that the factors should explain a large fraction of the variance of returns. For instance, in Fama and French (1993), the  $R^2$  of time series regressions of bond returns on stock factors are between 10% and 33%, while they easily reach 90% in stock return regressions.

### *Consumption-Based Models: Factors Should Proxy for the Growth in Marginal Utility*

This prescription follows from the consumption-based models such as those of Cochrane (2005) Chap. 1 in discrete time and Breeden (1979) in continuous time. Because consumption growth is likely to be related to macroeconomic fundamentals, it may seem reasonable to search for the factors among macroeconomic aggregates. The most obvious option is to take consumption growth itself as a factor. Unfortunately, empirical tests of the consumption-based

pricing models that rely on consumption data have not been empirically successful. One of the best known problems is the "equity premium puzzle" of Mehra and Prescott (1985): in a nutshell, the ex-post Sharpe ratio of US equities over a very long period such as the second half of the 20<sup>th</sup> century or the whole century is too large compared to the volatility of consumption unless risk aversion is taken equal to exceedingly large values. In other words, given the relatively low uncertainty in consumption, investors endowed with a reasonable risk aversion should not have required such a large compensation for investing in equities rather than rolling over T-bills. While this puzzle can be attributed to misspecification of the model, it should also be recognised that consumption is subject to various aforementioned measurement problems that make consumption data delicate to use in empirical tests of consumption-based models (see Grossman et al. (1987) and Breeden et al. (1989)).

### *APT: Factors Should Explain Time Variation in Individual Returns*

The APT takes the statement that asset pricing factors should explain common variation in returns one step further by suggesting that these factors should also explain a large fraction of the time variation in individual returns. Indeed, this condition ensures that the pay-off of an asset can be approximated as the pay-off of a portfolio of factors, which in turn allows one to proxy the price of the pay-off as a linear combination of factor prices. Hence, for approximate factor pricing to be accurate, the idiosyncratic terms in (2.19) should be as small as possible.

In the work of Fama and French (1993), this condition is not verified when bond factors are used as candidate pricing factors and stock portfolios as test assets,

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although the coefficients of the bond factors are statistically significant. The same observation can be made for stock factors and bond portfolios. The restriction of having small residuals implies that statistical analysis can be used to search for factors. Indeed, one way of writing a decomposition for the covariance matrix of the form (2.20) with a "small" matrix  $\Sigma_\varepsilon$  is to perform a principal component analysis and to retain as factors the principal components that correspond to the largest eigenvalues. A formal justification of this procedure is the Eckart-Young theorem (Eckart and Young, 1936), which implies that the best approximation of a  $N \times N$  positive definite symmetric matrix by a matrix of lower rank  $K$  is obtained by keeping the largest  $K$  eigenvalues and setting the  $N - K$  other ones to zero. Connor and Korajczyk (1993) use this approach and they introduce a statistical test to find the number of factors to retain from the PCA: implementing their method on stocks of the NYSE and AMEX markets, they find that at most six factors are sufficient to describe the cross section of expected returns. This result is worth noting because it implies that not all the potential factors identified by the empirical literature (there are over 300 of them in the equity class according to Harvey et al. (2013)) are needed to construct an SDF. An improvement to the method of Connor and Korajczyk (1993) was proposed by Jones (2001), to account for the presence of heteroscedasticity in returns. Regardless of the exact method used, it is now widely accepted that a single factor is not sufficient to describe the movements of all individual stocks: for instance, Campbell et al. (2001) argue that the explanatory power of the single-factor model has been decreasing over time, so that idiosyncratic volatility has grown in magnitude.

By treating factors as unobservable quantities, a statistical extraction is subject to the usual "rotation problem": any invertible linear transformation of the factors  $\mathbf{f}$  will explain returns equally well as the original factors, i.e. the idiosyncratic terms in (2.19) are insensitive to such a transformation. This remark is used by Pukthuanthong and Roll (2014) to identify potential pricing factors in a list of candidates: a necessary condition for a variable to be a pricing factor is that it be close to a combination of the principal factors, a condition that can be tested by using canonical correlations between candidate factors and principal components.

### *ICAPM: Factors Should Characterise the Conditional Distribution of Asset Returns*

Equation (2.14), derived from the ICAPM, implies that the risk premium on an asset is not simply proportional to its market beta, but that it is a linear function of the market beta and the betas with respect to the  $K - 1$  state variables that drive the dynamics of risk and return parameters. Hence, one should be looking for factors that govern changes in investment opportunities. As pointed by Merton (1973), an obvious example of such a variable is the risk-free interest rate. But the list of state variables that impact expected returns and volatilities of risky assets is potentially long, and it would be interesting to identify only those variables that will eventually appear in the right side of (2.14). A result of Nielsen and Vassalou (2006) can help in this perspective: the authors show that the only risks that multi-period investors are willing to hedge are those that affect the location and shape of the intertemporal capital market line (ICML). In other words, they are only concerned with the intercept (equal to the nominal risk-free rate) and the slope (equal to the maximum Sharpe ratio,

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which itself depends on the Sharpe ratios and correlations of the individual assets). Any state variable that does not have an effect on these two parameters does not induce a hedging demand. The pricing implication is that only the state variables that affect the intertemporal capital market line may appear in the beta decomposition of expected returns. Building on this result (presented by Nielsen and Vassalou in an earlier working paper), Brennan et al. (2004) propose an empirical implementation of the ICAPM where the state variables are the short-term risk-free rate and the slope of the ICML and find that these two factors are priced in the cross section of stock returns for the period 1952-2000, although the pricing restrictions implied by the model are not all validated by the data.

Having said this, the ICAPM does not provide explicit guidance as to how to choose the factors. This can be understood as permissiveness and seems to open the door to any multi-factor model that performs reasonably well in the data (the "fishing license" of Fama (1991)). In order to avoid data mining, Fama (1991) claims that researchers should look for variables that have ability to predict asset returns. Thus, for any candidate factor, the factor should have predictive power in the time series and the factor exposure should have explanatory power in the cross section. This requirement is of course stronger than simply explaining the cross section, and as noted by Campbell (1996), this limits the risk of finding spurious relations. In her implementation of the ICAPM, Petkova (2006) follows this prescription by choosing as pricing factors a set of variables that have been shown to proxy for the time variation in investment opportunities, namely the T-bill rate, the term spread, the aggregate dividend yield and the term spread.

### *Consumption-Based Models and ICAPM: Factors Should Have Little Predictability*

The factors given by the ICAPM (see Section 2.3.3) are the unexpected returns on the market portfolio and the hedging portfolios (i.e. the realised returns minus the expected returns). Hence, they are uncorrelated from any contemporaneous information.

This prescription is also consistent with Cochrane's observation (Cochrane (2005) Chap. 9) that since consumption growth exhibits little predictability in practice, the factors that proxy for the growth in marginal utility of consumption should have little predictability too. This motivates the use of portfolio returns or growth rates in macroeconomic indices as opposed to portfolio values or index levels, but the predictability can be further lowered by filtering out innovations from the state variables that forecast asset returns. Examples of the use of innovations in the time series regression include Campbell (1996) and Petkova (2006), who both extract the innovations from a VAR model.

### *Affine Term Structure Models: Factors Should Explain the Movements of the Yield Curve*

Since it takes the SDF as given, an affine term structure model cannot explain the origin of risk premia. However, the implications of the model can be taken as guidelines for the choice of factors. In particular, the bond pricing equation (2.22) implies that the zero coupon yield of maturity  $\tau$  is an affine function of the factors:

$$y_{t,\tau} = \frac{D(\tau)' \mathbf{X}_t}{\tau} - \frac{A(\tau)}{\tau}. \quad (2.25)$$

This decomposition suggests that the pricing factors should be sought among the factors that determine the movements

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of the yield curve. The empirical literature has identified three factors that explain a dominant fraction of the sovereign yield curve dynamics, namely changes in level, slope and curvature of the yield curve (see Litterman and Scheinkman (1991)).

In the presence of default or credit risk, the vector  $\mathbf{X}_t$  in the right-hand side of (2.25) has to include factors that matter for the determination of credit spreads in addition to factors that drive risk-free rates. The nature of these factors is discussed by Duffee (1999) and Driessen (2005). Duffee argues that the credit spread should include a combination of the default-free term structure factors, and, of course, an idiosyncratic term reflecting the financial health of the firm. Driessen decomposes the spread as the sum of a default-related contribution – which itself contains a firm-specific term – and a liquidity component – which is common to all firms.

It is possible to treat the factors  $\mathbf{X}_t$  as latent variables, which must be extracted from the data by filtering techniques (see Duffee and Stanton (2012) for a survey). Economic reasoning may also suggest observable variables as candidates: since the short-term rate is managed by the central bank, it is likely to be related to macroeconomic variables such as real activity and inflation (see Taylor (1993)). Ang and Piazzesi (2003) combine the latent variable approach and the explicit factor approach by choosing as  $\mathbf{X}_t$  a combination of the aforementioned two macro factors and three unobservable state variables.

### *Other Recommendations*

In addition to the previous recommendations, recent papers have identified other conditions that a variable should satisfy in order to be a pricing factor:

- *ICAPM: Model-implied risk aversion should be in a plausible range*

This implication of the ICAPM is highlighted by Maio and Santa-Clara (2012). It hinges on the decomposition (2.16) of asset risk premia: this equation involves a coefficient  $\gamma_t$ , which can be interpreted as the relative risk aversion of the investor at the optimum. Assuming that the investor maximises a constant relative risk aversion utility function,  $\gamma_t$  is a constant, which can be estimated together with the other model parameters. The authors point that the estimate should take plausible values. In particular, some of the intertemporal asset pricing models that they test (namely those of Campbell and Vuolteenaho (2004), Hahn and Lee (2006), Petkova (2006) and Koijen et al. (2010)) imply negative values for  $\gamma$ , which leads to consider them as inconsistent with the ICAPM.

- *ICAPM: If a state variable is positively correlated with expected market return, an asset with a higher exposure to this variable should earn a higher expected return*

If a state variable covaries positively with expected market return, then an asset with a higher exposure to this variable will tend to perform better when the expected market return is high than an asset with a lower exposure. As a result, it does not provide a hedge for reinvestment risk, which should be compensated by a higher expected return. Maio and Santa-Clara (2012) use this condition to test the compatibility of various multi-factor asset pricing models with the ICAPM. It should be noted that they measure the exposure of an asset to a state variable in terms of the covariance between the return and the variable: the sign is the same as that of the univariate beta, but not necessarily the same as that of the multivariate beta with respect to the factor. Hence, their criterion is that if a state variable  $X_k$  covaries positively with expected

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market return, then its coefficient  $\gamma_{Xk}$  in (2.16) should be positive.

- *APT: The sign of the factor premium determines the direction of the relation between conditional expected returns and conditional variances of factor-mimicking portfolios*

Charoenrook and Conrad (2008) note that the conditional expected excess return on a factor-mimicking portfolio should be connected to the conditional variance, and the sign of the relationship should be the same as that of the factor premium. Their argument applies to a portfolio whose return is an exact combination of the factors: in the terminology of APT, this portfolio is sufficiently well diversified to have zero idiosyncratic risk. By (2.19), the return on this portfolio satisfies:

$$r_{t+1} = \mathbb{E}_t[r_{t+1}] + \beta'_{Pt} f_{t+1}, \quad (2.26)$$

where  $\beta_{Pt}$  is the vector of portfolio betas. Hence, the conditional variance is  $\mathbb{V}_t[r_{t+1}] = \beta'_{Pt} \Sigma_{ft} \beta_{Pt}$  where  $\Sigma_{ft}$  is the covariance matrix of factors. Moreover, the SDF has the form  $m_{t+1} = a + \mathbf{b}' f_{t+1}$ , so that the expected excess return on the portfolio is:

$$\begin{aligned} \mathbb{E}_t[r_{t+1}] - r_{dt} &= -(1 + r_{dt}) \text{Cov}_t[m_{t+1}, r_{t+1}] \\ &= -(1 + r_{dt}) \mathbf{b}' \Sigma_{ft} \beta_{Pt}. \end{aligned} \quad (2.27)$$

Putting (2.26) and (2.27) together, we obtain:

$$\mathbb{E}_t[r_{t+1}] - r_{dt} = (1 + r_{dt}) \frac{\mathbf{b}' \Sigma_{ft} \beta_{Pt}}{\beta'_{Pt} \Sigma_{ft} \beta_{Pt}} \mathbb{V}_t[r_{t+1}].$$

Thus, the conditional risk premium is proportional to the conditional variance. Moreover, if the scaling coefficient  $\mathbf{b}' \Sigma_{ft} \beta_{Pt} / \beta'_{Pt} \Sigma_{ft} \beta_{Pt}$  is constant over time, then it must have the same sign as the unconditional risk premium. In summary, the conditional premium must be positively

related to the conditional variance if the unconditional premium is positive. The authors apply this test to the long-short portfolios SMB, HML and WML (the winners minus losers portfolio) and find that it is only for the first two of these portfolios that the condition is verified.

### 2.4.2 Identifying Priced Factors

The previous section has looked into the problem of finding factors that are relevant for pricing. In this section, we take a different perspective: taking a given factor  $X$ , which may or may not be a pricing factor, we want to measure its risk premium, defined as

$$\Lambda_X = -\frac{1}{\mathbb{E}[m]} \text{Cov}[m, X].$$

In particular, we are interested in testing whether  $\Lambda_X$  is different from zero, which will mean that the factor is priced. The identification of priced factors is useful because it leads to the identification of the factors that explain the cross section of expected returns: if a factor is not rewarded (i.e. has zero price), then it does not generate any dispersion in expected returns. On the other hand, such a factor may still be useful as a pricing factor: the correct measurement of multivariate betas in (2.6) requires that all pricing factors be taken account, and omitting one of them leads to bias in the other beta estimates, hence to incorrect assessment of expected returns even if the factor premia are correct.

#### Sorting Procedure

Suppose that the true factor model is  $m = a + \mathbf{b}' \mathbf{f}$ , with all elements of  $\mathbf{b}$  being non zero (that is, only the pricing factors are included in the right-hand side). Given a factor  $X$ , there are two possibilities: either it is equal to an affine combination of the factors, or it is not in the span of the factors. In the first case,  $X$  is of the form

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$g + \mathbf{h}'\mathbf{f}$  with at least one element of  $\mathbf{h}$ , say  $h_k$ , being non zero. Then, the SDF can be rewritten as  $m = c + \mathbf{d}'\mathbf{f}_{\neq k} + eX$ , where  $\mathbf{f}_{\neq k}$  is the vector of factors excluding the  $k^{\text{th}}$  one. In sum,  $f_k$  is replaced by  $X$  in the decomposition of the SDF. In the second case,  $X$  is not perfectly correlated to any affine combination of the factors, but the model can be equivalently rewritten as  $m = a + \mathbf{b}'\mathbf{f} + 0X$ .

In both situations, we have a beta representation of expected returns of the form:

$$\mathbb{E}[r_i] = \bar{\alpha} + \bar{\Lambda}'\bar{\beta}_i, \quad (2.28)$$

where the notations are defined as follows:

- in the first case,  $\beta_i$  is the vector of multivariate betas of  $R_i$  with respect to the vector  $\bar{\mathbf{f}} = (\mathbf{f}_{\neq k}, X)$ ;
- in the second case,  $\beta_i$  is the vector of multivariate betas with respect to  $\bar{\mathbf{f}} = (\mathbf{f}, X)$ ;
- in both cases,  $\bar{\Lambda}$  is the vector of premia of the factors  $\bar{\mathbf{f}}$ .

By (2.28),  $\Lambda_X$  is the sensitivity of the expected return with respect to the multivariate beta with respect to  $X$ , denoted with  $\bar{\beta}_{iX}$ :

$$\Lambda_X = \frac{\partial \mathbb{E}[r_i]}{\partial \bar{\beta}_{iX}}. \quad (2.29)$$

Thus, if we could plot the expected return as a function of  $\bar{\beta}_{iX}$  keeping all other betas constant, the factor premium would be the slope of the line. If the factor is priced (i.e. if  $\Lambda_X \neq 0$ ), then the line is not flat.

In practice, it is common to sort assets on their exposure to the factor  $X$  and the cross section of expected returns is examined. But two caveats are in order. First, it should be noted that the measure of sensitivity that matters here is not the univariate beta, but the multivariate beta with respect to  $X$  and the other pricing factors. It is only if  $X$  is

orthogonal to all other pricing factors that the two betas coincide. Second, if some other pricing factors are rewarded, then the observed differences in expected returns observed may result be attributed in part to differences in exposures to these factors:  $\Lambda_X$  is the sensitivity of the expected return with respect to an increase in  $\bar{\beta}_{iX}$ , other things being equal. Thus, to estimate  $\Lambda_X$  as a difference between expected returns in a sort, one should ideally consider assets (or portfolios of assets) with uniform exposures to the other priced factors. In order to bypass the construction of portfolios with controlled exposures, one can use the two-pass regression technique described below.

### Two-Pass Regression

The representation (2.6) lends itself to an empirical estimation of the various model parameters. With  $N$  assets and  $K$  factors, there are  $NK$  betas and  $K$  factor premia to estimate. But the constant  $\alpha$  is not always to be estimated: if there is a risk-free asset, it is equal to the risk-free rate, which is observable. In the absence of such a risk-free asset, the constant must be treated as an unknown parameter.

As explained in Cochrane (2005) Chap. 5.1, the first approach is to run two series of regressions:

1. Run  $N$  times-series regressions of returns on factors

$$r_{it} = c_i + \sum_{k=1}^K \beta_{ik} f_{kt} + \varepsilon_{it}, \quad t = 1, \dots, T, \quad (2.30)$$

to obtain estimates for the betas;

2. Run a cross-sectional regression of expected returns on betas

$$\mathbb{E}[r_i] = \gamma + \sum_{k=1}^K \beta_{ik} \lambda_k + \eta_i, \quad i = 1, \dots, N,$$

to obtain estimates for the risk premia.

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When the factor is a return or excess return, one can compare the risk premia estimates obtained in the second step with those directly measured from the data. This is one of the tests that the CAPM of Sharpe and Lintner fails to pass in the data: the fitted market risk premium is too low compared to the average excess return on a broad market stock index.

From a statistical perspective, one has to be careful that the cross-section regression raises an error-in-variables issue. The regressors are estimates for the betas, not the true betas, so they contain measurement errors. The concern is that any correlation between the errors and the regressors will bias the least-squares estimates of risk premia. In their tests of the CAPM, Fama and MacBeth (1973) address this problem by proposing to group stocks in portfolios and to estimate portfolio betas: the idea is that measurement errors tend to offset across stocks, so that estimates are more accurate at the portfolio level than at the individual stock level. Their procedure replaces Step 1 above by the following sequence:

- 1.a Estimate stock market betas at the individual level over a portfolio formation period  $[1, T_1]$  (3 years in the work of Fama and MacBeth);
- 1.b Sort stocks on their estimated beta and group them into portfolios (the authors use equally-weighted portfolios);
- 1.c Estimate the portfolio betas over a post-formation period  $[T_1 + 1, T_2]$  (they use 5 years here).

The motivation for ranking stocks by betas is to keep the spread of betas across portfolios as large as possible.

Moreover, Fama and MacBeth (1973) do not directly regress average returns on estimated betas as in Step 2 above, and

they instead regress returns on the betas. Step 2 is thus replaced by:

- 2.a For each month in a period  $[T_2 + 1, T_3]$  (which Fama and MacBeth take equal to 3 years), regress the cross section of portfolio returns on the portfolio betas. This yields  $K$  time series of estimates;
- 2.b The mean of each time series is interpreted as the expected return on a factor-mimicking portfolio.

# 3. Theoretical Analysis of Factor Investing



### 3. Theoretical Analysis of Factor Investing

Factor investing is an investment approach where the building blocks are proxies for factor-mimicking portfolios in the sense of Section 2.2.1, as opposed to being arbitrary constituents such as asset class indices. We refer to the building blocks as *factor indices* (the *factors* are the underlying asset pricing factors, which, unlike factor index returns, are not observed), which number is denoted by  $K$ .

A test of the relevance of factor investing is a joint test of the relevance of the chosen factors and the chosen allocation methodology. In this section, we start by presenting various allocation methods to the factor indices or the traditional indices. Next, we discuss the theoretical advantages from investing in factor-mimicking portfolios. Finally, we describe the theoretical impact of introducing new assets (here, the factor indices) on the mean-variance efficient frontier.

#### 3.1 Risk Allocation Methods

The allocation schemes that we introduce below are not specific to factor investing and can be applied to any set of constituents, even though some of them (the factor risk parity and maximum ENUB portfolios) are oriented towards a factor perspective, since they focus on the contributions of underlying risk factors. On the other hand, all of these methodologies fall within the general risk allocation category, a term that we use to denote any portfolio construction methodology that does not involve any expected return estimate, which are notoriously noisy (see Merton (1980)).

Throughout this section, we consider an investment universe of  $K$  constituents with

vector of realised returns  $\mathbf{r}$ , covariance matrix  $\Sigma$ , expected returns  $\boldsymbol{\mu}$  and expected excess returns  $\tilde{\boldsymbol{\mu}}$ . We denote with  $\mathbf{x}$  the  $K \times 1$  vector of dollar weights in these assets (the letter  $\mathbf{w}$  is used to denote a vector of weights of the individual constituents).

#### 3.1.1 Diversification in Terms of Dollar Contributions: Maximum Deconcentration Portfolios

The most obvious (sometimes called "naive") way to diversify a portfolio is to allocate equal weights to the constituents. Mathematically, the equally-weighted (EW) portfolio maximises the dispersion of the dollar contributions, measured as the inverse of the sum of squared weights. This quantity is known as the "effective number of constituents" (ENC):

$$ENC(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|^2}, \quad (3.1)$$

where  $\|\cdot\|$  denotes the Euclidian norm. An application of Cauchy-Schwartz inequality shows that  $ENC \leq K$ , and that equality holds if, and only if, all weights are equal.<sup>6</sup>

The equally-weighted (EW) portfolio does not require any unobservable risk or return parameter and is thus completely free of estimation error. Because of this property, it often outperforms optimised portfolios constructed as proxies for efficient portfolios but plagued by estimation errors in covariances and expected returns (see Bloomfield et al. (1977), Jorion (1991), DeMiguel et al. (2009) and Martellini et al. (2014)). Naive and scientific diversification techniques, however, do not have to be regarded as competing methodologies. For example, DeMiguel et al. (2009) show

6 - Cauchy-Schwartz inequality states that for any two vectors  $(x_1, \dots, x_k)$  and  $(y_1, \dots, y_k)$ , we have  $(\sum_i x_i y_i)^2 \leq (\sum_i x_i^2) \times (\sum_i y_i^2)$ . Equality is achieved if, and only if, one of the two vectors is zero or the two vectors are parallel. Here, we take  $y_i = 1$ .

### 3. Theoretical Analysis of Factor Investing

that the introduction of a minimum ENC target level (which they call "norm constraint") – that is, a minimum level of "naive" diversification – in an optimisation program can be seen as a form of statistical shrinkage, which alleviates the effects of estimation errors.

If various constraints are imposed (weight constraints, liquidity or turnover constraints, tracking error constraints, etc.), the maximum ENC of  $K$  may not be attainable, but it is still possible to define a "maximum deconcentration" (MDC) portfolio (also known as max ENC portfolio) as the portfolio that maximises the ENC subject to the constraints:

$$\mathbf{x}_{MDC} = \arg \max_{\mathbf{x} \in \mathcal{C}} ENC(\mathbf{x}),$$

where  $\mathcal{C}$  is the set of constraints. In the out-of-sample tests that we conduct in Section 5, we require weights to be nonnegative, because most investors face long-only constraints. Thus,  $\mathcal{C}$  will be a subset of the set of nonnegative vectors that sum up to 1.

#### 3.1.2 Risk Minimisation: Minimum Volatility and Tracking Error Portfolios

Unlike the MDC portfolio, the global minimum variance (GMV) portfolio is always mean-variance efficient if we abstract away from estimation errors. It is a remarkable point of the efficient frontier because it is the only efficient portfolio that does not require expected returns. Given that, as recalled earlier, first-order moments are extremely hard to estimate by purely statistical techniques (see Merton (1980)), this property makes the GMV attractive. The portfolio is mathematically defined by:

$$\mathbf{x}_{GMV} = \arg \min_{\mathbf{x} \in \mathcal{C}} \mathbf{x}' \boldsymbol{\Sigma} \mathbf{x},$$

where we recall that  $\mathcal{C}$  denotes the set of weight constraints.

Although it does not depend on expected returns, the GMV is still exposed to estimation risk through the covariance matrix. The precision of covariance estimators can be improved by increasing the sampling frequency while keeping the sample length constant (see Merton (1980)), but estimation errors can be sizable if the number of constituents is large compared to the sample. As shown by Kan and Zhou (2007), the ratio  $K / T$ , where  $T$  is the number of data points, is a key determinant of the accuracy of the estimator, and when it is large, errors in covariances can contribute as much as errors in expected returns to the loss of efficiency with respect to an ideal portfolio with no estimation risk. Therefore, in large universes such as the S&P 500, dedicated statistical techniques, such as shrinkage or principal component analysis, must be used to reduce the impact of sample noise (see Coqueret and Milhau (2014) for a survey). However, in an asset allocation context, the number of constituents is typically small (for example, we will have a maximum of eight factor indices at a time in Section 5), so that a robust estimation procedure is not critically needed, and we will take the sample covariance matrix as an estimate, and verify that  $T$  remains sufficiently greater than  $K$  in order to avoid having a singular estimate.<sup>7</sup>

Another well-known property of the GMV is that it tends to be highly concentrated in the least volatile constituents and, provided long-only constraints are imposed, to assign zero weights to the most volatile ones. In order to ensure a minimum level of naive diversification,

7 - The sample covariance matrix is non-singular if, and only if,  $T > K$ , but it can be ill-conditioned if  $T$  is close to  $K$ .

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it is necessary to impose bounds on weights tighter than 0 and 1 (alternatively one may also impose a minimum ENC, as discussed in the previous Section). As shown by Jagannathan and Ma (2003), imposing such weight constraints amounts to working with a shrunk estimate of the covariance matrix, and when the universe is large compared to the sample size, this contributes to reduce the out-of-sample volatility. Given that our universes have moderate sizes, limiting the effects of estimation risk is not our main motivation for introducing constraints.<sup>8</sup> Instead, we seek to avoid having a highly concentrated portfolio, which would be damageable to the Sharpe ratio. Maintaining a minimum level of naive diversification is likely to have a positive impact on the performance and the Sharpe ratio, though it can slightly increase the volatility. Specifically, in order to control the distance with respect to the EW portfolio, we express the constraints as:

$$\frac{1}{\delta K} \leq x_k \leq \frac{\delta}{K}, \quad (3.2)$$

Taking  $\delta = 1$  gives the EW portfolio, and taking  $\delta = \infty$  amounts to merely imposing long-only constraints. In the empirical section of this paper, we take  $\delta = 3$ .

The GMV minimises absolute risk, but in some context a relative risk target is more relevant, e.g. when investors are seeking to outperform a given commercial index or reference multi-asset portfolio. This risk budget in this case is defined as the tracking error with respect to the reference portfolio. The minimum tracking error (MTE) portfolio is defined by the minimisation of the tracking error with respect to the exogenous benchmark. To write its mathematical definition, we introduce the relative covariance matrix,  $\tilde{\Sigma}$ , which

is the covariance matrix of excess returns of assets with respect to the benchmark. It is related to the covariance matrix and the column vector of covariances between the constituents and the benchmark,  $\Sigma_B$ , through:

$$\tilde{\Sigma} = \Sigma - \Sigma_B \mathbf{1}'_K - \mathbf{1}_K \Sigma'_B + \sigma_B^2 \mathbf{1}_K \mathbf{1}'_K,$$

$\sigma_B^2$  being the benchmark variance. The tracking error of the portfolio with respect to the benchmark is then  $\sqrt{\mathbf{x}' \tilde{\Sigma} \mathbf{x}}$ , which shows that  $\tilde{\Sigma}$  is non-singular (i.e. is positive definite) if, and only if, no portfolio of the  $K$  constituents achieves a zero tracking error. The MTE is defined as:

$$\mathbf{x}_{MTE} = \arg \min_{\mathbf{x} \in \mathcal{C}} \mathbf{x}' \tilde{\Sigma} \mathbf{x}.$$

As for the GMV, we impose bounds on weights (see (3.2)) to avoid over-concentration in the constituents with the lowest tracking errors.

#### 3.1.3 Diversification in Terms of Risk Contributions: Risk Parity and Factor Risk Parity Portfolios

It is well known that in portfolios that are well diversified in terms of dollar allocations, constituents may have very different contributions to risk (see the example of a 60%/40% stock-bond policy portfolio in Qian (2005)). For this reason, Maillard et al. (2010) propose to think of diversification not in terms of dollar budgets but in terms of risk budget, an approach known as risk budgeting.

##### *Effective Number of Correlated Bets*

The (normalised) contributions of constituents to portfolio volatility are defined as:

$$c_k = \frac{x_k [\Sigma \mathbf{x}]_k}{\mathbf{x}' \Sigma \mathbf{x}}, \quad (3.3)$$

8 - In fact, Jagannathan and Ma (2003) show that if the universe is small, the constraints can increase the out-of-sample volatility with respect to an unconstrained portfolio.

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so that their sum equals 1. We stack them in the column vector  $\mathbf{c}$ . Qian (2006) gives a financial interpretation of the absolute contributions (i.e. the normalised contributions times the portfolio volatility) by showing that they are related to the contributions of constituents to large losses.

The risk parity (RP) portfolio is defined by Maillard et al. (2010) as the portfolio that equalises the contributions  $c_1 = \dots = c_K$ . The existence and uniqueness of a long-only portfolio that satisfies these equalities is proved by Spinu (2013), and numerical methods to compute it are described in Maillard et al. (2010), Spinu (2013) and Griveau-Billion et al. (2013). By analogy with the EW and MDC portfolios, one can incorporate constraints in the procedure by searching for the portfolio that minimises the dispersion of the normalised contributions subject to the constraints:

$$\mathbf{x}_{MENC B} = \arg \max_{\mathbf{x} \in \mathcal{C}} \frac{1}{\|\mathbf{c}\|^2}.$$

The objective function is the "effective number of correlated bets" (ENCB), a term chosen to emphasise that the constituents are in general correlated. By Cauchy-Schwartz inequality, we have  $ENCB \leq K$ , and the maximum is attained by the RP portfolio. In the presence of constraints, equality of contributions may not be achievable, so the optimisation program searches for the portfolio that minimises the dispersion. This methodology can be applied to individual correlated assets or to correlated factor indices.

#### *Effective Number of Uncorrelated Bets*

A shortcoming of the ENCB is that it can give a misleading picture for highly correlated assets. For instance, an equally-weighted portfolio of two highly correlated bonds

with similar volatilities is well diversified in terms of dollars and volatility contributions, but portfolio risk is extremely concentrated in a single interest rate risk factor exposure. To better assess the contributions of underlying risk factors, Meucci (2009) and Deguest et al. (2013) propose to decompose the portfolio returns (which can be seen as combinations of correlated asset returns or correlated factor returns) as a combination of the contributions of  $K$  uncorrelated implicit factors. Formally, consider a  $K \times 1$  vector of factor returns  $\mathbf{r}_f = \mathbf{A}'\mathbf{r}$ , where  $\mathbf{A}$  is a non-singular matrix such that the factor covariance matrix

$$\Sigma_f = \mathbf{A}'\Sigma\mathbf{A}, \quad (3.4)$$

is diagonal. Then, the factor weights are given by  $\mathbf{x}_f = \mathbf{A}^{-1}\mathbf{x}$ , and the portfolio volatility can be decomposed as:

$$\sigma_p = \sqrt{\mathbf{x}'\Sigma\mathbf{x}} = \sqrt{\mathbf{x}'_f \Sigma_f \mathbf{x}_f} = \sum_{k=1}^K \frac{x_{fk}^2 \sigma_{fk}^2}{\sigma_p}.$$

Note that unlike the decomposition of volatility across assets, this decomposition involves no attribution of correlated components. The (normalised) contribution of each factor is thus simply defined as:

$$c_{fk} = \frac{x_{fk}^2 \sigma_{fk}^2}{\sigma_p^2}, \quad k = 1, \dots, K.$$

The factor risk parity (FRP) portfolio is defined by Deguest et al. (2013) by the conditions  $c_{f1} = \dots = c_{fK}$ . Hence, it maximises the "effective number of uncorrelated bets" (ENUB), which is defined exactly as the ENC and the ENCB, except that the factor contributions replace the dollar or volatility contributions of correlated assets. More generally, the maximum ENUB portfolio subject to a set of weight constraints is defined as:

$$\mathbf{x}_{MENUB} = \arg \max_{\mathbf{x} \in \mathcal{C}} \frac{1}{\|\mathbf{c}_f\|^2}.$$

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We emphasise that the constraints, e.g. long-only constraints, are expressed on asset weights (individual assets or factor indices), and not on implicit uncorrelated factor weights.

#### Extracting Uncorrelated Factors

The construction of FRP or MENU portfolios relies on uncorrelated factors. Because these factors are implicit, they are not uniquely defined. In fact, it is easy to see that if  $\mathbf{A}$  is a change of basis matrix from constituents to factors and  $\mathbf{Q}$  is any orthogonal matrix, then the matrix  $\mathbf{M} = \mathbf{A}\Sigma_f^{-\frac{1}{2}}\mathbf{Q}$  is also a change of basis matrix such that  $\mathbf{M}'\Sigma\mathbf{M}$  is diagonal.<sup>9</sup> Thus, one has to specify an orthogonalisation procedure. An option is to perform principal component analysis on the covariance matrix, so as to sequentially extract uncorrelated factors that have the maximum marginal explanatory power with respect to asset returns. This decomposition, however, has some undesirable properties: Carli et al. (2014) note that the principal factors lack interpretability and Meucci et al. (2013) show that if all assets have the same volatility and the same pairwise correlation, then an equally-weighted portfolio of the assets is fully invested in the first principal factor (i.e. the one with the largest variance) and that the other factors have zero weight. As a result, the portfolio risk is entirely explained by the first factor, regardless of the value of the common correlation. This property is counter-intuitive, as one would expect uncorrelated factors to have more balanced contributions when the correlation across assets shrinks to zero.

To overcome these problems, Meucci et al. (2013) introduce an alternative method for extracting uncorrelated factors, known as "Minimum Linear Torsion". The MLT algorithm seeks to extract the matrix  $\mathbf{A}$  such that the factor covariance matrix (3.4) is diagonal while keeping the distance

between the factors and the asset returns as small as possible, thus involving the smallest deformation of the original components. In detail,  $\mathbf{A}$  is solution to the following program:

$$\min_{\mathbf{A}} \sum_{k=1}^K \mathbb{V}[r_{fk} - r_k],$$

subject to  $\mathbf{r}_f = \mathbf{A}'\mathbf{r}$  and  $\mathbf{A}'\Sigma\mathbf{A} = \text{diag}(\Sigma)$  (3.5)

where  $\text{diag}(\Sigma)$  denotes a diagonal matrix with diagonal elements equal to those of  $\Sigma$ . Note that the second constraint implies two properties: first, the factor covariance matrix is diagonal, and second, the factor variances are equal to the asset variances, which is a natural requirement since the factors should "resemble" asset returns as much as possible. In the previous example (uniform volatilities and correlations and equally-weighted portfolio), it can be shown that all uncorrelated factors have the same weight, which is more in line with the intuition.

Fortunately, the solution to the optimisation problem (3.5) is known in closed form, up to the singular value decompositions of and an auxiliary matrix: expressions for the change of basis matrix  $\mathbf{A}$  can be found in Meucci et al. (2013) and Carli et al. (2014), and are recalled in Appendix B. Since efficient numerical algorithms exist for computing singular value decompositions, the solution to (3.5) is straight-forward to obtain numerically. In what follows, we will use this "least intrusive" orthogonalisation procedure to extract uncorrelated factors starting from correlated factor indices.

#### 3.1.4 Diversification in Terms of Relative Risk Contributions: Relative Risk Parity and Relative Factor Risk Parity Portfolios

The concept of risk contribution can also

9 - A matrix  $\mathbf{Q}$  is said to be orthogonal if it satisfies  $\mathbf{Q}'\mathbf{Q} = \mathbf{Q}\mathbf{Q}' = \mathbf{I}_N$ .

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be applied to relative risk, measured by tracking error. The portfolio tracking error is equal to  $\sqrt{\mathbf{x}'\tilde{\Sigma}\mathbf{x}}$ , which is formally similar to the volatility. Thus, one can define the relative contributions of constituents to tracking error as:

$$c_{rel,i} = \frac{x_i[\tilde{\Sigma}\mathbf{x}]_i}{\mathbf{x}'\tilde{\Sigma}\mathbf{x}}, i = 1, \dots, K$$

and these  $K$  quantities add up to 1. The "maximum effective number of relative correlated bets" (MEN-RCB) is defined as the portfolio that minimises the dispersion of relative contributions subject to a set of weight constraints:

$$\mathbf{x}_{rel,MENRCB} = \arg \max_{\mathbf{x} \in \mathcal{C}} \frac{1}{\|\mathbf{c}_{rel}\|^2}.$$

Following the same arguments as for the RP portfolio (see Spinu (2013)), it can be shown that if the relative covariance matrix is non-singular (i.e. if the benchmark is not a combination of the assets), then a unique "relative RP" portfolio exists, i.e. a portfolio such that  $c_{rel,1} = \dots = c_{rel,K}$ . This portfolio coincides with the MENRCB when the only constraints applied to weights are nonnegativity constraints.

Finally, the "maximum ENRUB" portfolio is defined as the maximum ENUB one, but the orthogonal factors are extracted from the relative covariance matrix, as opposed to the absolute matrix:

$$\mathbf{x}_{rel,MENRUB} = \arg \max_{\mathbf{x} \in \mathcal{C}} \frac{1}{\|\mathbf{c}_{rel,F}\|^2},$$

where  $\mathbf{c}_{rel,F}$  denotes the vector of contributions of the relative factors. The  $K$  relative factors are obtained from the covariance matrix of excess returns, and their contributions are expressed as fractions of the tracking error.

#### 3.2 Theoretical Benefits from Factor Investing

In this section, we present two theoretical advantages of formulating allocation decisions in terms of factor indices as opposed to arbitrary groups of securities or individual securities themselves. The first result (see Proposition 4 below) is important because it provides the main theoretical justification for factor investing: the loss of efficiency in a two-step process (first efficiently grouping individual securities in benchmarks portfolios, and then efficiently allocating to the benchmark portfolios) with respect to a one-step process is zero if, and only if, the benchmark portfolios are constructed as replicating portfolios for asset pricing factors.

##### 3.2.1 Cancelling the Loss of Efficiency in the Two-Step Process

Investment decisions in delegated money management are typically made in two stages:

- A chief investment officer decides how to allocate money to various building blocks, which are most often asset classes benchmarks;
- Asset managers are in charge of making decisions in individual securities for each asset class.

This two-step process is to be compared with a one-step process, in which a single asset owner or asset manager would decide how to allocate efficiently to each individual security. This leads to ask whether there could be better ways of grouping the securities in the benchmark construction step.

##### *Measuring the Loss of Efficiency*

By the two-fund separation theorem, an investor with mean-variance preferences and facing  $N$  individual risky securities and

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a risk-free asset should hold a combination of two portfolios: the MSR portfolio of the  $N$  risky constituents and the risk-free asset. The efficient frontier is a straight line whose intercept is the risk-free rate and whose slope is the Sharpe ratio of the MSR portfolio,  $\lambda_{MSR}$ . Consider now an investor facing  $K$  indices constructed as groups of the  $N$  individual securities. The efficient frontier is again a straight line starting from the risk-free rate, but the slope is the Sharpe ratio of the MSR portfolio of indices,  $\lambda_{MSR,I}$ . Thus, the second investor – who has access to the building block indices but not to their individual components – faces a deteriorated opportunity set if, and only if,  $\lambda_{MSR,I} < \lambda_{MSR}$ .

It is always true that a weak inequality holds, i.e. that  $\lambda_{MSR,I} \leq \lambda_{MSR}$ . This follows from the fact that any portfolio of indices can be regarded as a portfolio of securities while the converse is not true, so that the grouping of securities implies a loss of degrees of freedom in the maximisation of Sharpe ratio. A measure of the loss of efficiency in the two-step process is given by the quantity

$$\Delta\lambda^2 = \lambda_{MSR}^2 - \lambda_{MSR,I}^2,$$

which is always nonnegative. The following proposition (Proposition 4) gives an expression for the loss of efficiency as a function of the alphas of constituents with respect to indices. To introduce the notations, we let  $\mathbf{W}$  be the  $\mathbf{N} \times \mathbf{K}$  matrix whose columns  $\mathbf{W}_1, \dots, \mathbf{W}_K$  give the compositions of the  $\mathbf{K}$  indices. For each index fully invested in the underlying securities, the corresponding column of  $\mathbf{W}$  sums up to 1. On the other hand, for indices defined as dollar-neutral portfolios, the corresponding columns of  $\mathbf{W}$  sum up to 0. We also let  $\tilde{\mathbf{I}}$  be the vector of indices excess returns, which is related to the vector of asset excess returns,  $\tilde{\mathbf{r}}$ , through

$\tilde{\mathbf{I}} = \mathbf{W}'\tilde{\mathbf{r}}$ . If  $\mathbf{W}$  is square and non-singular, then the excess returns of the constituents can be recovered from those of the indices. In this case, the indices are simply an invertible linear transformation of the constituents, and the grouping creates no loss of Sharpe ratio. But the relevant case is when a "large" set of individual constituents is summarised by a "small" set of building blocks. Thus, we have  $K < N$ , and  $\mathbf{W}$  cannot be inverted.

The regression equations read:

$$\tilde{r}_i = \alpha_i + \beta_i' \tilde{\mathbf{I}} + \eta_i, \quad (3.6)$$

The alphas are stacked in the  $N \times 1$  vector  $\boldsymbol{\alpha}$ . With these notations, the loss of Sharpe ratio can be expressed as follows.

#### Proposition 4 (Loss of Sharpe Ratio in Two-Step Process)

Assume that the  $N \times K$  matrix of the weights of the indices in the individual securities is such that  $\mathbf{W}'_k \mathbf{1}_N = \mathbf{1}$  or  $\mathbf{0}$  for all  $k = 1, \dots, K$ . Then, the loss of efficiency is given by:

$$\lambda_{MSR}^2 - \lambda_{MSR,I}^2 = \boldsymbol{\alpha}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\alpha}.$$

**Proof.** See Appendix A.4.

Thus, the loss of Sharpe ratio is zero if, and only if, all alphas are zero, that is, if, and only if, the asset risk premia are fully explained by the indices risk premia. This is the case if the excess returns on indices act as pricing factors in the sense that an SDF exists in the form  $m = a + \mathbf{b}'\tilde{\mathbf{I}}$ : indeed, with such an SDF, we have (see Section 2.2.3):

$$r_i - r_d = \beta_i' \tilde{\mathbf{I}} + \varepsilon_i, \quad i = 1, \dots, N$$

If index returns do not price all individual assets, non-zero alphas exist, and the loss of Sharpe ratio is a measure of the size of the vector  $\boldsymbol{\alpha}$ .

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#### 3.2.2 A Parsimonious Characterisation of the MSR Portfolio

Proposition 4 implies that the MSR of the  $N$  individual securities can be replaced by the portfolio of  $K$  factor indices without loss of Sharpe ratio if these factor indices collectively represent an exhaustive collection of asset pricing factors. The second justification for factor investing that we present now is related to this property. Proposition 5 below shows that provided the asset pricing factors have been identified, the weights of the MSR of the  $N$  securities can be computed without knowing all individual expected returns and covariances. Before we state the proposition, we recall that  $\Sigma$  and  $\Sigma_f$  denote respectively the covariance matrix of the  $N$  assets and that of the  $K$  factors, that  $\tilde{\mu}$  denotes the vector of expected excess returns over the risk-free rate and that  $\Lambda$  is the vector of factor risk premia.

#### Proposition 5 (Factor Exposures of MSR Portfolio)

Assume that a factor model exists where the SDF can be written as  $m = a + \mathbf{b}'\mathbf{f}$ ,  $\mathbf{f}$  being a  $K \times 1$  vector whose elements are portfolio excess returns or dollar-neutral portfolio returns (or a mixture of both). Denote with  $r_{MSR}$  the return of the long-short MSR portfolio of the  $N$  assets. Then, the MSR excess return is a linear combination of the factors, with zero specific risk:

$$r_{MSR} - r_d = \beta'_{MSR}\mathbf{f},$$

and the factor exposures of the MSR are given by:

$$\beta_{MSR} = \frac{\Sigma_f^{-1}\Lambda}{\mathbf{1}'_N \Sigma^{-1}\tilde{\mu}}.$$

**Proof.** See Appendix A.5.

This proposition contains two results: (1) it shows that the idiosyncratic risk of the MSR portfolio is equal to zero, which

exemplifies the efficiency of the MSR portfolio as being a portfolio containing no unrewarded risk, and (2) it gives the factor exposures of the MSR. Hence, the MSR portfolio of the assets can be formally regarded as maximising the "Sharpe ratio of a portfolio of factors". This result implies the previously identified consequence of Proposition 4: since  $\Lambda = \mathbb{E}[\mathbf{f}]$ , the expected excess return and the volatility of the MSR portfolio of the  $N$  constituents are

$$\begin{aligned}\tilde{\mu}_{MSR} &= \beta'_{MSR}\Lambda = \nu\Lambda'\Sigma_f^{-1}\Lambda, \\ \sigma^2_{MSR} &= \beta'_{MSR}\Sigma_f\beta_{MSR} = \nu^2\Lambda'\Sigma_f^{-1}\Lambda,\end{aligned}$$

where  $\nu = \mathbf{1}'_N \Sigma^{-1}\tilde{\mu}$ . Hence, the squared Sharpe ratio is:

$$\lambda^2_{MSR} = \Lambda'\Sigma_f^{-1}\Lambda,$$

and the right-hand side is exactly the squared Sharpe ratio of the MSR portfolio of the factors. Hence, we recover the result that there is no loss of efficiency in the two-step process.

According to Proposition 5, the MSR portfolio is entirely described by its factor exposures and can be regarded as a portfolio of factors. In order to construct the MSR portfolio, one can proceed as follows:

1. Estimate the covariance matrix and the factor premia: these are the parameters  $\Sigma_f$  and  $\Lambda$ ;
2. Construct a portfolio of assets whose factor exposures are  $\Sigma_f^{-1}\Lambda$ ;
3. Normalise the weights in assets so that they sum up to one.

The number of parameters to estimate can be decomposed as follows:

1. Step 1 requires  $\frac{K(K+1)}{2}$  covariances and  $K$  factor premia. Note that these numbers are independent from the size of the universe;
2. Step 2 requires the knowledge of the  $NK$

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betas of assets with respect to the factors; 3. Step 3 requires no additional parameters.

Hence, the total number of parameters required to compute the weights of the MSR is  $NK + \frac{K(K+3)}{2}$ , which is an affine function of  $N$ . On the other hand, if the MSR portfolio was constructed without using the factor structure, there would be  $\frac{N(N+1)}{2}$  covariances and  $N$  expected returns to estimate, so that the number of parameters involved would be a quadratic function of  $N$ . Clearly, when  $N$  is large, this procedure is less sensitive of estimation risk. We emphasise that this comparison neither assumes that the factor structure is "exact" in the sense of the APT (i.e. that there is no idiosyncratic risk), nor that idiosyncratic risks are uncorrelated across assets.

The assumption that factors are returns or dollar-neutral returns in Proposition 5 is not as restrictive as it may seem since it is always possible to replace a set of pricing factors by portfolio returns. Indeed, Proposition 1 shows that if  $m = a + \mathbf{b}'\mathbf{f}$  is an SDF, then  $m^* = ac^* + \mathbf{b}'\mathbf{f}^*$ , where  $c^*$  is the constant-mimicking pay-off and  $\mathbf{f}^*$  the vector of factor-mimicking pay-offs is an SDF too. Rewriting pay-offs as prices multiplied by returns, we obtain the existence of a linear combination of returns that prices all securities. As a conclusion, if the SDF is not given as a combination of portfolio returns, the result of Proposition 5 remains valid provided the factors  $\mathbf{f}$  are replaced by the factor-mimicking returns, the factor covariance matrix by the covariance matrix of the returns and the factor risk premia by the expected excess returns to the portfolios.

### 3.3 Factor Indices as Additional Assets

Broadly speaking, there are two main approaches to factor investing. The first approach is a *substitution* approach, where the factor indices replace a set of traditional indices (e.g. asset class, country or sector indices). The second approach is an *extension* approach, where the factor indices are used as complements to existing indices. In the former case, the universe is changed ("substitution"), while in the latter the universe is extended ("extension"). By construction, a universe extension can only improve the efficient frontier, but to assess whether the new assets add value, it is necessary to know whether the frontier is *strictly* improved or whether it stays identical. We adopt the following notations:  $\mathcal{I}_0$  is the original universe (a set of traditional indices),  $\mathcal{I}_a$  is the set of additional assets (factor indices), and  $\mathcal{I}_1$  is the extended universe, i.e. the union of the two sets. The numbers of elements in these universes are denoted respectively with  $n$ ,  $K$  and  $p$ , so that  $p = n + K$ .

#### 3.3.1 Testing for Redundancy

The new assets (factor indices) are said to be *redundant* with respect the existing assets (asset class indices) if the former can be regarded as portfolios of the latter. In this case, the efficient frontier would not be modified (improved) at all by the introduction of the factor indices. Hence, a necessary condition for the new indices to add value is that they should not be redundant. From a technical standpoint, redundancy is an undesirable property because it makes the covariance matrix of the extended universe singular, so that portfolios that require inversion of this matrix, such as the GMV or the MSR, are ill defined.

The following proposition gives three equivalent characterisations of redundancy.

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The third one relates this property to the canonical correlation, which is defined as:

$$\max_{\substack{\mathbf{x}_0 \neq \mathbf{0}_n \\ \mathbf{x}_a \neq \mathbf{0}_K}} \left| \frac{\mathbf{x}'_0 \Sigma_{0,a} \mathbf{x}_a}{\sqrt{\mathbf{x}'_0 \Sigma_0 \mathbf{x}_0} \times \sqrt{\mathbf{x}'_a \Sigma_a \mathbf{x}_a}} \right|, \quad (3.7)$$

where  $\mathbf{x}_0$  and  $\mathbf{x}_a$  denote respectively a  $n \times 1$  vector and a  $K \times 1$  vector of weights. When a risk-free asset exists, the canonical correlation can be interpreted as the maximum absolute correlation between two leveraged portfolios of constituents from  $\mathcal{I}_0$  and  $\mathcal{I}_a$ .

#### Proposition 6 (Characterisation of Redundancy of New Assets)

Consider an initial universe  $\mathcal{I}_0$  and an additional universe  $\mathcal{I}_a$  with returns stacked respectively in the vectors  $\mathbf{r}$  and  $\mathbf{R}$ . Assume that the assets within each universe are not redundant, i.e. the covariance matrices  $\Sigma_0$  and  $\Sigma_a$  are non singular. Then, the following three statements are equivalent:

1. The covariance matrix of the extended universe  $\mathcal{I}_0 \cup \mathcal{I}_a$  is singular;
2. Two non-zero vectors  $\mathbf{x}_0$  and  $\mathbf{x}_a$  and a scalar  $c$  exist such that  $\mathbf{x}'_0 \mathbf{r} = \mathbf{x}'_a \mathbf{R} + c$ ;
3. The canonical correlation of the universes  $\mathcal{I}_0$  and  $\mathcal{I}_a$  equals 1.

**Proof.** See Appendix A.6.

The next proposition gives a practical way of computing the canonical correlation, which alleviates the need to solve the maximisation program (3.7). This result can also be found in Pukthuanthong and Roll (2014).

#### Proposition 7 (Canonical Correlation)

The value of the maximum in (3.7) is the square root of the maximum eigenvalue of the matrix  $\mathbf{M} = \Sigma_0^{-\frac{1}{2}} \Sigma_{0,a} \Sigma_a^{-1} \Sigma'_{0,a} \Sigma_0^{-\frac{1}{2}}$ . It is achieved when  $\mathbf{w}_0$  is proportional

to an associated eigenvector and  $\mathbf{w}_a$  is proportional to

$$\Sigma_a^{-1} \Sigma'_{0,a} \Sigma_0^{-\frac{1}{2}} \mathbf{w}_0.$$

**Proof.** See Appendix A.7.

Of course, when  $n = K = 1$ , the canonical correlation is simply the absolute value of the linear correlation coefficient between the two constituents. When  $n = 1$  but  $K \geq 1$ , it can be verified that the matrix  $\mathbf{M}$  is actually a scalar number,  $\frac{1}{\sigma_0^2} \Sigma_{0,a} \Sigma_a^{-1} \Sigma_{a,0}$ , and that this number is the  $R^2$  in the regression of the original asset on the new ones. Thus, the canonical correlation is  $\sqrt{R^2}$ , and the new assets are redundant with the original one if, and only if, the  $R^2$  of the regression is equal to 1.

#### 3.3.2 Impact on the Efficient Frontier without a Risk-Free Asset

By the two-fund separation theorem, the (long-short) efficient frontier without a risk-free asset is generated by any two portfolios, e.g. the GMV and the maximum risk-return ratio (MRR) portfolios, where the risk-return ratio is defined as the expected return divided by the volatility (this is identical to a Sharpe ratio computed with a zero risk-free rate). Geometrically, in the standard deviation - expected return diagram, the frontier is (a quarter of) a hyperbola, whose equation is:

$$\sigma = \sqrt{\sigma_{GMV}^2 + \frac{1}{\zeta_{MRR}^2 - \zeta_{GMV}^2} (\mu - \mu_{GMV})^2}, \quad \mu \geq \mu_{GMV}. \quad (3.8)$$

Hence, the efficient frontier is entirely described by the following three parameters: the minimum volatility,  $\sigma_{GMV}$ ; the maximum risk-return ratio,  $\zeta_{MRR}$ ; and the risk-return ratio of the  $\zeta_{GMV} = \mu_{GMV}/\sigma_{GMV}$ . The minimum volatility and the maximum

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risk-return ratio are two diversification indicators that have straightforward interpretations: the minimum variance is a pure risk indicator, and the maximum ratio measures the highest reward achieved per unit of risk. A straightforward property is that in a universe extension,  $\sigma_{GMV}$  decreases and  $\zeta_{MRR}$  increases. But the effect on  $\mu_{GMV}$  is undefined ex-ante. Proposition 8 below relates the changes in these parameters to the coefficients of the regressions of the new assets on the original ones:

$$R_k = \alpha_k + \beta'_k \mathbf{r} + \varepsilon_k, \quad k \in \mathcal{I}_a. \quad (3.9)$$

We collect the alphas and the betas in the  $n \times 1$  vector  $\alpha$  and the  $n \times K$  matrix  $\beta$ . At this stage, we emphasise the difference between regressions (3.9) and (3.6). In (3.6), the individual securities are regressed on the factors, so the factors are in the left-hand side and zero alphas are expected. In (3.9), the factors are regressed on the original assets, so they are in the right-hand side, and non-zero alphas are expected.

**Proposition 8 (Impact on the Efficient Frontier without a Risk-Free Asset)**

Consider the regressions of the returns of the new assets on the original ones (Equations (3.9)). When the universe  $\mathcal{I}_0$  is expanded to  $\mathcal{I}_1$ :

- the change in the inverse of the minimum variance is:

$$\frac{1}{\sigma_{GMV,1}^2} - \frac{1}{\sigma_{GMV,0}^2} = [\beta' \mathbf{1}_n - \mathbf{1}_K]' \Sigma_\varepsilon^{-1} [\beta' \mathbf{1}_n - \mathbf{1}_K];$$

- the change in the squared maximum risk-return ratio is:

$$\zeta_{MRR,1}^2 - \zeta_{MRR,0}^2 = \alpha' \Sigma_\varepsilon^{-1} \alpha;$$

- the change in the ratio of expected return to variance for the GMV is:

$$\frac{\mu_{GMV,1}}{\sigma_{GMV,1}^2} - \frac{\mu_{GMV,0}}{\sigma_{GMV,0}^2} = -[\beta' \mathbf{1}_n - \mathbf{1}_K]' \Sigma_\varepsilon^{-1} \alpha.$$

**Proof.** See Appendix A.8.

These expressions are reminiscent of Proposition 4. Similar results can be found in Jobson and Korkie (1984), who computes the change in the squared Sharpe ratio in a universe extension. Proposition 8 enables to state a necessary and sufficient condition for the new assets to be irrelevant, i.e. for the efficient frontier to be unaffected by their introduction. From (3.8), it appears that the efficient frontier is unchanged if, and only if, we have:

$$\sigma_{GMV,1}^2 = \sigma_{GMV,0}^2,$$

$$\zeta_{MRR,1}^2 = \zeta_{MRR,0}^2,$$

$$\mu_{GMV,1} = \mu_{GMV,0}.$$

On the other hand, by Proposition 8, if the first two equalities hold, it must be the case that  $\beta' \mathbf{1}_n = \mathbf{1}_K$  and  $\alpha = \mathbf{0}_K$ , which implies in turn that the third equality also holds. As a conclusion, a necessary and sufficient irrelevance criterion is:

$$\sigma_{GMV,1}^2 = \sigma_{GMV,0}^2$$

and

$$\zeta_{MRR,1}^2 = \zeta_{MRR,0}^2,$$

or equivalently:

$$\beta' \mathbf{1}_n = \mathbf{1}_K \text{ and } \alpha = \mathbf{0}_K.$$

**3.3.3 Impact on the Efficient Frontier with a Risk-Free Asset**

If there is a risk-free asset, the efficient frontier in the standard deviation - expected return space is a straight line with equation:

$$\mu = r_d + \lambda_{MSR} \sigma, \quad \sigma \geq 0.$$

Hence, it is shifted upwards if, and only if, the maximum Sharpe ratio increases. Proposition 9 below gives an expression for the change in squared Sharpe ratio: it

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is identical to the one established by Jobson and Korkie (1984) and similar to the one written in Proposition 4. It uses the alphas from regressions involving excess returns (we use tildas to denote returns in excess over the risk-free rate):

$$\tilde{R}_k = \tilde{\alpha}_k + \tilde{\beta}'_k \tilde{r} + \varepsilon_k, \quad k \in \mathcal{I}_a. \quad (3.10)$$

**Proposition 9 (Impact on the Efficient Frontier with a Risk-Free Asset)**

*Consider the regressions of the returns of the new assets on the original ones (Equations (3.9)). When the universe  $\mathcal{I}_0$  is expanded to  $\mathcal{I}_1$ , the change in the squared maximum Sharpe ratio is:*

$$\lambda_{MSR,1}^2 - \lambda_{MSR,0}^2 = \tilde{\alpha}' \Sigma_\varepsilon^{-1} \tilde{\alpha}.$$

**Proof.** The derivation of this equality relies on the identity  $\lambda_{MSR,j}^2 = \tilde{\mu}'_j \Sigma_j^{-1} \tilde{\mu}_j$ , which holds for  $j = 0$  and  $1$ , and is similar to the proof of Proposition 8.

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The objective of this section is to provide a description of the regularities identified in the returns of three important asset classes, namely equities, bonds and commodities. It is beyond the scope of our paper to provide further verification of the existence and the robustness of these patterns, or to search for new patterns, so we limit ourselves to those relatively consensual factors whose existence has been documented in several academic papers, relies on plausible economic justification and is widely accepted by the investors' community. For each of these patterns, we describe the empirical evidence in favour of, or against, the existence and we review the possible explanations. Among these, the risk-based rational explanation states that the pattern is due to a pricing factor, while other justifications may invoke the presence of behavioural biases or market frictions.

We also give an overview of the possible multi-class pricing factors, some of which have only been recently introduced in the literature and are not yet standard in the investment practice. We conclude with a list of the factors that we will empirically study in Section 5.

### 4.1 Equity Related Factors

Equities is the asset class on which the largest number of asset pricing models have been tested. Since the recognition that the market beta alone is not sufficient to explain the cross section of expected returns, a growing number of empirical regularities, or "anomalies", have been identified in returns. This has resulted in a "factor zoo" populated by over 300 potential factors according to the recent survey of Harvey et al. (2013).

#### 4.1.1 Insufficiency of the Market Factor

The CAPM (see Section 2.3.2) identifies the market factor as the unique pricing factor: under the model assumptions, expected excess returns on stocks are proportional to their market beta. The theoretical factor is a portfolio of all assets weighted by their market capitalisation, including assets that are privately held and do not have observable market prices (e.g. social security claims, human capital, etc.). As a result, the factor is not itself observable, and its approximation is a matter of debate (see Roll (1977)). The most common approach is to use a broad cap-weighted equity index as a proxy for the true unobservable market factor.

Beyond the observability issue, it has early been recognised that the CAPM prediction regarding expected returns is not empirically validated for two main reasons. First, patterns exist, which would be regarded as "anomalies" if the CAPM was the true asset pricing model, which cannot be explained by the exposure to a broad index. Among the most famous are the size, value and momentum effects, which are described in detail below. This finding suggests that one factor only is not sufficient to explain all possible dimensions of expected returns. The second reason is that the positive linear relation between expected returns and betas is not well-verified in the data. Black et al. (1972) observe that after the period 1931-1939, high-beta portfolios of stocks have negative alphas, while low-beta portfolios have positive alphas. The relationship between alpha and beta tends to become even more clearly decreasing in the most recent period of their study. This means that the capital

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market line is too flat to explain the cross section of stock returns. This observation has led to the introduction of the "betting-against-beta" factor of Frazzini and Pedersen (2014), which is a portfolio with a long position in low-beta assets and a short position in the high-beta ones, both legs being leveraged or deleveraged to have a beta of one. The authors find that this factor has a positive long-term Sharpe ratio, which can be theoretically rationalised in a model where investors have limited access to leverage. The CAPM thus overstates the returns on high-beta stocks and understates the returns on low-beta ones. In addition, it misses a number of notorious patterns that have been documented in stock returns, which we review now.

### 4.1.2 Size Effect

The size effect refers to the empirical finding that small cap stocks tend to outperform large cap stocks. An extensive survey of the empirical evidence in favour of or against the size effect, as well as the related explanations, is provided in Van Dijk (2011).

#### *Empirical Evidence of the Size Effect*

The first empirical evidence of a size effect was provided by Banz (1981), who showed that over the period 1936-1975, common stocks of large firms earn lower average returns than those of small firms after controlling for the market beta. Other studies published at this time with time spans covering approximately 1962-1979 point to the same effect (Reinganum (1981), Brown et al. (1983), Keim (1983)). As shown by Keim (1983), the outperformance of small stocks may be largely explained by a "turn-of-the-year effect": the relation between size and average return is slightly negative in non-January months and steeply

negative in January. Over a longer sample (1973-1985), Lamoureux and Sanger (1989) add to the evidence by demonstrating the existence of a size effect in firms traded over the counter. The use of non-listed firms addresses the selection bias concern that potentially affects the dataset of listed firms only, and gives access to smaller capitalisations. Among these studies, both Banz (1981) and Brown et al. (1983) suggest that the relation between market capitalisation and return is not linear and is more pronounced on the side of small firms. Brown et al. (1983) actually uncover a close to linear relation between return and the logarithm of market value.

The size effect is also present in the work of Fama and French (1992): size proves to be a more discriminant attribute than the market beta for expected returns in the cross section, and after controlling for size, there is no significant beta effect. This finding reinforces an earlier observation made by Schwert (1983) about the work of Banz: the relation between size and return is as statistically significant as that between market beta and return in Fama and MacBeth (1973), which poses a serious challenge to the CAPM. Fama and French (1992) also conduct Fama-MacBeth regressions of stock returns on size and find that the slope is significantly negative over the periods 1963-1990, 1941-1990 and 1941-1965.<sup>10</sup> The latter period is the one used in the early empirical tests of the CAPM, which tended to validate the positive relationship between expected return and beta, but even over this period, the beta effect is not robust to a size control.

Evidence for other developed markets than the US is provided in Liew and Vassalou

10 - Technically, Fama and French (1992) regress stock returns on the logarithm of the market value, as opposed to the market value itself.

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(2000), but the effect is not significant everywhere: if small cap stocks outperform large cap stocks in all studied markets (with the exception of Switzerland), there are many markets in which the excess return is not statistically significant. Van Dijk (2011) reviews evidence in data on developed or emerging countries: in the 21 tested regions except Korea, there is a positive size premium, which gives strong arguments in favour of the geographical robustness of the size effect.<sup>11</sup> Fama and French (2012) complete the international evidence by studying the Global, North America, Europe, Japan and Asia Pacific zones over the period 1990-2011. They find that the size pattern is driven by the extreme two size deciles, which is in line with earlier findings showing that the impact of size on average return is not strictly monotonic across size deciles (see Banz (1981)).

11- However, Van Dijk (2011) points that some of the studies that he cites do not meet the standard academic quality criteria, so that their results should be taken cautiously.

Some studies have looked at the size effect within segments of the equity market. For instance, Daniel and Titman (1997) and Fama and French (2012) show that it is concentrated within the high book-to-market segment. Vassalou and Xing (2004) construct a measure of default risk from the model of Merton (1974) and they show that over the period 1971-1999, the excess return of small cap stocks over large cap stocks is only significant for the firms within the highest quintile of default risk. In the other quintiles, the size effect is present but not statistically significant.

### *Questions on Robustness*

As for other patterns, a series of papers have questioned the existence of a genuine size effect by showing that it is not robust to a correction for statistical issues or changes in implementation details such

as the choice of the measurement period and the incorporation of transaction costs.

From a statistical perspective, Roll (1981) examines the test methodology of Banz (1981) and Reinganum (1981) and argues that since small stocks are less frequently traded than large ones, their estimated risk measures (volatility or beta with respect to a market portfolio proxy) are biased downwards, which results in the misleading impression that they earn higher average returns while not being riskier. However, his study does not directly compare the returns on large cap and small cap stocks, and focuses instead on the performance and the risk of a cap-weighted index (taken as a proxy for a large-stock portfolio) and an equally-weighted index (taken as a proxy for a portfolio without a large stock bias). Reinganum (1982) concludes that Roll's statistical artefact does play a role, but is unable to fully explain the size effect. Handa et al. (1989) also point that market betas are sensitive to the length of the estimation window, and that size is no longer statistically significant to explain returns when betas are computed with annual data. Another statistical criticism comes from Lo and MacKinlay (1990), and applies more generally to any empirical test of an asset pricing model: they demonstrate that such tests are biased towards rejection of the tested model (e.g. the CAPM) when the test portfolios are sorted on an attribute (e.g. size) identified by previous empirical research as a driver of average returns. Correcting for the effects of this "induced ordering" turns out to considerably reduce the statistical significance of the negative link between the CAPM alpha and the market capitalisation.

Roll (1983) also shows that the excess return of small cap stocks over large cap stocks depends on how returns are measured:

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taking simple arithmetic averages or computing the returns to equally-weighted portfolios sorted on size with daily rebalancing produces a larger return spread than looking at the returns to annually rebalanced equally-weighted portfolios. More importantly, the spread measured from the latter, more realistic, portfolios, is not statistically significant. Overall, he concludes that the size effect is not robust to the way returns are measured.

Other studies have documented the role of transaction costs. Stoll and Whaley (1983) estimate that these costs are higher for small stocks and find that small cap stocks do not earn higher net returns compared to large cap stocks. Nevertheless, Schultz (1983) find that the excess returns survive the introduction for transaction costs, so that these frictions are only an incomplete explanation.

Another controversy around the size effect concerns its robustness over time. An aspect of this time variation is the seasonality: the results of Keim (1983) and Daniel and Titman (1997) show that the size pattern is largely explained by January returns. One explanation for the abnormal January returns is the tax-loss selling hypothesis, but Reinganum (1983) argues that an anomalous behaviour is observed even in those of the small stocks that are unlikely to be affected by tax-loss sellings. In the long run, the size premium also appears to vary over time. For instance, Handa et al. (1989) report a positive (albeit non statistically significant) relationship between market value and return over the period 1941-1954. One of the most salient features is the "disappearance" of the size effect after the early 1980s, several studies finding no outperformance of small stocks during the following two decades (see for example

Horowitz et al. (2000a), Horowitz et al. (2000b) and, for the UK market, Dimson and Marsh (1999)). A possible explanation for this disappearance or reversion of the size effect is an investors' rally to profit from the anomaly after it has been uncovered (see Dimson and Marsh (1999) and Schwert (2003)).

But as emphasised by Van Dijk (2011), a temporary reversion of the size effect is not inconsistent with an explanation based on rational asset pricing: if small stocks earn higher average returns because they are more exposed to a priced risk factor, then one should expect them to perform poorly in the "bad times" characterised by a high marginal utility of consumption (see Section 2.2.2). Hou and Van Dijk (2014) also warn that realised returns are very imprecise estimates of expected returns. This point was made by Elton (1999), who argues that even if they are averaged over long periods of time, as is typically done in asset pricing tests, realised returns may substantially deviate from expected returns: the reason is that unexpected returns contain a contribution of unanticipated economic shocks, and these surprises do not necessarily cancel out when they are averaged if they are sufficiently large, or serially correlated. The discrepancy between expected returns and realised returns was also highlighted in the context of the bond term premium (see Fama and Bliss (1987) and Elton (1999)), but it seems that only few equity studies have explicitly taken it into account. An example is Campello et al. (2008), who relate expected stock returns to expected corporate bond returns using the model of Merton (1974), and model the latter expected returns as functions of the yield spread, expected yield changes and expected default probabilities. As far as the size effect is concerned, Hou and Van Dijk (2014) present evidence suggesting that the disappointing returns

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of small firms between 1984 and 2005 were due to unexpected negative profitability shocks, and that removing the effect of these shocks enables to recover a significant size premium.

### *Explanations for the Size Effect*

Risk-based explanations for the size effect posit that small cap stocks outperform large cap stocks because they are more exposed to a systematic risk factor (see Proposition 1), but the exact nature of this factor has not yet been identified. Fama and French (1993) relate size to profitability: small firms are more likely than large firms to experience long periods of depressed earnings, which represents a risk factor for which investors would require a compensation. Vassalou and Xing (2004) propose another interpretation in terms of firm default risk: they note that the size effect observed at the broad market level is in fact driven by the 20% firms with the highest default risk, and within this segment, size varies from very small to medium-to-large, with average returns being higher for the former firms. Thus, the excess return of small firms would be explained by their higher financial distress. This argument is in line with the results of Fama and French (1993), who report that small firms are more exposed to the default factor (DEF), defined as the excess return of corporate bonds over sovereign bonds.

Another explanation is that small cap stocks are less liquid: in support for this idea, Amihud and Mendelson (1986) show that size has no significant effect on returns when a more obvious proxy for liquidity, namely the bid-ask spread, is included in the regression. Other explanations involve the higher downside risk of small stocks (Chan et al., 1985), or their higher exposure to the market in good times, when the expected market return is high (Jagannathan and Wang, 1996).

Behavioural explanations have also been proposed. Banz (1981) suggests that the size effect arises because of a lack of information about small firms: investors are more reluctant to invest in their stocks because they have little information about the distribution of returns. The idea that there are less investors for small firms than for large ones is also present in the CAPM with limited information of Merton (1987): depending on parameter values, the model is able to predict a negative relationship between the alpha with respect to the market factor and the size. Other related explanations involve investors' over-confidence (Daniel et al., 2001) or the delay in the incorporation of new information in stock prices (Hou and Moskowitz, 2005). Some behavioural explanations are not optimistic about the persistence of the effect in the future: for instance, Daniel and Titman (1997) argue that if investors have believed in the past that small or value firms were exposed to systematic factors, this may have justified high returns for these stocks, but the anomaly will vanish as agents become aware of their mistake.

### *Size Factor*

The size factor is essentially defined as the excess return of small cap stocks over large cap stocks. If the return spread between small and large stocks is due to differences in exposures to a systematic risk factor (yet to be identified), then the long-short return should proxy for the variations in this unknown factor. But it should be noted that this return may not have a (multivariate) beta of 1 with respect to this factor and 0 with respect to the other factors, so that the size premium is not necessarily equal to the risk premium of the underlying factor (see Proposition 1).

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Fama and French (1993) introduce a "small-minus-big" (SMB) portfolio, defined as the difference between the average return on three small-stock portfolios and that on three large-stock portfolios. In each size group, three portfolios are defined by sorting stocks on book-to-market: this is meant to avoid biasing the small (resp., the large) portfolio towards high (resp., low) book-to-market stocks. By definition, the SMB premium is the expected return on the SMB portfolio (see Section 2.2.2), which Fama and French find to be 0.27% per month (3.24% per year) over the period 1963-1991. Variants of this double-sort procedure are encountered in the literature. For instance, Liew and Vassalou (2000) control for momentum in addition to book-to-market.

Various tests have been conducted to check whether the SMB factor is both a pricing factor and a priced factor (see Section 2.2.2). Since asset pricing theory predicts that a factor commands a positive premium if, and only if, it covaries negatively with the SDF (see (2.10)), one may expect the size factor to be higher in "good" states of the world, where marginal utility of consumption is low, compared to "bad" states of the world. In this perspective, Liew and Vassalou (2000) show that the SMB factor is positively related to the future GDP growth in all of the developed markets that they test: if good states are defined as states with high future GDP growth, this supports the assumption that the SMB factor performs better in good states, which justifies a positive premium for this factor. Vassalou and Xing (2004) estimate the size premium defined as in (2.10) in a factor model where the SDF is a linear combination of the three Fama-French factors. They find the premium to be negative (thus having the wrong sign), but non significant, although the coefficient of the size factor

in the SDF is significant. Thus, their results imply that the size factor is a "pricing factor" in the sense that it is relevant to price assets, but not a "priced factor" in the sense that it does not carry a significant premium.

Finally, some recent papers check the compatibility between the SMB factor and some of the recommendations drawn from theoretical factor models: Charoenruek and Conrad (2008) verify that the conditional expected return on the SMB portfolio is proportional to the conditional variance with a positive slope, which is one of the conditions that they identify for the factors in Merton's ICAPM. Pukthuanthong and Roll (2014) show that the SMB factor is related to the principal components of the covariance matrix of returns, so that it satisfies the necessary conditions to be a factor in the sense of APT.

### 4.1.3 Value Effect

Value investing has become a popular investment policy since the seminal writings of Graham and Dodd (1934) and Graham (1959). Broadly speaking, a stock is said to be "value" if it is inexpensive according to some criterion. The most widespread measure of value in the academic literature is the book-to-market ratio (BE/ME), but other measures have been proposed, including in particular past long-term returns.

#### *Empirical Evidence of the Value Effect*

Early evidence of a relationship between expected stock returns and BE/ME dates back to Stattman (1980) and Rosenberg et al. (1985). Fama and French (1992) document a similar effect for US stocks over the period 1963-1990. In their study, the value effect turns out to be stronger than the size effect, the spread between the 5% deepest value stocks and the 5%

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deepest growth stocks (1.53% per month) being larger than the spread between the 5% smallest and the 5% largest stocks (0.74%). Fama-MacBeth regressions show that the (logarithm of the) BE/ME ratio has a positive and significant impact on stock returns in the cross section over the period 1963–1990, an impact which is robust to the introduction of size as a regressor.

Several articles have looked at the robustness of the value effect across geographical zones and time periods. For instance, as they do for the size effect, Liew and Vassalou (2000) show that high BE/ME stocks outperform low BE/ME ones in several developed markets, and the spread is significant in most of them. However, the findings of Vassalou and Xing (2004) suggest that it is not uniformly strong in all segments of the equity market. Specifically, they show that it is concentrated within firms with the highest default risk (where default risk is assessed through their default likelihood measure). The difference with respect to the size effect is that the BE/ME effect is observed within the two quintiles with highest default risk, while the size premium is only significant within the top quintile. Israel and Moskowitz (2013) report a related result, namely that the value premium monotonically decreases with firm size. If one believes that small cap firms have higher default risk compared to large cap stocks, this finding is line with the results of Vassalou and Xing (2004). Looking at the value effect across industries, Novy-Marx (2011) show that value stocks outperform growth stocks within a given industry, while it is not true that value stocks of a growth industry outperform the growth stocks of a value industry. Instead of studying value profits across various groups of stocks, Asness et al. (2013) investigate the robustness of the effect across market liquidity or funding liquidity conditions, and they find that the returns

on value strategies are lower in periods of high liquidity.

Some recent studies have proposed to refine or complement the standard BE/ME measure. Asness and Frazzini (2013) show that measuring the book and market values on the same date, namely the previous fiscal year end, is not optimal: while a lag in the book value is inevitable due to publication delays, using the current price produces higher returns for the high-minus-low BE/ME portfolio. Novy-Marx (2013a) emphasises that the essence of value investing is to purchase high quality assets at a low price, not simply to purchase any assets at a low price. Thus, he proposes to combine sorts on gross profitability and BE/ME to identify value stocks.

Other measures of value combining accounting and market information have been proposed. Among them is the price-earnings ratio (P/E): Basu (1977) shows that over the period 1957–1971, low P/E portfolios outperformed the high P/E ones, and Basu (1983) shows that this effect subsists even after controlling for firm size. Lakonishok et al. (1994) and Fama and French (1996) report similar patterns in more recent data (1963–1993 in the work of Fama and French), using E/P and the cash flow-to-price ratio C/P. They also propose a measure which uses no market information at all and is defined as the rank of the firm based on the past five-year sales growth: firms with the highest past sales growth appear to underperform the others. While all these variables work as measures of value, it is BE/ME that produces the largest spread. Following Lakonishok et al. (1994), Fama and French (1996) also perform double sorts on sales rank and one of the variables BE/ME, E/P and C/P and find that the use of sales rank in combination with one of the ratios allows us to increase the spread.

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The value effect has also been connected to the long-term reversal effect. DeBondt and Thaler (1985) show that when stocks are sorted on their past thirty-six month performance, portfolios of winners tend to underperform the losers. Fama and French (1996) show that the three-factor model explains the long-term reversal because the alphas of portfolios sorted on the past five-year return are statistically insignificant. Moreover, the beta with respect to the HML factor – defined as the excess return of high BE/ME stocks over low BE/ME ones – is clearly decreasing in the past return, which supports the theory of a link between BE/ME and the past return. More recently, Gerakos and Linnainmaa (2012) have argued that the five-year change in market value drives out BE/ME from a multivariate regression and, more importantly, that most of the variation in BE/ME that gives rise to a return spread comes from changes in equity market value. This leads them to suggest that the five-year change in market value can be a valuable substitute for BE/ME. The past five-year return is also used in recent studies on asset pricing across asset classes, like Israel and Moskowitz (2013) and Asness et al. (2013), as a measure of value in asset classes for which there is no equivalent to BE/ME.

### *Explanations for the Value Effect*

As for the size effect, risk-based explanations postulate the existence of an underlying systematic risk factor to which value stocks are more exposed than growth stocks. For instance, this factor could be related to financial distress. In support of this argument, Fama and French (1992) report that firms with negative BE/ME have high average returns. They interpret this finding as an indication that the ratio BE/ME proxies for financial distress: a negative BE means that earnings are persistently negative, and a high BE/ME that ME has fallen down compared to BE, because of negative market

anticipations.<sup>12</sup> Vassalou and Xing (2004) make a similar association between a high BE/ME and increased financial distress. Another explanation suggested by Fama and French (1992) is that firms with high BE/ME have high market leverage relative to book leverage: this interpretation is based on the identity

$$\frac{BE}{ME} = \frac{A}{ME} \times \left(\frac{A}{BE}\right)^{-1},$$

where  $A$  is the book asset value. Fama and French (1993) also explain that BE/ME may be related to profitability: firms with high BE/ME have persistently low earnings, which result in low current price and high expected returns.

Zhang (2005) develop a model with a distinction between assets in place and growth options, costly reversibility of capital and time-varying risk premia. In bad times, all firms try to reduce their capital, but value firms experience more difficulties doing so because they have more assets in place and scaling them down is costly. When the environment improves, growth firms exercise their growth options and invest more than value firms. Overall, value firms are riskier in bad times, which justifies the value premium. Petkova and Zhang (2005) provide supporting empirical evidence for this model: value betas are positively correlated with the expected market premium, and since the expected premium tends to be high in bad times, this implies that value betas are higher in bad times.

In addition to the rational explanations, there also exist behavioural justifications for the existence of a value premium. Lakonishok et al. (1994) invoke excessive optimism of investors with respect to stocks that have performed well in the past, and excessive pessimism with respect to those that have performed poorly. Investors are reluctant to incorporate mean reversion

12 - The relationship between BE and earnings is given by the accounting identity  $CF_t = E_t - (BE_t - BE_{t-1})$ , where  $CF_t$  is the cash flow distributed to equity holders, and  $BE_t$  and  $E_t$  are the book value of equity and the earnings at the end of period  $t$ . Since  $CF_t \geq 0$ , we have  $BE_t \leq E_t + BE_{t-1}$ , which shows that book value can go negative if substantially negative earnings occur for a long period of time.

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in their predictions and rely heavily on past performance, so that they neglect the least "glamour" stocks, which have done poorly in the past, which lowers their prices and increases their subsequent returns. Conversely, they extrapolate high past growth rates, so that they are ready to pay high prices for the corresponding stocks, which depresses the subsequent returns. On the other hand, the authors claim that value strategies are not riskier than growth strategies, so they reject a risk-based explanation. In particular, they did not perform particularly poorly in NBER recessions, when the marginal utility was presumably high. Daniel and Titman (1997) also express doubts as to the explanation that relates BE/ME to a distress factor: they argue that expected returns depend on BE/ME or size as a characteristic, but not on the exposures to the Fama-French value or size factors. This leads them to favour behavioural explanations of the anomaly.

### *Value Factor*

Similar to the size factor, the value factor is defined as the return to a portfolio going long the deepest value stocks and short the deepest growth stocks. Fama and French (1993) define the "high-minus-low" (HML) portfolio as the excess return of a portfolio of high BE/ME stocks over low BE/ME stocks. Just like for the SMB portfolio, the HML portfolio is meant to be the mimicking portfolio for an unobservable factor, the exposure to which are proxied by the ratio BE/ME. The HML premium, equal to the excess return of the high BE/ME portfolio over the low BE/ME portfolio, is found to be 0.40% per month (4.8% per year) over the period 1963-1991, which is larger than the SMB premium.

In most of the developed markets that they study (including the US), Liew and Vassalou

(2000) find that the HML factor is positively correlated with future GDP growth, which supports the idea that this portfolio performs better in good states and thus commands a positive risk premium. In the same spirit, Petkova (2006) study whether the HML factor can be related to investment opportunities. They indeed find that the innovations to the term spread, a variable that is a good descriptor of business conditions (Fama and French, 1989), are strongly related to the HML factor.

Recent work by Gerakos and Linnainmaa (2012) and Gerakos et al. (2013) questions the use of the HML factor, arguing that it consists of a priced component and an unpriced one, so that the HML beta factor may not fully reflect the covariation with the priced part. The authors explain that this results in fallacious success of the three-factor model in pricing portfolios sorted on C/P or E/P.

Other papers have checked whether the HML factor satisfies necessary conditions imposed by the ICAPM or the APT (see Section 2.4). Charoenrook and Conrad (2008) show that the conditional mean of HML returns is positively related to their conditional variance, as it should be for a factor-mimicking portfolio (although the relationship is not significant in all periods). Pukthuanthong and Roll (2014) perform a statistical analysis and find that as the SMB one, the HML factor is related to the principal components of the covariance matrix of returns.

### **4.1.4 Momentum Effect**

The momentum factor is arguably the most commonly adopted equity factor after the size and value factors. Momentum (or "trend-following") strategies aim to exploit a short-term continuation effect in returns by purchasing assets that have performed

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well in the recent past and selling those that have had poor returns.

### *Empirical Evidence of the Momentum Effect*

The autocorrelation of stock returns depends on the frequency at which these returns are measured. Early work on continuation or reversal phenomena in returns were concerned with the validity of the random walk hypothesis – which states that returns are unpredictable – and the statistical and economic significance of the predictability in stock returns. Examples of such studies include Fama (1970), DeBondt and Thaler (1985), French and Roll (1986), Fama and French (1988b) and Lo and MacKinlay (1988). Jegadeesh (1990) takes a cross-sectional perspective by looking at the performance of strategies that select stocks based on their past one-month return and hold them for one month: an equally-weighted portfolio of the 10% stocks with the past lowest return outperforms a portfolio of the 10% winners. This can be interpreted as evidence for short-term reversal. A similar result holds for horizons from 3 to 5 years (DeBondt and Thaler, 1985).

On the other hand, for intermediate horizons, a momentum effect is observed. Jegadeesh and Titman (1993) coin the term "relative strength strategies" for dollar-neutral strategies that purchase the stocks that have best performed over the past 3 to 12 months, sell the losers and hold the position for 3 to 12 months. Among US stocks, over the period 1965–1989 and for all choices of ranking and holding periods within the aforementioned range, these momentum strategies have statistically significant positive performance, around 1% per month (only one out of the 16 tested combinations yields a positive but non significant performance). This result subsists (and is in fact reinforced) after allowing for a one-week delay between

the measurement of past returns and the portfolio formation. It also holds within subsamples of size and market beta, so that the effect observed at the whole sample level is not driven by a particular subsample. Fama and French (1996) confirm the short-term continuation in returns for the period 1963–1993: sorting stocks in deciles based on their past one to three-year performance produces average returns that are higher for past winners than for past losers, even though the effect is not strictly monotonic across deciles. They also note that the pattern of returns is significantly different over the period 1931–1963: in this period, reversal is observed for both short-term and long-term returns. On the other hand, in the period 1963–1993, the momentum effect poses a puzzle to the three-factor model of Fama and French (1993): past short-term losers have higher exposures than past winners to the factors SMB and HML, so that they should earn higher average returns.

Although the momentum anomaly was reported well after the size and the value effects (see Banz (1981) and Stattman (1980)), the strategy of buying past winners has a long tradition among institutional investors. Grinblatt et al. (1995) study a sample of 274 US mutual funds over the period 1974–1984, and it appears that the majority of the funds studied had a tendency to purchase past winners, which suggests that they followed, at least implicitly, a momentum investing policy. The authors provide yet another evidence for momentum profitability: they report that the managers who followed "momentum-like" strategies, by tilting their portfolios towards past quarter or past semester winners and away from past losers, outperformed managers who followed "contrarian-like" strategies by doing the opposite. They conclude that part of the success of these funds might be attributed to a systematic investment

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rule rather than a skill in picking stocks. Another study at the mutual fund level is done by Carhart (1997), who ties persistence in performance amongst mutual fund managers to persistence in performance of underlying stocks.

The profitability of momentum strategies has also been documented in international equity markets. Rouwenhorst (1998) reports a momentum effect in 12 European countries, with ranking and holding periods of 6 months. He also shows that the pattern exists in all size groups, thereby extending the US results of Fama and French (1996). He finally unveils a positive correlation between the returns to US and European momentum strategies, which is suggestive of the presence of a common source of risk. Similar comovements between momentum strategies across equity markets are reported in a more recent contribution from Asness et al. (2013). Further evidence for momentum in European, UK, Japan and global stocks is given in Israel and Moskowitz (2013) and Asness et al. (2013). Importantly, these studies were written 20 years after the seminal paper of Jegadeesh and Titman (1993) uncovering the anomaly, so it appears that the momentum effect survived its publication. Looking at different segments of the equity market, a relation between momentum profits and firm size seems to emerge in both Fama and French (1996) and Rouwenhorst (1998), with the profits being larger for small stocks than for large ones.

However, recent studies (Israel and Moskowitz (2013) and Asness et al. (2014)) have questioned this finding and re-affirmed that momentum is present in all size groups. Overall, momentum profitability has proved to be remarkably consistent over time and across markets.

*Explanations for the Momentum Effect* Jegadeesh and Titman (1993) focus on a statistical decomposition of the momentum effect. In a simple one-factor model for returns (take  $K = 1$  in (2.19)), they identify three effects that can contribute to generate a positive cross-sectional covariance between the returns of period  $[t - 1, t]$  and those of period  $[t - 2, t - 1]$ . The first effect is the presence of systematic risk factors that affect expected returns and consequently affect realised returns as well. The second effect is a positive autocorrelation in the common factor return, and the third one is a positive autocorrelation of the idiosyncratic component of the return. The authors conclude that the first two elements cannot play the dominant role. In particular, winner stocks are not smaller and do not have higher beta than the losers, so that size and market beta do not explain momentum profits. Jegadeesh and Titman (1993) also examine and reject the possibility that profits are driven by a lead-lag effect, i.e. an impact of the  $t - 1$  factor value on the  $[t - 1, t]$  return. They conclude that the momentum effect is mainly due to the third element, which itself is the result of a delayed incorporation of firm-specific news in the price. The insufficiency of systematic risk factors such as market, size and value is also pointed by Fama and French (1996), who admit that the short-term continuation of returns is the "main embarrassment" for their three-factor model.

The idea of a gradual diffusion of information in prices has been explored in subsequent behavioural models. A possible behavioural explanation is that investors under-react to short-term information because of inertia in their investment choices. As a result, new information is only progressively taken into account in prices. Hong and Stein (1999) develop a behavioural model with heterogeneous agents (named

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"newswatchers" and "momentum traders") that predicts both short-term under-reaction and long-term overreaction. The key assumption is that information (whether public or initially private) is only gradually reflected in prices.

Many of the explanations based on rational asset pricing models have focused on the role of macroeconomic variables as systematic factors. Chordia and Shivakumar (2002) point that the persistence of the momentum anomaly after its discovery poses a difficulty to behavioural models, and they note that momentum strategies perform better in expansions than in recessions, which is consistent with (2.2): to deserve a positive premium, an asset has to deliver a higher pay-off when the marginal utility is lower. Unlike Jegadeesh and Titman (1993), they conclude that the pay-offs to momentum strategies are more related to the cross-sectional variation in time-varying expected returns than to idiosyncratic returns. In their model, expected returns are functions of traditional macroeconomic predictors for aggregate returns, namely the dividend yield, the 3-month Treasury bill rate, the term spread and the default spread. This supports the idea that momentum profits are linked to systematic risk factors. However, Cooper et al. (2004) show that the macroeconomic model loses its explanatory power when microstructure effects are taken into account and they also show that the lagged market return is a better predictor of momentum pay-offs. Bansal et al. (2005) examine another model, more directly inspired by consumption-based models, where the risk factor is aggregate consumption. They report that the cash flows of winner stocks have higher consumption beta than those of loser stocks, a finding which re-assigns a role to macroeconomic risk in explaining momentum profitability. Liu and Zhang (2008) study another macroeconomic

aggregate, the growth rate in industrial production. They allow for time-varying exposures in this variable and find that winner stocks have higher betas than losers in the first months following the ranking period, but that the difference in betas progressively vanishes in the following months. They also find that the growth rate is a priced factor, and that half of the momentum profits can be explained by this sole factor.

### *Momentum Factor*

A momentum factor was first introduced by Carhart (1997) as an additional factor in the model of Fama and French (1993) to explain the momentum effect, which represents an anomaly for this model. It is defined as the excess return of an equally-weighted portfolio of the 30% past year winners over an equally-weighted portfolio of the 30% past year losers. To avoid any potential bias towards another characteristic, it is possible to combine this sort with another sort: Charoenrook and Conrad (2008) do this by considering size as the other attribute.

There has been a debate on whether the positive performance of the long-short winners-minus-losers (WML) portfolio was more due to the short side than to the long side. If it were, momentum would be of less interest to long-only investors. For instance, Hong et al. (2000) and Avramov et al. (2007) argue that most of the momentum profits can be attributed to selling losers. In the context of the under-reaction model of Hong and Stein (1999), an interpretation is that bad news get reflected in prices more slowly than good news. But more recent research has questioned the asymmetry of the momentum effect: Israel and Moskowitz (2013) show that both arms of the portfolio contributed almost equally to the performance.

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As for the size and the value factors, Charoenrook and Conrad (2008) and Pukthuanthong and Roll (2014) conduct formal statistical tests of compatibility with the ICAPM and the APT. Pukthuanthong and Roll (2014) do find that the WML portfolio has a significant link to the principal components of the covariance matrix, which suggests that it affects all returns. However, unlike the size and the value portfolios, it does not pass the test that Charoenrook and Conrad (2008) perform to check whether it is a priced factor in the sense of the ICAPM. More precisely, the conditional premium appears to be negatively related to the variance, while the positive sign of the momentum premium would command the opposite).

### 4.1.5 Liquidity Effect

Liquidity is defined by Pastor and Stambaugh (2003) as "a broad and elusive concept that generally denotes the ability to trade large quantities quickly, at low cost, and without moving the price." A comprehensive survey on the evidence for liquidity effect and the pricing of liquidity risk can be found in Amihud et al. (2006).

#### *Empirical Evidence of the Liquidity Effect*

The first evidence for a liquidity effect in the stock market was provided by Amihud and Mendelson (1986). Specifically, they develop an asset pricing model with liquidity risk that predicts an increasing relationship between expected returns and the bid-ask spread. The model also predicts a clientele effect: specifically, the least liquid assets (i.e. those with the highest spread) will enter the portfolios of investors with long horizons, which reduces in turn the compensation that these investors require for holding them. As a result, the model-implied relationship is concave. Amihud and Mendelson (1986) find

empirical support for this functional form in data on NYSE stock returns for the period 1961-1980. Eleswarapu (1997) provide further robustness checks: they find that the spread effect also exists in the NASDAQ market, both in January and non-January months.

Similar studies have been conducted using other measures than the bid-ask spread. For instance, Brennan and Subrahmanyam (1996) use the "marginal cost of trading" and the "relative fixed cost of trading", two quantities that rely on Kyle's measure of the impact of a single trade on a stock price (Kyle, 1985). They also report a positive relationship between expected returns and illiquidity (measured by a high cost of trading) when stocks are sorted in quintiles. Overall, numerous liquidity measures have been studied: they include for instance the dollar trading volume (Brennan et al., 1998), the stock turnover (Datar, 2001), the price change per dollar volume traded (Amihud, 2002) or the amortized spread – a measure that combines the turnover and the spread – (Chalmers and Kadlec, 1998). To address the concern raised by the availability of microstructure data, Hasbrouck (2002) propose three alternative measures which rely on daily observations only. Loderer and Roth (2005) return to the bid-ask spread as a measure of liquidity, but instead of focusing on expected returns, which are notoriously difficult to estimate, as an explained variable, test for the impact on prices. They uncover a negative relationship between the price and the spread, which is in line with the other studies. Their study is on the Swiss Stock Exchange, which provides evidence for the liquidity effect outside the US market.

#### *Explanations for the Liquidity Effect*

The risk-based explanation for the liquidity effect is that the various liquidity measures

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at the stock level are proxies for the sensitivities to a liquidity factor. Pastor and Stambaugh (2003) argue that such a factor is likely to be of concern for many investors. Indeed, any investor facing a solvency constraint will prefer to hold assets that have low exposures to aggregate liquidity risk: assets with high exposures are more likely to fall down as liquidity decreases, so that they will have to be sold to respect the solvency constraint, and the need for liquidation will arise precisely when liquidation is slow or costly. As a result, investors require higher expected returns to hold these assets.

### *Liquidity Factor*

Pastor and Stambaugh (2003) define an aggregate liquidity measure by averaging individual liquidity measures across stocks. At the stock level, they estimate liquidity as the magnitude in the "volume- reversal effect". They find that over the period 1966-1999, their liquidity factor is priced, even after taking into account the three Fama-French and the momentum factors. Charoenrook and Conrad (2008) define a liquidity factor in a way reminiscent of the definition of the Fama-French factors: they take the excess return of a portfolio of the 10% stocks with the lowest liquidity over a portfolio of the 10% stocks with the highest liquidity, both portfolios being equally weighted. At the individual stock level, the liquidity measure is the ratio of a stock-specific illiquidity over the average market illiquidity, and the stock-specific illiquidity is a measure that compares past realised returns to volumes traded. For the period 1963-2003, the authors find that their liquidity factor, defined as the return on a long-short portfolio, satisfies the necessary condition for being a priced risk factor. Indeed, there is a significantly positive relationship between

the conditional mean and variance of the portfolio.

### **4.1.6 Volatility Effects**

Ang et al. (2006) identify two volatility effects in stock returns: first, stocks with high exposures to aggregate volatility tend to underperform those with low exposures; second, stocks with high total or idiosyncratic volatility (as estimated in the three-factor Fama-French model) underperform those with low volatility.

### *Empirical Evidence of Volatility Effects*

Many studies on equity options suggest that the market assigns a negative price to aggregate volatility risk. For instance, Bakshi and Kapadia (2003) interpret the fact that implied volatility is almost always greater than historical volatility (see Jackwerth and Rubinstein (1996) for empirical evidence) as the sign of a negative volatility price. Moreover, they show that the performance of delta-hedging strategies – which purchase a call and sell the underlying stock as a hedge – provides a robust means to assess the sign of the volatility price. These strategies have negative returns, which points to a negative volatility premium. Carr and Wu (2009) also study the variance risk premium for five S&P, NASDAQ and Dow Jones indices and thirty-five individual stocks, but use a different estimation method: the premium is estimated as the expected market variance minus a variance swap rate (the authors verify that this definition is consistent with the general definition of a factor premium in (2.10)). The conclusion is that market assigns a significantly negative price to aggregate volatility risk, and that this premium is not explained by the three Fama-French factors or by momentum. Ang et al. (2006) study the pricing of volatility risk in the cross section of stock returns without using option data. They

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proxy volatility as the VIX published by the Chicago Board of Option Exchange and find that a portfolio of the 20% stocks with the lowest (algebraic) exposures earns higher returns than one formed with the 20% stocks with the highest exposures, the excess return being statistically significant. Moreover, the effect is monotonic across quintiles of volatility beta, which suggests that the differences can be attributed to a priced risk factor. It is also robust to the introduction of a control for other characteristics which are known to drive expected returns, such as size, book-to-market, momentum and liquidity: this means that the volatility effect does not simply amount to one of these known effects.

The second volatility effect reported by Ang et al. (2006) is the famous "volatility puzzle": a portfolio of the 20% stocks with the lowest volatilities outperforms a portfolio of the 20% stocks with the highest risk, a result which holds whether volatility is measured as the total standard deviation of returns or the idiosyncratic volatility from the three-factor Fama-French model. However, this effect is not monotonic since it only exists between the extreme two quintiles. The pattern, originally identified in US stocks, is also encountered in international developed equity markets, and the return differentials between low and high volatility stocks display comovements across markets, as if it was explained by common source of risk (see Ang et al. (2009)). In another recent study, Baker et al. (2011) report the same volatility effect. The studies of Baker and Haugen (2012) and Hsu et al. (2013) also document the existence of similar anomalies in international data, including emerging markets. It should be noted that they use other volatility measures than Ang et al. (2006): Baker and Haugen (2012) sort stocks on total

volatilities, and Hsu et al. (2013) define a volatility adjusted for size, country and industry, in order to avoid confounding a volatility pattern with any effect arising from these three characteristics.

These findings confirm prior suspicions for the existence of a negative relationship between volatility and expected return, which date back to early empirical studies of the CAPM. For instance, Soldofsky and Miller (1969) find a significant "risk premium" for low volatility securities, defined in their paper as the additional unit of return per unit of volatility, for the 1950-1966 period. However, their sample mixes stocks and bonds, and examination of their results suggests that the slope of the regression line and its significance are mostly driven by the existence of two groups: common stocks on the one hand, bonds and preferred stocks on the other hand, with the former having higher volatilities and average returns. The risk-return relationship is much less clear within the common stock class. Similar findings are reported in Haugen and Heins (1972) and Haugen and Heins (1975), who find evidence for a positive risk-return relationship in the stock-bond market over the 1931-1966 period, but question its existence within each market. Indeed, focusing the analysis on common stocks, they find that the relationship becomes either insignificantly positive or significantly negative. Haugen and Baker (1996) report concordant results for US stocks over the 1979-1993 period: stocks that have the highest expected returns (as estimated within a multi-factor model) have on average lower volatility, an effect which is almost monotonic across deciles.

The two volatility effects are not independent. The connection is explained by Ang et al. (2006) as follows. If aggregate

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volatility is a pricing factor which is not perfectly correlated with the Fama-French factors, then the residuals of the three-factor model contain both an exposure to the volatility factor and "true" idiosyncratic risk. The latter risk can be diversified away by collecting stocks in portfolios, so that the three-factor residuals for these portfolios will reveal their exposures to the volatility factor. As a result, one should find that portfolios with high three-factor idiosyncratic risk underperform those with low idiosyncratic risk, which is indeed what is found in the data. But as Ang et al. (2006) note, the low returns on high idiosyncratic volatility stocks represent a puzzle in the sense that they cannot be attributed to any bias towards a usual attribute (such as size or value) and though it is true that the high idiosyncratic volatility stocks have higher volatility exposures, this factor alone does not suffice to justify the low returns.

### *Questions on Robustness*

The existence of a low volatility anomaly has been questioned in numerous studies, which have pointed the lack of robustness of the puzzle with respect to methodological choices. Other papers do not deny the existence of the effect but argue that it interacts with other known regularities in stock returns in such a way that it does not uniformly exist across all stocks.

From a methodological standpoint, Bali and Cakici (2008) study the robustness of Ang et al. (2006)'s findings with respect to several implementation aspects. First, they show that using other weighting schemes than value weighting leads to a slight and insignificant outperformance for high idiosyncratic volatility stocks, a result which is observed both for equally-weighted portfolios and portfolios weighted by the inverse of specific volatility.

Second, the return spread is sensitive to the breakpoints chosen to delimit quintiles. For instance, if the breakpoints are defined in such a way that each quintile has the same capitalisation, as opposed to having the same number of stocks, the outperformance of low volatility stocks is no longer significant. Third, estimating volatility from monthly returns instead of daily returns produces no statistically significant difference between the high and the low volatility groups. Fourth, removing the smallest, most illiquid and lowest priced stocks makes the anomaly disappear. Fifth, the puzzle is observed within the universe of all NYSE/AMEX/NASDAQ stocks, but not in the NYSE universe. In sum, the results of Ang et al. (2006) could be attributed to the choice of the NYSE/AMEX/NASDAQ universe with particular sampling frequency, quintile definition and weighting scheme. More recently, Van Vliet et al. (2011) also put forward two methodological issues that potentially affect studies of the risk-return relationship. First, the use of geometric, as opposed to arithmetic, averages, reduces the average returns of high volatility stocks more than those of low volatility stocks, which has a substantial impact on the magnitude of the anomaly. This effect is also mentioned in Baker et al. (2011) and is examined in more detail below in the context of our dataset (see Section 5.3.1). Second, survivorship bias tends to exclude the high volatility with very low returns, which, in contrast, biases upwards the estimate of the volatility-return relationship.

Another methodological controversy has arisen regarding the volatility measure. Fu (2009) focuses on the relation between the conditional idiosyncratic volatility and expected returns, as opposed to considering the lagged volatility. The conditional volatility is extracted from

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an exponential GARCH model. The result is a significantly *positive* relationship to the expected return. Similar conclusions are reached by Spiegel and Wang (2006) and Brockman et al. (2009). However, two subsequent studies by Fink et al. (2012) and Guo et al. (2014) argue that the results of Fu (2009) are driven by a look-ahead bias in the likelihood maximisation technique used to estimate the parameters of the EGARCH model. They show that the positive relationship disappears when the conditional volatility is estimated only with the information available to date.

Other articles have pointed the role of the short-term reversal effect reported by Jegadeesh (1990): monthly returns are negatively serially correlated and, as verified by Fu (2009), the realised idiosyncratic volatility in one month is positively correlated with returns. Fu (2009) concludes that the combination of these two effects can explain the abnormal low returns of high specific volatility stocks in the month that follows the month when volatility is measured: those stocks are likely to experience high returns in the measurement month, and these high returns reverse to negative returns in the next month. As a matter of fact, Huang et al. (2010) show that the negative relation uncovered by Ang et al. (2006) between specific volatility and average returns in the following month is no longer significant when reversals are accounted for. They also argue that the reversal effect can explain the contradictory results reported by Bali and Cakici (2008) regarding equally-weighted and value-weighted portfolios.

Finally, the anomaly does not seem to exist in all segments of the equity market. Campbell et al. (2014) find that growth firms with large specific volatility underperform growth firms with low specific volatility, but

the relationship is positive among value stocks.

### *Explanations for Volatility Effects*

The first volatility effect is the underperformance of stocks with high exposure to aggregate volatility with respect to those with low exposure. This effect can be rationalised in the context of an asset pricing model if increases in volatility tend to coincide with bad times. Given that high volatility tends to be associated with negative market returns (French et al., 1987), this property is not unrealistic, and investors will be ready to accept lower returns for assets that pay off well when volatility rises. In the same spirit, Ang et al. (2006) note that a negative volatility premium can be justified in Campbell (1993)'s version of the ICAPM if a high volatility corresponds to worse investment opportunities: indeed, investors will seek to purchase assets that pay off well when the opportunity set deteriorates, which raises the price of assets that are more exposed to volatility risk, and decreases their subsequent returns.

The negative relationship between idiosyncratic volatility and return is more difficult to explain. First, several theoretical models predict a positive relationship. For instance, considering an extension of the CAPM model in a setting with limited information, Merton (1987) finds that the alpha of a stock in the one-factor model is increasing in the idiosyncratic volatility. The second difficulty posed by the idiosyncratic volatility puzzle is its relative lack of robustness. Campbell et al. (2014) propose to explain the absence of monotony across volatility deciles reported by Ang et al. (2006), as well as the contradictory findings of Bali and Cakici (2008), by an interaction between the volatility and the value effects. They argue that a high specific volatility

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has more impact on growth firms than on value firms, because it raises the value of the growth options held by the former and subsequently increases their volatility beta. Because the price of aggregate volatility is negative, it follows that growth firms with high specific volatility should earn lower returns than those with low specific volatility. On the other hand, value firms own less growth options, so their behaviour need not be the same. As a matter of fact, Campbell et al. (2014) find no evidence for a volatility anomaly among value stocks.

Behavioural arguments have often been put forward to explain why highly volatile stocks are overpriced, which depresses their subsequent returns. As noted by Blitz and Van Vliet (2007) and Baker et al. (2011), these stocks have 'lottery-like' pay-offs, with a small probability of very high outcomes, which are attractive to investors. This appetite for lottery tickets is a manifestation of investors' preference for positive skewness (see Mitton and Vorkink (2007)).

Several aspects of institutional investment may also contribute to explain the anomaly. Following the above lottery argument, Baker and Haugen (2012) suggest that the demand from money managers for volatile stocks may be explained by compensation rules: the typical compensation package has an option-like profile, with the probability of a bonus being higher if the manager chooses a high volatility portfolio. The authors also show that more volatile stocks are more covered by analysts and media, so that they are more likely to be recommended by analysts to CIOs. Finally, most managers are concerned with their relative performance with respect to a benchmark, and Blitz and Van Vliet (2007) note that investing in high volatility or high beta stocks appears to

them as the most straightforward way to generate outperformance. Baker et al. (2011) develop a similar argument, and conclude that the practice of benchmarking tends to exacerbate the demand for the most risky stocks.

### *Volatility Factor*

Ang et al. (2006) define the aggregate volatility factor as the daily change in the VIX. They note that this series has mean close to zero and a very small autocorrelation. Thus, this factor satisfies the condition of being almost unpredictable (see Section 2.4). The authors estimate the price of volatility risk to be about  $-1\%$  per year. This negative sign confirms the findings of previous studies, which measured the price of volatility risk from option prices.

### **4.1.7 Investment and Profitability**

Investment and profitability are two factors that have recently become popular in equity investing strategies. In what follows, we first review the theoretical motivation for considering these factors and then move on to the analysis of empirical findings and the various proxies used in empirical tests.

### *Theoretical Background*

The dividend-discount model (DDM) is used by Fama and French (2006) to provide a microeconomic justification for the existence of a link between expected stock returns and the profitability or the investment level of the firm. The expected stock return  $\rho$  is the discount rate such that the sum of expected future discounted dividends equals the market value of equity:

$$ME_t = \sum_{s=t+1}^{\infty} \frac{\mathbb{E}_t[D_{s-1}]}{(1 + \rho)^{s-t}}. \quad (4.1)$$

An accounting identity states that the dividend paid in period  $[s - 1, s]$  is equal

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to earnings  $Y_{s-1}$ , minus the change in book equity. Thus, the market-to-book ratio satisfies:

$$\frac{ME_t}{BE_t} = \sum_{s=t+1}^{\infty} \frac{\mathbb{E}_t[Y_{s-1} - \Delta BE_s]}{BE_t(1+\rho)^{s-t}}.$$

Fama and French note that this equality implies three forecasts regarding the link between expected returns and microeconomic variables:

- for fixed expected earnings and expected changes in book-to-equity, a higher book-to-market (BE/ME) ratio implies a higher expected return;
- for fixed BE/ME ratio and expected earnings, a higher expected change in book-to-equity implies a lower expected return;
- for fixed BE/ME ratio and expected change in book-to-equity, higher expected earnings imply a higher expected return.

The first effect is the value effect. The second and the third ones are referred to respectively as the "investment" and "profitability" effects. The DDM thus predicts that a low investment firm, i.e. a firm which re-invests a low fraction of its earnings, will have a higher expected stock return. Similarly, a high profitability firm, i.e. a firm whose returns are expected to grow fast in the future, should deliver higher average stock returns.

Asness et al. (2013) also use a version of the DDM, namely the Gordon model, to motivate their definition of the "quality" of a stock. In this model, the dividend growth rate is a constant  $g$ , so that the market-to-book ratio becomes (see (4.1)):

$$\frac{ME_t}{BE_t} = \frac{1}{BE_t} \times \frac{D_t}{\rho - g} = \frac{\frac{\pi_t}{BE_t} \times \frac{D_t}{\pi_t}}{\rho - g}, \quad (4.2)$$

where  $\pi_t$  denotes the profit in period  $[t, t + 1]$ . Hence, the stock price should be an increasing function of the profitability (measured by the profit-to-book ratio), the payout ratio (measured by the dividend-to-profit ratio) and the dividend growth rate. It should also be decreasing in the expected return, which is the return that investors require in view of the riskiness of the stock, no matter how risk is measured (market risk or risk measured from accounting data).

In a related effort, Hou et al. (2015) develop an interpretation based on an analysis of the production side of the economy, that is, the optimality conditions for a firm choosing the level of its investment in a two-date model so as to maximise the market value of equity. It turns out that the discount rate of future cash flows, which can also be regarded as the cost of capital, is increasing in the ROE and decreasing in the ratio of investment to book asset value. Hence, other things being equal, firms with higher cost of capital invest less or have a higher ROE. However, the authors dispute an interpretation of the investment and ROE factors as systematic risk factors, because they are defined from individual characteristics of firms, which have no obvious relation to common sources of risk that would affect all firms in the economy.

### *Empirical Evidence of Profitability and Investment Effects*

In line with the theoretical predictions of the DDM, Fama and French (2006) find a positive and significant link between average returns and lagged earnings-to-book ratios, taken as a proxy for profitability. They also confirm the existence of an investment effect, since average returns are decreasing in the lagged asset growth, which proxies for the level of investment.

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In an attempt to construct more forward-looking measures of profitability and investment, they also construct regression-based forecasts for future earnings-to-book and asset growth. It turns out that the former predictor is still positively related with average return, but not significantly, while the second one is positively related to returns, albeit in a non significant way. The authors attribute this effect to an error-in-variable problem and argue that the use of lagged variables is a way to address it.

It should be noted that evidence for investment and profitability effects can be found in earlier papers. Lakonishok et al. (1994) study past profitability of a firm as a measure of "value" based on the book-to-market ratio. Specifically, stocks with the lowest past sales growth are said to be "value", while those at the other end of the scale are said to be "glamour" because they are supposedly more attractive to investors. The intuition is that investors overpay for the appealing stocks, which lowers their subsequent returns. As a matter of fact, stocks with low past profitability, measured as sales growth over the past five years, outperform the glamour ones over the period 1968-1990. In the paper, the outperformance of stocks with low past five-year sales growth is seen as a special case of a more general stylised fact: value stocks outperform growth stocks, so that contrarian strategies, which purchase the seemingly least attractive stocks, outperform investment policies favouring the others. Thus, the same explanations as for the book-to-market effect apply here: investors overpay for glamour stocks because they are excessively confident that these stocks will continue to display high sales growth, and they neglect value stocks.

Haugen and Baker (1996) identify variables related to the growth potential of firms as candidates for explaining the cross section of expected returns. As a matter of fact, they find an increasing relation between various measures of growth potential and average returns. The tested variables are asset turnover, profit margin, return on assets, return on equity and their growth rates, and earnings growth. On the other hand, firms with high growth potential are likely to be those that are the most profitable. Hence, this finding provides evidence for a positive relation between profitability and expected returns.

Recent studies have confirmed these findings. Novy-Marx (2013b) provide empirical evidence that firms with high profitability (with profitability being measured as gross profits-to-assets) outperform the less profitable ones. Hou et al. (2015) show that a long-short strategy that buys stocks with high return on equity (ROE) and sells low ROE ones generates significant profit over the period 1972-2011, and that the factor models of Fama and French (1993) and Carhart (1997) do not explain these returns. A similar observation holds for a portfolio long the low investment stocks and short the high investment ones.

Investment and profitability are also inputs to the quality score introduced by Asness et al. (2013). Specifically, the authors use profits (several accounting definitions are tested) divided by book value as a measure of profitability, and the payout ratio as a measure of the propensity of the firm to pay dividends. They show that stocks that have the highest quality measure (where the quality score aggregates the previous two variables plus the growth in profits and the riskiness of the company) outperform the others. This effect is not only present

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in US stocks over the period 1956-2012, but also in 24 developed markets over the period 1986-2012.

### *Investment and Profitability Factors*

The definition of these factors is similar to that of the size and value factors in Fama and French (1993). Stocks are first sorted on a measure of investment or a measure of profitability. The investment factor is the excess return of low investment stocks over high investment ones, while the long and short legs of the profitability factor consist respectively of most profitable and least profitable stocks. Variations in the definitions across papers are essentially due to the choice of the variables used to measure profitability and investment. To construct their "robust minus weak profitability" and "low minus high investment" factors, Fama and French (2013) choose respectively the operating profit (minus interest expenses) and the growth of total assets.<sup>13</sup> Hou et al. (2015) also use the variation in total assets (denoted as  $\Delta A / A$ ) as the sorting criterion for the investment factor, but take the return on equity as a profitability measure.

Hou et al. (2015) also argue that using multiple sorts in the definition of factors is a way to reduce their correlation. To this end, they use sorts on size,  $\Delta A / A$  and ROE to compute the long-short factor returns: for instance, the long leg of the ROE factor contains the most profitable stocks from all size and  $\Delta A / A$  groups, and the same principle applies to the short legs and to the other factors. The authors define a four-factor model that includes the market factor, a size factor and the  $\Delta A / A$  and ROE factors, and they show that it is successful in pricing correctly (i.e. with insignificant alphas) a wide cross section of test portfolios. In particular, the model performs as well as the Fama-French and Carhart models in

explaining the returns to portfolios sorted on size, book-to-market or momentum, and it performs better in explaining the idiosyncratic volatility anomaly.

A related measure is the "quality-minus-junk" factor of Asness et al. (2013). This factor is defined as a score that aggregates four variables which determine the price of a stock in the Gordon model (4.2), namely the profit-to-book and the dividend-to-profit ratios, the stock risk and the dividend growth rate.

### **4.1.8 Macroeconomic Factors**

The use of macroeconomic variables to explain the cross section of stock returns can be justified by an APT reasoning: it is natural to think that important macroeconomic variables are common factors in stock returns.

### **Theoretical Background**

In the ICAPM, macroeconomic aggregates are relevant candidates if they drive changes in the opportunity set. In order to identify a set of potentially interesting variables, one can think of a stock price as the expectation (under the risk-neutral measure) of discounted future dividends. Under the (highly simplistic) assumptions that the discount rate  $k$  is the same across all maturities and that dividends grow at a constant rate  $g$ , we have the expression for the price in the Gordon model (Gordon, 1982):

$$p_0 = \sum_{t=1}^{\infty} \left( \frac{1+g}{1+k} \right)^t \delta_0 = \frac{1+g}{k-g} \delta_0,$$

where  $\delta_0$  is the dividend paid at date 0. Thus, stock returns are influenced by variables that impact either the discount rate or expected future dividends.

<sup>13</sup> - They acknowledge that the choice of equity book value would be more consistent with (4.2), but they indicate that this variable leads to similar results.

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A modern version of the model, which incorporates a clean treatment of uncertainty, has been developed by Campbell and Shiller (1988a) and Campbell and Shiller (1988b). The details of the derivation and assumptions can be found in Campbell et al. (1997), and the final formula shows that unexpected stock returns can be decomposed as:

$$r_{t+1} - \mathbb{E}_t[r_{t+1}] = \eta_{d,t+1} - \eta_{r,t+1},$$

where  $\eta_{d,t+1}$  is a contribution from the change in the expectation of future dividends and  $\eta_{r,t+1}$  is a contribution from the change in expected future real returns. This analysis suggests that relevant macroeconomic factors should be factors that impact future dividends and/or real capital returns. Term structure variables are examples of such factors, given that they have an impact on real interest rates.

### *Empirical Evidence*

Chen et al. (1986) measure the risk premia associated with various macroeconomic indicators, which are likely to affect dividends, discount rates or both: the growth in industrial production, expected and unexpected inflations, real interest rates, revisions in expected inflation, credit spread and term spread. They estimate the risk premia by running Fama-Mac Beth regressions. Innovations to production appear to be positively priced in the cross section of stock returns, which is in line with the expectation that factors that are high in good times (when the SDF is low) have positive premia. Interestingly, broad stock indices (whether equally- or value-weighted) carry insignificant premia when they are introduced jointly with the economic variables. Hence, they do not appear as pricing factors, though they have explain a large fraction of comovements in the time series.

GDP growth may also appear ex-ante as a useful factor. Indeed, equities are claims on dividends, and dividends depend on real economic activity. However, Ilmanen (2011) Chap. 16 reports mixed evidence for a relationship between GDP growth and stock returns. On the one hand, he finds a clear positive relationship between unexpected good news about growth and S&P 500 returns between 1997 and 2010. This result differs from the one that Flannery and Protopapadakis (2002) obtain for the previous period, 1980-1996: they report an insignificant negative effect of GNP growth on the NYSE-AMEX-NASDAQ equity index from CRSP for the previous period 1980-1996. On the other hand, Ilmanen finds no cross-sectional relationship between GDP growth and the equity market return in the period 1988-2009: for instance, in this period, the Asia ex-Japan market gained about 6.9% per year, which is hardly better than in the US and in Europe (respectively, 6.6% and 6.4%), while the GDP growth was much faster in the former area (6.4% per year, versus 2.6% in the US and 2.0% in Europe). In fact, the most solid relationship uncovered by Ilmanen is that equity market returns are positively correlated with next year GDP growth, so that equity returns have some ability to predict future GDP growth.

Another potentially relevant factor is the human capital, defined as the present value of future non-financial income. Indeed, it is a component of the aggregate wealth in the CAPM, but it is ignored if one uses the return on a broad stock index as the market factor in empirical tests (see the Roll (1977) critique). This omission is all the more important as for many investors, the human capital exceeds the financial wealth. In response to this concern, Campbell (1996) includes both sources of return in the return on the market portfolio, in addition to taking

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into account time variation in expected returns. The model is thus a version of the ICAPM of Campbell (1993). The covariance with financial wealth still appears to be the main driver of average stock returns, but this does not mean that labour income plays no role, because its return is correlated to the return on financial wealth. In fact, the incorporation of the human capital and the time variation in investment opportunities enables to recover a strong positive relation between average stock returns and their market beta, a relation which is missed by traditional empirical studies of the CAPM. The importance of including labour income in a consumption-based model is also pointed by Santos and Veronesi (2006).

### 4.1.9 Interest Rate Factors

Among the many patterns identified in average stock returns (see Harvey et al. (2013) for a recent survey), we mention here those related to interest rates. Because interest rate factors are the main drivers of bond returns, it is important to assess the exposures of different classes of stocks to these factors in a multi-class allocation exercise. Following the introductory discussion on macroeconomic factors, term structure variables appear as natural candidates because they impact stock prices through the discount rate.

Empirical studies show that the correlation with bond returns varies greatly across stocks (see e.g. Cornell (2000), Reilly et al. (2007) and Baker and Wurgler (2012)). Coqueret et al. (2014) survey the literature and perform theoretical and empirical analysis. The most salient conclusion is that stocks with low volatility and/or high dividend yield are those with the highest interest rate-hedging properties.

Regarding the existence of an "interest rate pattern" in stock returns, empirical evidence suggests that interest rate risk is negatively priced in stock returns, as it is in bond returns: that is, stocks with a more negative interest rate correlation earn higher average returns. Indeed, Sweeney and Warga (1986) find that an interest rate factor, defined as the change in a long-term interest rate, has a negative price in the cross-section of expected returns. More recently, a similar effect is documented by Coqueret et al. (2014): a portfolio of stocks with low past volatility, high past dividend yield or high past correlation with a bond index (that is, "strong" negative correlation with interest rate changes) has also higher returns than a portfolio without a selection. This evidence is consistent with the low volatility anomaly of Ang et al. (2006).

### 4.2 Bond Related Factors

The two traditional bond factors are term and credit, and can be regarded as interest rate factors. But more recently, new patterns have been reported, which focus on bond returns as opposed to interest rates. In particular, some recent papers have investigated the existence of momentum and low risk effects similar to those documented in the equity class. For the term and credit factors, we start by presenting mathematical decompositions for the corresponding premia.

#### 4.2.1 Term Premium

The term premium is also referred to in the literature as the "maturity premium" or the "bond risk premium", and is broadly defined as the excess return from holding a long-term bond over a short-term one. The usual distinction between the ex-post premium (i.e. the average excess return measured from historical data) and the

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ex-ante premium (i.e. the expected excess return conditional on the information available on the prediction date) applies here.

### Theoretical Background

As explained by Ilmanen (2011), there are two ways of quantitatively measuring the notion of term premium. The first one compares the short-term returns to long-term and short-term bonds. Formally, it is the one-year expected excess return of a long-term bond, say a  $n$ -year bond, with respect to a short-term bond, say a one-year bond. If  $b_{t,T}$  denotes the price of a zero coupon that pays \$1 (without default risk) at date  $T$  and  $y_{t,n}$  is the zero-coupon yield of maturity  $n$  on date  $t$ , then we have, by definition:

$$TP_{t,n,H} = \mathbb{E}_t \left[ \ln \frac{b_{t+1,t+n}}{b_{t,t+n}} \right] - y_{t,1}. \quad (4.3)$$

This quantity compares the return from holding a long-term bond for one year to the return from holding a one-year bond to maturity. The second definition compares the long-term returns to a long-term bond and a roll-over of short-term bonds. This second term premium is the  $n$ -year return to the  $n$ -year bond (which is known in advance) over a roll-over of one-year bonds. Mathematically, the definition reads:

$$TP_{t,n,Y} = y_{t,n} - \frac{1}{n} \mathbb{E}_t \left[ \sum_{i=0}^{n-1} y_{t+i,1} \right].$$

As a starting point, the pure expectation hypothesis (EH) of the term structure states that both premia are zero.<sup>14</sup> In order to give a precise statement with a list of sufficient conditions under which the term premium is zero, we consider the same version of the EH as Fama and Bliss (1987). It postulates that the  $n$ -period zero-coupon

rate is the average of the expected future one-period rates, that is the quantity  $TP_{t,n,Y}$  is zero. A straightforward computation shows that under this condition, the expected annualised  $m$ -period return on a pure discount bond maturing on date  $T$  equals the  $m$ -period rate:<sup>15</sup>

$$\frac{1}{m} \mathbb{E}_t \left[ \ln \frac{b_{t+m,T}}{b_{t,T}} \right] = y_{t,m}. \quad (4.4)$$

Taking  $m = 1$  in this equation, we obtain that the term premium  $TP_{t,n,H}$  is equal to zero for any  $n$ . Hence, there is no reward from holding a long-term bond over a short-term bond. A similar result obtains under the more general formulation of the EH of Campbell and Shiller (1991). Another immediate consequence of the above form of the EH is that the one-period excess returns of long-term bonds over one-period bills are not predictable, since they are zero in any market conditions.

However, a few mathematical manipulations suggest that some observable variables extracted from the yield curve have ability to predict excess bond return, in contradiction with the EH. For instance, Fama and Bliss (1987) note that the term premium  $TP_{t,n,H}$  admits the following model-free decomposition:

$$\begin{aligned} TP_{t,n,H} &= \mathbb{E}_t \left[ \ln \frac{b_{t+1,t+n}}{b_{t,t+n}} \right] - y_{t,1} \\ &= F_{t,t+n-1,1} - y_{t,1} \\ &\quad - (n-1) \mathbb{E}_t [y_{t+1,n-1} - y_{t,n-1}], \end{aligned} \quad (4.5)$$

where  $F_{t,t+n-1,1}$  is the forward rate of maturity one year locked up at date  $t$  for loans starting in  $n-1$  years, equal to  $[ny_{t,n} - (n-1)y_{t,n-1}]$ . Fama and Bliss argue that yields are close to a random

14 - In fact, a variety of forms of the EH exist, some of which are mutually inconsistent (see Cox et al. (1981) and Campbell (1986)).  
15 - There is an equivalence between the two forms of the property. If  $TP_{t,n,Y} = 0$  for any  $n$ , then (4.4) also holds for any  $m$  and the converse is true.

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walk, so that the expected change in the  $(n - 1)$ -period rate is close to a constant, and the expected excess return on the bond is close to an affine function of the forward-spot spread,  $[F_{t,t+n-1,1} - y_{t,1}]$ . Hence, the forward-spot spread should have predictive power. A related decomposition is proposed by Ilmanen (2011) Chap. 9.1. Under the approximation that  $y_{t+1,n-1} \approx y_{t+1,n}$  (that is, the yield curve is locally flat between maturities  $n - 1$  and  $n$ , an assumption which is reasonable if  $n$  is much larger than one), we have the approximate decomposition:

$$TP_{t,n,H} \approx y_{t,n} - y_{t,1} - (n - 1)\mathbb{E}_t [y_{t+1,n} - y_{t,n}]. \tag{4.6}$$

This equality highlights the role of the yield spread,  $[y_{t,n} - y_{t,1}]$ , as a predictor of excess bond returns. Both decompositions of the term premium convey the same qualitative insight: if yields are close to random walks, then the term premium should be positively related to a steepness measure of the yield curve (the yield spread or the forward rate). However, these decompositions do not predict that the term premium will necessarily be positive, even if the term structure is upward sloping: indeed, the term premium depends positively on the slope, but expectations of rising interest rates can cause it to be negative.<sup>16</sup>

The previous two decompositions for the term premium are model-free, but it is also interesting to look at the predictions of an affine term structure model (see Section 2.3.7). Under convenient assumptions on the dynamics of factors, the expected excess return on a zero coupon is given by Equation (2.24), which we rewrite here for the reader's convenience:

$$\mathbb{E}_t \left[ \frac{db_{t,t+n}}{b_{t,t+n}} \right] - r_t dt = - \sum_{k=1}^K \frac{1 - e^{-na_k}}{a_k} \delta_{1k} \Lambda_{kt}.$$

The quantities  $a_k$  are the speeds of mean reversion in the  $K$  factors under the equivalent martingale measures, the  $\delta_{1k}$  are the loadings of the short-term interest rate on the factors and the  $\Lambda_{kt}$  are the factor premia. This equation clearly predicts that bonds of different maturities have different sensitivities to the factors, which should result in differences between expected returns. But again, it does not predict alone that the expected return is an increasing function of the maturity. Specifically, the impact of each factor on the expected return depends on the sign of the quantity  $\delta_{1k} \Lambda_{kt}$ .

### *Empirical Evidence of the Maturity Effect*

To see the existence of a maturity effect, one can start by comparing the historical returns of long-term bonds and Treasury bills. For instance, Fama (1984a) tests whether the bond and bill term premia, defined as the one-month expected excess return over the one-month T-bill rate, are zero. Their analysis over the period 1953–1982 leads to reject this hypothesis, which empirically invalidates the EH. But the examination of historical excess returns of long-term Treasuries over Treasury bills shows that the premium has not always been positive. Fama (1984a) find that in the period 1953–1982, portfolios of bonds with maturities longer than four years had lower monthly returns than the one-month T-bill. A similar observation is reported in Ilmanen (2011) Chap. 9.2 for the close period 1952–1980: on average, the bonds with maturities exceeding approximately five to seven years underperformed the T-bills. Only the bonds with shorter maturities earned a premia over the T-bills. Similarly, Fama and Bliss (1987) report negative ex-post term premia for bonds of maturity 2 to 5 years over the period 1964–1985. Over the more recent period 1981–2009, Ilmanen finds that the premium is positive for all bonds.

16 - Therefore, the property that the term premium is positive should not be confounded with the property that the yield curve is upward sloping. An increasing yield curve can be exploited in the context of a "yield curve carry" strategy, which consists to lend money at the long rate and to borrow at the short rate.

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But as Fama (1984a) notes, the higher volatility of long-term bond returns compared to short-term bonds makes historical averages imprecise estimates of expected returns, so that it is useful to look at ex-ante term premia. This argument is reminiscent of the point made by Elton (1999) for equities: historical average returns are not always good proxies for expected returns. In line with Equation (4.5), Fama and Bliss (1987) find that the forward-spot spread of maturity 2, 3, 4 or 5 years has power to predict the one-year return on a bond of the same maturity, which provides yet another evidence against the EH. The predictor shows substantial variability over time, changes sign, and seems to have some relationship to the business cycle. This suggests that the ex-ante term premium depends on market conditions and is not always positive. Their regressions of excess bond returns over the forward-spot spread lead to a slope coefficient which is not significantly different from one – as predicted by (4.5) – and in any case significantly different from zero, which confirms the forecasting ability of the regressor. The findings of Cochrane and Piazzesi (2005) reinforce the evidence for predictability by showing that a single factor, defined as a linear combination of the forward rates of maturities from 1 to 5 year(s), can predict the five excess returns. The  $R^2$  obtained over the period 1964–2003 are even higher than those of Fama-Bliss' regressions over the same period. Another recent approach to the estimation of the ex-ante term premium, is to use surveys to measure market expectations. For instance, it is possible to estimate the term premium  $TP_{t,n,Y}$  as the difference between the current  $n$ -period yield and the average of future expected short-term rates for the next  $n$  years.

A related question related is whether the term premium is increasing in the maturity. The results of Fama (1984a) lead to reject the hypothesis of a flat term structure in the period 1953–1982, but his analysis suggests that the term structure is not monotonically increasing. For instance, the average return on a T-bill peaks around the maturity nine months. The hump-shaped pattern is less clear for portfolios of bonds sorted by maturity, but it appears that the portfolios formed with the longest bonds are never the best performers. The figures of Ilmanen (2011) Chap. 9.2 are in line with these findings: the term structure of expected bond returns is downward sloping in the sub-period 1952–1980. In the sub-period 1981–2009, the term structure appears to be increasing over the whole maturity spectrum. In the whole period, 1952–2009, the term structure of average returns over the period 1952–2009 is steeply increasing for maturities from 1 month to 1 year, then more slowly increasing from 1 year to 5 years and roughly flat after 5 years.

Ludvigson and Ng (2009) explore a different route for forecasting bond returns by going beyond the predictors constructed from the sole yield curve. They use a combination of macroeconomic and financial series, which relate to output, employment, inflation and the stock market, and show that factors constructed from such series have ability to predict bond returns. This highlights in particular the fact that the term premium varies with the business cycle.

As a conclusion, the data leads to reject the EH, which would imply that the term premium is always zero, but it indicates that this premium has not always been positive and that it is, at least in some periods, a hump-shaped function of the maturity. Moreover, the premium appears to depend on market conditions.

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### *Explanations for the Maturity Effect*

The mathematical decompositions for expected bond returns provide justifications for the use of forward-spot spreads or yield spreads to predict excess bond returns, but they are model-free and do not explain the economic origin of the term premium. Some explanations can be found in early theories of the term structure. For instance, in the liquidity preference theory of Hicks (1939), a term premium arises because lenders and borrowers have different preferences regarding the maturities of loans.

An explanation based on modern asset pricing theory requires the identification of a pricing factor to which long-term bonds are more exposed than short-term ones. If the factor has positive premium, then owners of long-term bonds expect to receive a higher return than if they had invested in a short-term bond. The Arbitrage Pricing Theory suggests that the pricing factors can be chosen among the factors that statistically explain bonds returns. Precisely, it is widely recognised that three factors suffice to explain close to 100% of the variance of Treasury bond returns: Litterman and Scheinkman (1991) show that these factors can be interpreted as level, slope and curvature, and that the first of these factors actually explains about 80% or more of the variance. Moreover, they find that the exposure of a zero-coupon bond to the level factor is approximately a linear decreasing function of the maturity. Deguest et al. (2013) focus on coupon bonds and confirm that the exposure to the level factor is negative and decreasing in the maturity. In fact, this exposure can be proxied by an observable quantity. Indeed, by definition of the modified duration, the return to a bond for a small change in the yield to maturity is close to the negative of the change multiplied by the duration. Thus, the beta with respect to the level factor

should be close to the negative of duration. This is empirically verified by Reilly et al. (2007), who regress the returns to a bond index on the negative of yield changes, and find that the regression coefficient (which they name "empirical duration") is close to the modified duration. Hence, the exposure to the level factor is accurately proxied by the negative of duration. Since duration is positive and is in general increasing in the maturity,<sup>17</sup> it follows that long-term bonds have a more negative exposure to the level factor than short-term bills. Thus, exposure to the level factor appears as a good candidate for explaining the return spread between bonds of different maturities. But to have a complete justification of a positive term premium, we need a negative premium for the level factor, the factor premium being defined as in (2.10) is negative. This is the case if, and only if, the level factor covaries positively with the marginal utility of consumption. If the covariance has the opposite sign, the factor premium will be negative.

Another possibility is that the term premium embeds an inflation risk premium, and investors require an additional reward for investing in a long-term nominal bond, which by definition offers no protection against the risk of an inflation surge. In this perspective, inflation uncertainty should be an important determinant of the premium. Empirical evidence in this direction is provided by D'Amico and Orphanides (2008) and Wright (2011).

### *Term Factor*

Fama and French (1993) define their TERM factor as the excess return of a long-term Government bond over the one-month Treasury bill. This definition is close to (4.3): if the long-term bond was a zero-coupon, then the expectation of TERM would be exactly the term premium as it is defined in

<sup>17</sup> - The latter property is not universally true and depends in fact on whether the bond trades above or below par (see Hopewell and Kaufman (1973)).

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(4.3). Some algebra shows that the ex-ante equality (4.6) also holds for ex-post returns, that is, the realised excess return on a  $n$ -year zero-coupon bond can be written as:

$$\ln \frac{b_{t+1,t+n}}{b_{t,t+n}} - y_{t,1} = F_{t,t+n-1,1} - y_{t,1} + (n-1) [y_{t+1,n-1} - y_{t,n-1}].$$

Hence, the ex-post excess return is the sum of the forward-spot spread (which is known ex-ante) and the change in the  $(n-1)$ -period rate. With the approximation that  $(n-1)$ -period yields are equal to  $n$ -period yields, we have the following approximation for the factor value:

$$\ln \frac{b_{t+1,t+n}}{b_{t,t+n}} - y_{t,1} \approx y_{t,n} - y_{t,1} + (n-1) [y_{t+1,n} - y_{t,n}]. \quad (4.7)$$

This equation suggests that the TERM factor is related to the slope of the yield curve, measured by the term  $[y_{t,n} - y_{t,1}]$ .

Fama and French show that their TERM factor is a common factor in bond returns, in the sense that all bond portfolios, regardless of their rating, have a statistically significant loading on the factor. The same conclusion holds for stock portfolios, but the explanatory power of the two factors is much lower than for bonds: the  $R^2$  hardly exceeds 20% in the best case, while the two factors capture more than 79% of the variance of bonds, except in the low grade class.

Because the realised excess return of a long-term zero-coupon is related to the term spread (see (4.7)), an alternative possible definition for the term factor is as the spread between a long and a short rate (see e.g. Petkova (2006)). This option is attractive because a strand of the literature has documented that the term spread has predictive power for stock and bond returns

(see Fama and French (1989) and Campbell and Shiller (1991)). The ICAPM would conclude that the term spread is a pricing factor. However, it is not investable, because it is not defined as a portfolio return. Therefore, we will stick to Fama and French's definition in our empirical study.

### 4.2.2 Credit Premium

A well established stylised fact is that corporate bonds have had historically higher discount rates than Government bonds, at least as long as the latter bonds are default-free. However, whether these higher interest rates have translated into superior performance is less clear. In this section, we use the expressions "credit spread" and "credit premium" to refer, respectively, to the spread between defaultable and default-free interest rates, and to the excess returns of defaultable bonds over default-free ones. It should be noted that the terminology is not uniform across the literature. In particular, the "credit spread" may be understood in a narrower sense, to refer only to the fraction of the spread that arises because of the risk of downgrades.

#### *Theoretical Background*

In the reduced-form approach to default modelling of Duffie and Singleton (1999), the default of an issuer occurs at a random date  $\tau$ .<sup>18</sup> To this default time are associated two "intensity processes",  $\xi$  and  $\tilde{\xi}$ , respectively under the physical and the risk-neutral probability measures. Heuristically, saying that  $\xi$  is a default intensity means that the probability for default to occur in the small time interval  $[t, t + dt]$  conditional on the absence of default prior to date  $t$  is  $\xi_t dt$ . As shown by Duffie and Singleton (1999), the price of a defaultable zero-coupon that promises the payment of \$1 on date  $T$  is:

$$p_t = \mathbb{E}_t^Q \left[ e^{-\int_t^T (r_u + \tilde{s}_u) du} \right],$$

18 - See Duffie (2001) Chap. 11 for the technical conditions that the default time must satisfy.

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where  $\mathbb{Q}$  is the risk-neutral probability measure,  $r$  is the short-term risk-free rate and  $\tilde{s}$  is the credit spread. This quantity can be expressed as  $\tilde{s}_t = (1 - \delta_t)\xi_t'$  where  $\delta_t$  is the recovery rate of the zero-coupon in the event of default, that is the ratio of the bond price just after default to the price just before.

The derivation of the expected return on the bond from the previous formula is carried out in Lando (2004) and Yu (2002). In order to keep notations simple, let us assume that the risk-free rate and the spread follow mean-reverting processes of the type of Vasicek (1973), with constant prices of risk  $\lambda_r$  and  $\lambda_{\tilde{s}}$ . Then, it can be shown that the expected return on the defaultable bond under the physical measure is:

$$\begin{aligned} \mu_{\text{def},t} = & r_{dt} - D_r(T-t)\sigma_r\lambda_r \\ & - D_{\tilde{s}}(T-t)\sigma_{\tilde{s}}\lambda_{\tilde{s}} + \tilde{s}_t - s_t, \end{aligned} \quad (4.8)$$

where  $s_t = (1 - \delta_t)\xi_t$ , and  $D_r(T-t)$  and  $D_{\tilde{s}}(T-t)$  are the negative of the elasticities of the bond price with respect to the state variables  $r_t$  and  $\tilde{s}_t$ . The expected return on an otherwise identical default-free bond is given by a similar expression, without the contributions from the spread:  $\mu_{rf,t} = r_t - D_r(T-t)\sigma_r\lambda_r$ . Thus, the credit risk premium, defined as the excess return on the defaultable bond over the default-free one, is:

$$\mu_{\text{def},t} - \mu_{rf,t} = \delta_t(\xi_t - \tilde{\xi}_t) - D_{\tilde{s}}(T-t)\sigma_{\tilde{s}}\lambda_{\tilde{s}}. \quad (4.9)$$

In this simple model, the credit premium consists of two terms: the first one is a "default event premium", which would be zero if investors were able to diversify away default risk and therefore did not ask for a premium to bear the default risk of a particular issuer (Jarrow et al., 2005). In

general, however, defaults tend to occur more frequently in "bad times", that is in times where marginal utility from consumption is high, so that default risk has a systematic component and cannot be completely eliminated. Thus, the default event premium is likely to be positive. The second term in the right-hand side of (4.9) compensates investors for being exposed to the risk of a change in the credit spread of the issuer. Here, the "bad risk" is the risk of a downgrade, which means an increase in the spread, hence a negative shock on price. The sign of this second term can be predicted with the help of the consumption-based asset pricing model (see Section 2.1.2). If the spread tends to increase in bad times, then it covaries positively with marginal utility, so that  $\lambda_{\tilde{s}}$  must be negative. As a result, the term  $-D_{\tilde{s}}(T-t)\sigma_{\tilde{s}}\lambda_{\tilde{s}}$  is positive, which makes sense: an increase in the spread means a decrease in the bond price, and investors require a positive premium for holding an asset that has low pay-off in bad times.

An important consequence from (4.9) is that the expected excess return of a defaultable bond over a default-free one does not coincide with the spread. Specifically, the difference between the credit premium and the spread can be written as:

$$\mu_{\text{def},t} - \mu_{rf,t} - \tilde{s}_t = -s_t - D_{\tilde{s}}(T-t)\sigma_{\tilde{s}}\lambda_{\tilde{s}}.$$

If we assume for a moment that the spread is constant, so that the second term in the right-hand side is zero, then the credit premium is less than the spread: this is precisely because the defaultable bond can default, or, in more formal terms, because the historical intensity of default is positive. In this case, the spread overestimates the reward from holding a defaultable bond over a default-free one.

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In this model, the spread is determined by the default intensity and the recovery rate:  $\tilde{s}_t = (1 - \xi_t)\tilde{\xi}_t$ . But this specification can be extended to accommodate non-default sources of spread. For instance, Driessen (2005) introduces a liquidity contribution in the spread, in order to reflect the lower liquidity of corporate bonds with respect to Government bonds. Empirically, disentangling default and liquidity contributions in spreads is a challenging task (see Longstaff et al. (2005)). The spread may also be affected by the differences between the tax treatments of corporate and Government bonds and by the presence of embedded options in some corporate bonds: for instance, Duffee (1998) finds that when Treasury yields fall, the yield spreads increase less for non-callable than for callable bonds (which contain a short position in an option because the issuer has the possibility to redeem them prior to maturity). In the presence of multiple sources of risk in the spread (factors common to all firms, firm-specific factors, liquidity factor), Driessen (2005) provides the derivation of the expected return on defaultable bonds, thereby extending (4.9).

### *Empirical Evidence of the Credit Effect*

The ex-post credit premium is defined as the excess realised return of defaultable bonds over default-free bonds. Evidence in favour of a positive historical premium is mixed, and appears to depend on the rating class. As noted by Ilmanen (2011), the ex-post premium is difficult to measure over a long period, because ancient data is difficult to collect and the durations of corporate and sovereign bonds do not always match. Moreover, because some bonds have defaulted, care must be taken to avoid any survivorship bias in the estimation.

Over the period 1900-2000, Dimson et al. (2008) find a premium of about 50 bp

for corporate bonds over Treasuries in the US market. Of course, this figure hides disparities across grading classes. Over the period 1973-2009, Ilmanen (2011) Chap. 10.2 finds that a portfolio of AAA/AA-rated bonds actually underperformed Treasuries, and the highest premia were recorded by the sector just below investment grade, namely the BB-rated class. The outperformance of this sector appears relatively robust, since it is observed in both periods 1973-2009 and 1985-2009. Finally, CCC-rated bonds had the lowest cumulative performance across all classes over the period 1985-2009. This comparison highlights the fact that credit spread are not good predictors of the excess returns to defaultable bonds over default-free ones, and that they actually tend to overestimate the expected excess returns.

### *Explanations for the Credit Effect*

Default and credit risks have a systematic component, and default and downgrade events are more likely to occur in bad times. Thus, rational investors should require a positive risk premium for holding securities that are exposed to these risks. In other words, the existence of a credit risk premium is straightforward to justify ex-ante.

Ilmanen (2011) Chap. 10.2 asks instead why ex post excess returns were so low compared to historical credit spreads. An obvious explanation is that default and downgrading losses caused realised returns to be substantially lower than expected. But the explanation is incomplete, due to the "credit spread puzzle": in a nutshell, standard structural models of default predict spreads that are too low compared to historical spreads (see e.g. Cremers et al. (2008)). Moreover, the low ex-post returns are also observed in IG bonds, for which default matter less. Ilmanen (2011) proposes two main explanations for the disappointing excess returns to

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corporate bonds. First, they are subject to a "downgrading bias": an IG bond is more likely to be downgraded than upgraded, and the negative impact of a downgrade on the price is larger than the positive impact of an upgrade. Second, corporate bond indices have typically rating criteria, and bonds that fall below a certain grade are automatically excluded from the universe. This can lead to sell bonds that otherwise showed good performance, thereby depressing index returns.

### *Credit Factor*

Fama and French (1993) define the DEF factor as the excess return of a portfolio of corporate bonds over a long-term Government bond, all securities being long-term bonds. This factor is intended as a proxy for a common factor that affects the likelihood of default of all issuers. As the TERM factor, DEF is a common factor in bond returns. Rather unsurprisingly, the DEF loading of the bond portfolio displays a tendency to decrease when the rating improves, even if the effect is not strictly monotonic. Together with the TERM factor, this factor explains more than 79% of the variance of bond returns, except for the low grade sector. In this class, the  $R^2$  falls to 49%, which suggests that the returns on low grade bonds are more affected by idiosyncratic factors than those of investment grade securities. DEF also behaves like a common factor with respect to stocks, since all stock portfolios have a significant loading on it.

### **4.2.3 Momentum Effect**

Compared to the equity universe, the evidence for a momentum effect in the bond market is more limited: empirical studies are less numerous, and the results appear to depend on the rating class.

### *Empirical Evidence of the Momentum Effect*

A distinction must be made between sovereign bonds, investment-grade (IG) and high-yield corporate bonds. As Khang and King (2004), Gebhardt et al. (2005) find that there is no momentum effect in IG bonds, and that the empirical evidence is more in favour of a short-term reversal effect: bonds that best performed in the past three to twelve months tend to underperform the past losers over the same subsequent period of time. But Gebhardt et al. (2005) document a spillover effect from equities: for a given firm, high past one-year equity returns are associated with high bond returns over the next year.

More recently, Israel and Moskowitz (2013) and Asness et al. (2013) have studied the existence of a momentum effect in ten country bond indices, representing the bonds of ten sovereign issuers credited to have little default risk.<sup>19</sup> Asness et al. find a positive momentum premium for the period 1982–2011, but it appears to be lower than in the equity universe: a portfolio of the 33% indices that best performed over the past twelve months outperforms a portfolio of the past losers only by 0.4% per year, which is well below the values greater than 5% observed in most equity markets, and the spread is not even statistically significant. Using the same dataset, Israel and Moskowitz compute the alpha of a long-short momentum bond strategy with respect to an equally-weighted combination of all bonds, and they find a negative abnormal performance. This result is suggestive of a short-term reversal effect, but the alpha is not even significant. Overall, evidence for a momentum effect in sovereign bonds is not as clear as in equities.

19 - The countries are Australia, Japan, Canada, Denmark, Germany, Japan, Norway, Sweden, Switzerland, the United Kingdom and the United States.

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Momentum seems to be stronger in non-investment grade (NIG) bonds. Pospisil and Zhang (2010) report a momentum effect in high-yield bonds over the period 1998-2009, when past returns are computed over periods ranging from 1 to 24 month(s). The conclusions of Jostova et al. (2013) for the period 1973-2011 are in line with these findings: when bonds are ranked and held over periods of 3, 6, 9 and 12 months, momentum profits are low and insignificant among IG bonds, but are significant in the worst credit group, which roughly corresponds to BB+ or worse. This result is robust to controls for characteristics such as duration, age or outstanding amount and holds in sub-periods. Moreover, as in the equity universe (see Fama and French (1996)), the effect is missed by the common factors of Fama and French (1993), namely market, SMB, HML, TERM and DEF. At the population level, the authors find that the returns to momentum strategies are positively and significantly related to the proportion of NIG bonds in the population.

### *Explanations for the Momentum Effect*

As shown by Jostova et al. (2013), the momentum effect in bonds is driven by NIG bonds. Given that high-yield bonds have many equity-like features, one might have suspected that the momentum in NIG bonds simply reflects equity momentum. However, the authors argue that this view is not supported by the data: for instance, bond winners (losers) are in general not stock winners (losers). They also reject the hypothesis of trading frictions, but retain the possibility that the momentum effect is a consequence from the gradual diffusion of information in prices, as in the model of Hong and Stein (1999). In support for this interpretation, they point that diffusion of information is likely to be slower in NIG bonds of privately held firms, and it is

precisely in this segment that momentum profits are largest.

### *Momentum Factor*

Asness et al. (2013) define an international bond momentum factor as a long-short portfolio of ten country indices. The indices are ranked by their past 12-month return, and the weight of each constituent is proportional to the difference between its rank and the mean rank. Thus, the five winner indices receive a positive weight while the five losers have a negative weight, and the weights add up to zero, so that the portfolio is dollar-neutral. The factor premium for the period 1982-2011 is positive (1.0% per year), but is not significant.

### **4.2.4 Other Effects**

Term and credit are by far the two most standard factors in the bond universe, and the momentum effect has only been reported more recently. In this section, we briefly review other patterns, which also belong to the recent literature and have in general not led to the introduction of a dedicated factor (with the exception of the value factor in Asness et al. (2013)).

### *Liquidity Effect*

As noted by Amihud et al. (2006), the impact of liquidity on returns is easier to study in the Treasury bond market, because to the extent that these bonds are default-free, there is no need to disentangle the default premium and the liquidity premium. Moreover, one advantage of default-free bonds over stocks is that the return for holding the bond until maturity is observable and known in advance (it is the yield-to-maturity).

Amihud and Mendelson (1991) compare the yields of Treasury notes and bills, and find that the former have higher yields. This supports the view that a liquidity

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premium exists since notes are less liquid: as pointed by the authors, notes are subject to higher brokerage fees and display higher bid-ask spreads than bills. But liquidity is of special concern for investors in the corporate bond markets, because corporate bonds are far less liquid than Government bonds. Using various liquidity measures, either model-free or model-implied, Chen et al. (2007) find a negative relationship between liquidity and the yield spread.

### *Value*

There is no direct counterpart for the book-to-market ratio used in equity investing for bonds. Therefore, one has to define a measure of "cheapness" for bonds. Asness et al. (2013) define the value attribute for a portfolio of a Government bonds as the past five-year change in the ten-year yield. This measure is strongly related to the past five-year return. The other measures of value tested by the authors are the five-year change in the ten-year yield, the real ten-year yield, the term spread and an average of the three measures. A value effect is observed because portfolios of the 33% stocks with the highest score outperform those that contain the 33% stocks that are the least value. Moreover, the outperformance is significant for three measures out of four.

### *Low Risk Anomaly*

Unlike for stocks, historical volatility makes little sense for bonds because the volatility of a bond shrinks to zero as maturity date approaches. In response to this shortcoming, a number of alternative risk measures have been introduced. Five of them are studied in de Carvalho et al. (2014): the duration-times-yield (DTY), the modified duration, the yield-to-maturity, and the duration-times-spread and the option-adjusted spread for corporate bonds. The authors show that if

corporate bonds are sorted in quintiles on their DTY, the average return over the period 1997-2012 is inversely related to the risk measure. This effect is observed both for investment grade and high yield bonds, in each of the three studied currencies (USD, EUR, GBP and JPY). The other risk measures give a similar picture, but it is only for the DTY sort that a strict monotony is obtained.

### *Equity Factors*

Fama and French (1993) show that excess returns on Government and corporate bonds (from IG to high yield) are significantly correlated with the market excess return. The factors SMB and HML have also significant loadings in a three-factor regression of bond returns. However, the three factors capture a much lower fraction of the variance of bond returns than stock returns: most of the  $R^2$  range between 10% and 30%, while they easily exceed 80% in equity regressions. Interestingly, the  $R^2$  is decreasing in the credit quality: the equity factors explain better the returns on low grade bonds than on Government and IG bonds. This suggests that NIG bonds are more equity-like than IG bonds.

### **4.3 Commodity Related Factors**

Most often, investors do not hold commodities physically, if only because of the storage costs, and they invest in commodities through futures contracts. Thus, this section focuses on the analysis of the performance of passive strategies rolling over commodity futures.

#### **4.3.1 Theoretical Background**

We start by recalling the definition of the roll return and we briefly review the theoretical determinants of the long-term returns of passive strategies. A comprehensive survey of the latter topic can be found in Till (2006).

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### *Roll Return*

Standard commodity futures contracts do not have a well-defined expiration date and have instead a maturity identified by a month in a year. The contract with the nearest term is in general the most liquid of contracts written on a given commodity, so the usual form of the passive strategy consists of a long position in this contract. When it approaches maturity, the contract is rolled into the next nearest contract. For the sake of simplicity in a theoretical exposure, it can be assumed that contracts have a specified maturity date, that the nearest contract is the one with the nearest future maturity date, and that the roll over takes place on the maturity day. In practice, the nearest contract is the one with the nearest future maturity month, and the roll procedure is implemented over a period of several days. Examples of roll periods are given in the methodology of the S&P GSCI (available on S&P Dow Jones website) or in Feldman and Till (2006).

As explained by Ilmanen (2011) Chap. 11.3, the realised total return to such a strategy between two dates consists of three components. The first is the return on the collateral asset (in our empirical section, we take the collateral to be a Treasury bill, so this component is known in advance). The second is the "spot price change", where the spot price is defined in this particular context as the price of the nearest contract. The third component is the roll return, which is captured by the investor when rolling the nearest contract to the next one. By definition, the roll return is zero if no roll over has taken place between the two dates. If a roll over was implemented, the roll return is positive if the second contract traded at a lower price than the first one, and negative otherwise. This motivates the definition of the "roll yield", as the relative

difference between the prices of the nearest contract ( $F_{1t}$ ) and the next nearest contract ( $F_{2t}$ ):

$$RY_t = \frac{F_{1t} - F_{2t}}{F_{2t}}. \quad (4.10)$$

Note that if the nearest contract were to mature at date  $t$  exactly, we would have  $F_{1t} = S_t$ , where  $S_t$  is the spot price, defined here as the price of a contract with infinitesimal maturity.<sup>20</sup> In general, the spot price is difficult to observe, hence it is proxied as the price of the nearest contract.

The roll yield depends on the slope of the term structure of futures prices: an upward sloping term structure means a negative roll yield and a downward sloping one implies a positive roll yield. Thus, if the term structure is decreasing when the investor rolls over into the next contract, he enjoys a positive roll return, which contributes positively to the total return. In case of an increasing term structure, the roll return is negative. The total return is the sum of the aforementioned three components, and the excess return is defined as the sum of the second and third ones, so that the effect of the collateral is eliminated. It turns out that in the long run, roll returns are an important determinant of the strategy return: Feldman and Till (2006) find that the correlation between the (excess) return and the roll return is increasing in the investment horizon, and in the same vein, Ilmanen (2011) Fig. 11.8 shows that the total returns over a relatively long period (1992–2009) are highly correlated with roll returns. Moreover, Erb and Harvey (2006) show that over a long period, roll returns are much closer to cumulative excess returns than spot returns. Hence, even though the spot price of a particular commodity has risen, the passive strategy may have lost money

20 - The futures price at the maturity date must equal the spot price to rule out arbitrage opportunities.

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if the roll returns were consistently negative.

By definition, a commodity is said to be in backwardation if the futures price is less than the expected spot price at maturity. This is often associated with a downward sloping term structure, where the futures price is decreasing in the maturity. In this situation, the futures price is less than the spot price, so that the roll return is positive. Hence, roll returns are more likely to be positive if the commodity is in backwardation. The converse situation, where the futures price is higher than the expected spot price, is known as contango, and it often corresponds to an upward sloping term structure. In these conditions, roll returns are more likely to be negative.

### *Theoretical Determinants of Risk Premia*

Keynes (1930) argues that the term structure of commodity prices is normally backwardated due to the hedging pressure from producers: producers are "hedgers", who prefer to lock in the price at which they will sell commodities rather than selling at an uncertain price. Hence, they offer futures contracts to hedge against the risk changes in the spot price, which causes the futures price to fall below the expected spot price. The other market participants, known as "speculators", purchase the contracts precisely because they bet on futures prices rising in the future. If the risk premium is defined as the futures price minus the expected spot price, then it is normally negative, and its absolute value represents the amount that producers are ready to give up in order to sell their production at a fixed, as opposed to random, price. A consequence of low futures prices is that on average, the futures price should be less than the spot price, and roll returns should be positive.

The theory of normal backwardation is a special case of the "hedging pressure theory", in which the risk premium is determined by a competition between hedgers and speculators, but hedgers are not always net short. For instance, Cootner (1960) argues that in the case of wheat, the influence from hedgers-producers on futures prices depends on the season, with the downwards pressure on prices being larger when inventories are high, that is typically at harvest time. As a result, the sign of the net position of hedgers can vary over time. If hedgers are net long, then speculators-consumers are net short, which can raise futures prices above expected spot prices.

An alternative theory of commodity risk premia is the theory of storage, which was introduced by Kaldor (1939). It is based on the idea that the holder of the physical commodity bears storage costs but has the advantage of being able to sell it when the context is favourable: inventory holders virtually benefit from their reserves in case of a temporary price increase or a shortage. According to the formulation of Fama and French (1987), the basis, defined as the difference between the futures and the spot prices, can be written as:

$$F_t - S_t = (T - t)S_t[y_{t,T}^a + W_t - d_t], \quad (4.11)$$

where  $y_{t,T}^a$  is the arithmetic zero-coupon rate of maturity  $T - t$ ,  $W_t$  is the marginal storage cost and  $d_t$  is the marginal convenience yield (the three quantities being expressed on an annual basis). The sign of the basis depends on the magnitudes of the various terms in the right-hand side of (4.11). If the convenience yield is high compared to the interest rate and the storage costs, then the basis is negative. This is the case of agricultural

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commodities before the harvest season, because inventories are low and the spot price is high. From (4.11), one can relate the risk premium to the convenience yield net of storage costs and the expected spot price change as:

$$\begin{aligned}\Lambda_t &= F_t - \mathbb{E}_t[S_T] \\ &= (T - t)S_t[y_{t,T}^a + W_t - d_t] - \mathbb{E}_t[S_T - S_t].\end{aligned}$$

The theory also predicts a decreasing relationship between the convenience yield and the level of inventories (see Brennan (1958), Telser (1958) and Routledge et al. (2000)). In line with these predictions, Casassus and Collin-Dufresne (2005) find that the convenience yield for crude oil, copper and silver is increasing in the spot price.

Another explanation for risk premia is provided by asset pricing theory: commodities should earn a positive risk premia if, and only if, they tend to perform well in good times and poorly in bad times. As noted by Ilmanen (2011) Chap. 11.3, if good and bad times are characterised by inflation and the performance of traditional asset classes such as equities, it is unclear whether a positive premium should exist. On the one hand, the increasingly high correlation between commodities and the stock market suggests that commodities act less as a diversifier, which should increase the risk premium. On the other hand, the notorious ability of some commodities, such as energy, to hedge inflation risk, should have a negative impact on the premium. As noted by Breeden (1980), the proposition that commodity prices are correlated with aggregate macro indicators makes all the more sense because many commodities enter production process or are themselves consumption goods. Because Breeden (1979)'s ICAPM implies that risk premia are proportional to consumption betas,

Breeden (1980) develops a model that expresses consumption betas as functions of structural factors. A comprehensive asset pricing model is constructed by Richard and Sundaresan (1981), who also build on Merton's and Breeden's ICAPM and relate futures and forwards risk premia to consumption beta.

However, Hirshleifer (1989) argues that aggregate consumption is not the relevant indicator for pricing commodity futures, given that many individuals do not participate in the futures market. There exists another market imperfection, which is the presence of nonmarketable positions in hedgers' endowments. The author develops a futures pricing model that incorporates these two imperfections and takes into account the correlation between supply and demand shocks.

### 4.3.2 Backwardation and Hedging Pressure Effects

As explained in Section 4.3.1, roll returns are important determinants of the long-run returns of commodity futures strategy. Hence, one can expect commodities that are typically in backwardation to deliver positive long-term returns. Similarly, going long backwardated contracts and short contangoed contracts should be a profitable strategy. Because the shape of the term structure is driven by the relative pressure from consumers and producers in the hedging pressure theory, we present the term structure and hedging pressure effects together.

#### *Empirical Evidence of the Term Structure Effect*

Feldman and Till (2006) study soybeans, corn and wheat over the period 1949-2004, and they verify that the total return over the 55 years is aligned with the average backwardation (measured as  $\frac{S_t - F_{t,T}}{F_{t,T}}$ ).

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They also find that the five-year return is highly correlated with the percentage of months where the commodity was in backwardation. In particular, soybeans have been frequently backwardated during the sample period, and a roll-over of soybeans contracts earned a positive return. On the other hand, corn and wheat were more often contangoed, which gave rise to negative returns. These findings support the conclusion that passive strategies that select backwardated commodities lead on average to higher returns. As further evidence, the authors also report that the average backwardation and the percentage of time in backwardation explain a growing fraction of returns when the horizon increases.

From a prospective standpoint, investors will only find these results useful if they are able to identify which commodities will trade in backwardation in the future. But the shape of the term structure depends on structural factors such as the characteristics of the physical market (see Feldman and Till (2006) for a case study on soybeans), and as often, the past is not a reliable predictor of the future. More generally, the identification of backwardated commodities is not a straightforward task because the expected spot price is not observable, so that evidence can only be indirect, through the long-term returns of passive futures strategies (see Kolb (1992)).

### *Empirical Evidence of the Hedging Pressure Effect*

The hedging pressure theory predicts that a commodity market is backwardated or contangoed depending on the hedging pressure from producers and consumers, so that this pressure should eventually have an impact on the profitability of roll-over strategies.

Empirically, Bessembinder (1992) confirms that for most underlying assets (including commodities but also currencies), the returns are higher in months when hedgers are net short than when they are net long. If the net hedging pressure is defined as the excess of the number of short hedge positions over the number of long hedge positions, divided by the total number of hedge positions (Till, 2006), this implies that average returns are positively related to the hedging pressure. De Roon et al. (2000) also find that the hedging pressure has a significant role in explaining the returns on futures strategies for many agricultural commodities and minerals. Moreover, each commodity futures return has a positive loading on the corresponding measure of hedging pressure: hence, the expected return is higher when hedgers are net short than when they are net long, which is in line with theoretical predictions. The authors further show the existence of cross-hedging pressure effects across commodities: the expected returns on several futures strategies is also significantly impacted by the measure of pressure in other commodities.

### *Term Structure and Hedging Pressure Factors*

Miffre et al. (2014a) introduce two commodity factors motivated by the previous empirical evidence. The term structure factor is defined as the return on a portfolio that goes long the most backwardated contracts (i.e. those with the highest roll returns) and short the most contangoed ones (i.e. those with the lowest roll yields). The signal used to rank the contracts is the past 12-month average roll yield, defined as in (4.10). Similarly, the hedging pressure factor is the return on a portfolio that goes long (short) the commodities with the lowest (highest) hedging pressure. The data is obtained

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from the Commodity Futures Trading Commission, and as the roll yield, the signal is averaged over the past 12 months.

### 4.3.3 Momentum Effect

Like equities and many other asset classes, commodities display a tendency to short-term momentum. This can be exploited through the construction of strategies that purchase the past winners and sell short the past losers.

#### *Empirical Evidence of the Momentum Effect*

For a single commodity, Ilmanen (2011) Chap. 14 implements a strategy that takes a long position in the futures contract maturing on the next month if its price is above the past 12-month moving average, and a short position if the price is below the average. His results for the period 1990-2009 indicate that the strategy was most successful for energy (e.g. crude oil or natural gas): the Sharpe ratios are between 0.45 and 0.54. Results are more mitigated for metals, with the trend-following strategy invested in silver futures experiencing a negative excess return of -9.5%. Agricultural goods also shows substantial dispersion in Sharpe ratios, which vary from 0.05 to 0.5. We recall that since these portfolios are long-short, a positive excess return, or equivalently a positive Sharpe ratio, means that a collateralised strategy with the risk-free asset as a collateral outperformed the collateral.

The performance of momentum strategies in commodity markets is also documented by Miffre and Rallis (2007). The authors test strategies similar to those implemented by Jegadeesh and Titman (1993) in the equity class, purchasing the 20% contracts that displayed the best performance over a ranking period and selling the 20% losers.

They test different ranking and holding periods comprised between 1 and 12 month(s), and they find that most of the momentum strategies have statistically significant performance and have attractive Sharpe ratios, close to 0.4 or 0.5. However, the statistical significance of the profit depends on the choice of the roll-over date for the futures contracts.

#### *Explanations for the Momentum Effect*

The behavioural explanations for the momentum effect in the equity class equally apply to commodities. On the other hand, rational asset pricing explanations can also be given that rely on the specific features of commodities. For instance, Miffre and Rallis (2007) find that momentum strategies tend to purchase backwarddated contracts and sell contangoed contracts. Since backwarddated commodities tend to generate profits over the long run (see Section 4.3.2), this can provide at least a partial explanation to the performance of momentum strategies. However, this rule is not universal: Miffre and Rallis show that for a commodity such as western plywood, the momentum strategy tends to buy in contango and sell in backwardation. Miffre et al. (2014a) provide further evidence that momentum profitability is not entirely explained by term structure effects by showing that the term structure and momentum factors (defined below) make opposite predictions of future market returns and variances: specifically, the former is negatively related with future returns and positively related with future variances, while the opposite holds for the latter.

Gorton et al. (2013) suggest a more fundamental explanation: the returns on momentum and roll strategies could be proxies for the level of inventories, a variable which is hard to observe accurately,

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especially at a high frequency. In support for this view, they show that direct proxies for inventories, despite their imperfections, generate significantly positive returns in the period 1971–2012: a low inventory portfolio outperforms a high inventory one. The authors also report that other strategies relying price-based signals are implicitly taking long positions in low inventory commodities and short positions in the high inventory ones: the past twelve-month spot return – which is another measure of momentum –, the current futures basis and the spot price volatility – which is negatively related to the level of inventories (low inventories are associated with higher price volatility since prices reflect all supply and demand shocks).

### *Momentum Factor*

Following the definition of the equity momentum factor by Carhart (1997), Miffre et al. (2014a) define a commodity momentum factor as the return on a portfolio that is long the contracts with the highest momentum and short the contracts with the lowest momentum.

### **4.3.4 Other Patterns**

Among the other patterns that have been reported in equity returns, the level of inventories is specific to the commodity class. The value and low volatility patterns have been studied following the extensive literature that has documented similar effects in the equity class. Finally, some papers have tested the ability of theoretical asset pricing models such as the CAPM or the ICAPM to explain the returns to commodity futures strategies.

### *Level of Inventories*

Gorton et al. (2013) point the role of the state of inventories in the determination of returns on futures strategies. They develop a theoretical model where the basis

$(F_{t,T} - S_t)$  and the risk premium  $(F_{t,T} - \mathbb{E}_t[S_T])$  are functions of the level of inventories, and they empirically verify that portfolios invested in contracts written on commodities with low inventories outperform the high-inventory portfolios. However, the data is not easy to collect, as it is scattered across many sources (see the Appendix of their paper for details).

### *Value*

Asness et al. (2013) document the existence of a value effect in commodity futures for the period 1972–2011, value being measured as the negative of the spot price return over the past five years. This is an estimator of "cheapness" inspired by the work of DeBondt and Thaler (1985) use to show the existence of long-term reversal in stock returns.

### *Low Volatility*

In the commodity class, Miffre et al. (2014b) study the existence of a low idiosyncratic volatility anomaly, and compare different models for extracting volatility. A similar comparison is carried in Fuertes et al. (2015). Following Ang et al. (2006) in the equity class, they construct strategies that go long the low volatility futures and short the high volatility ones. The performance of these long-short portfolios appears to depend strongly on how idiosyncratic volatility is assessed. If traditional benchmarks, such as broad stock, bond and commodity indices, SMB and HML, are used, then the strategy appears to deliver large expected returns, Sharpe ratios and alphas. But such regressors miss a large part of the systematic risk in commodities, so that the volatility of residuals is not really "idiosyncratic". The performance is much lower when commodity related factors are employed. These factors are the momentum factor or factors that capture the fundamental role of backwardation and contango in the

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commodity market, such as term structure and hedging pressure. With such factors, the t-statistics for the ex-post average returns are lower than when volatility is measured with respect to the traditional benchmarks.

### *Factors Inspired by CAPM or ICAPM*

Early work has focused on the CAPM. Dusak (1973) reports that both the systematic risk and the expected excess return for corn, wheat and soybeans are close to zero. This seems consistent with a CAPM-based explanation. On the other hand, Bodie and Rosansky (1980) report positive excess returns for portfolios of commodity futures, but find negative betas with respect to the S&P 500, which cannot be explained by the CAPM. De Roon et al. (2000) also show that the S&P 500 alone does not explain the returns on commodity futures strategies since other variables, such as the hedging pressure, remain significant after the stock market effect is taken into account.

Other studies have tested the ability of the ICAPM to price commodity futures. Jagannathan (1985) estimate a version of the ICAPM where the state variables that describe the opportunity set are the spot-futures ratio, the lagged consumption and the real return on nominal bonds. Their results indicate that the over-identifying restrictions imposed by the model are rejected at the 5% confidence level.

### 4.4 Multi-Asset Class Factors

The previous sections have reviewed the notorious patterns that have been reported in the average returns of the members of various asset classes. This leads to the identification of a number of factors or characteristics that drive the expected returns in a class. But in a multi-class allocation exercise, it would be highly useful to identify

characteristics or factors that influence returns in several classes. Each security can then be thought of as a package of characteristics or factor exposures, which facilitates the comparison across classes. The recent academic literature provides insights regarding the choice of the factors, by developing "global asset pricing models", that is models that apply to several classes, as opposed to pricing only one asset class (in general, equities). In the next sections, we review some of the factors that have been proposed as determinants of expected returns across classes.

#### 4.4.1 Return-Based versus Underlying Factors

Following the terminology of Ilmanen (2011) Chap. 16.1, we make a distinction between return-based factors and underlying risk factors:

- return-based factors are defined as the returns on strategy styles that have been shown to be profitable, at least in the long run. The choice of these factors is directly inspired by the single class analysis. When the factor is computed from the returns on securities that belong to the same asset class, we refer to it as a "single-class factor". We emphasise that this denomination refers to the definition of the factor but does not presume that it prices only the securities of the class in question;
- underlying risk factors are defined as factors which are not computed as returns or excess returns, but are intended as proxies for market conditions and do not refer to an asset class in particular.

From a theoretical perspective, searching for factors among pay-offs entails no loss of generality, since by Proposition 2, it is always possible to replace a given set of pricing factors by a set of pay-offs that price assets equally well. It turns out that most of the empirical factors that have

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been proposed for the pricing of securities such as equities, bonds and commodities are returns.

On the other hand, underlying factors are not necessarily given as returns. The implementation challenge raised by underlying factors is the construction of the corresponding factor-mimicking portfolios. Indeed, by Proposition 5, the maximum Sharpe ratio that investors can attain by combining these portfolios is equal to the maximum Sharpe ratio that can theoretically be achieved by combining the individual securities.

### 4.4.2 Examples of Return-Based Factors

As pointed by Ilmanen, these factors have the advantage of being investable, at least in theory. In practice, of course, not any linear combination of returns on individual assets represents a feasible portfolio return, since one has to take care of frictions such as short-sales constraints, transaction costs and market capacity. But at least, return-based factors verify minimal investability conditions. On the other hand, their economic interpretation can be problematic, as asset pricing theory requires us to justify their performance in terms of exposures to risk factors that matter to investors. We now make a list of return-based factors that have been applied to the pricing of multiple asset classes.

#### *Equity Related Factors*

These factors are defined as the returns on equity portfolios. The most famous set consists of the three factors of Fama and French (1993) (FF3 model): the excess return on a broad equity index over the risk-free rate, the excess return of small stocks over large ones and the excess return of high book-to-market stocks over low book-to-market ones.

Fama and French (1993) not only test the ability of the FF3 model to price equity portfolios, but they also apply it to the bond class. To be bond pricing factors, the first requirement that the three equity factors should meet is to be common risk factors in bond returns. As a matter of fact, Fama and French find that Government and corporate bond returns are significantly related to the three factors, except if one controls for the term structure factors TERM and DEF (see also Section 4.2). Next, they test whether the FF3 model prices bond portfolios by looking at the intercepts in the regression of bond returns on the factors. If the cross section of expected bond returns is explained by the differences in factor exposures, then the intercepts should be zero (see Section 2.2.3). As a matter of fact, the regressions yield insignificant alphas. However, Fama and French note that this may be because the average returns on bond portfolios and the TERM and DEF factors are close to zero in their sample. Moreover, one can note that the estimated alphas are all negative for corporate bond portfolios, and their absolute value is decreasing in the credit quality. This suggests that the FF3 model overstates the expected returns on corporate bonds, especially for the low grade ones.

The FF3 model is applied to the joint pricing of equities and bonds in Maio and Santa-Clara (2012) and to the joint pricing of equities, bonds and commodities in Miffre et al. (2014a). The latter article also tests the four-factor model of Carhart (1997) (which adds equity momentum to the FF3 factors), and finds that the second model yields lower pricing errors on average.

#### *Bond Related Factors*

Fama and French (1993) also look at whether their bond factors TERM and DEF can price equity portfolios. Except when

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the three equity factors are included in the regressions, the 25 size- or value-sorted equity portfolios load significantly on the two bond factors. But the two factors are unable to explain the cross section of average stock returns: many of the 25 alphas are statistically significant, and the alpha seems to be positively related to the book-to-market. This suggests that the two factors miss a large part of the systematic risk in these equity returns. Fama and French attribute the significant alphas to the very low returns on the TERM and DEF portfolios compared to equity returns in their sample.

### *Commodity Related Factors*

These factors are constructed from commodity returns. Miffre et al. (2014a) estimate a four-factor ICAPM where the factors are the market factor and the unexpected returns on three long-short portfolios of commodity futures: the three commodity strategies exploit respectively the term structure, hedging pressure and momentum effects. The market index is itself defined as a weighted average of stocks, bonds and commodity indices.<sup>21</sup> The result is that the model prices the 25 stock portfolios sorted on size and book-to-market of Kenneth's French library, six US Treasury bond indices and the S&P GSCI as well as (and even slightly better than) eight alternative multi-factor models: the competing models are the three-factor Fama-French model (Fama and French, 1993), the three-factor model augmented with the equity momentum factor (Carhart, 1997) or with a liquidity factor (Pastor and Stambaugh, 2003), and six models that use predictors of stock and bond returns as state variables.

### *Global Value and Momentum Factors of Asness et al. (2013)*

As noted by Ilmanen (2011) Chap. 12.6,

value and momentum strategies should be complement to each other since the latter purchases assets that have performed well in the recent past, while the former purchases assets that have performed poorly over a past long period. Therefore, it is not surprising to find negative correlations between value and momentum returns. Asness et al. (2013) (henceforth, AMP13) show that this is true not only in a given asset class, but also across asset classes and geographical areas. Moreover, value and momentum are positively correlated across classes, which is suggestive of the presence of common underlying sources of risk.

AMP13 also introduce a three-factor model where the market factor is completed with two value and momentum factors. Importantly, the model is tested not only to price equity portfolios, but also portfolios of other asset classes (namely bonds, commodities and currencies). In detail, the model reads:

$$\begin{aligned} r_{it} - r_{dt} &= c_i + \beta_{MKT,i}MKT_t \\ &+ \beta_{VAL,i}VAL_t \\ &+ \beta_{MOM,i}MOM_t + \varepsilon_{it}, \\ j &= 1, \dots, N \end{aligned} \quad (4.12)$$

$N$  being the number of test assets and  $r_{dt}$  being the risk-free rate. The market factor is a global equity index (the MSCI World Index), and the "value everywhere" and "momentum everywhere" factors are the equal-volatility-weighted averages of the single-class value and momentum factors. The model performs better than the CAPM and the FF3 model augmented with the momentum or the TERM and DEF factors in explaining the cross section of expected returns on portfolios of the asset classes sorted on value and momentum. The models are compared on 48 test assets, since there

21 - The authors also test stock-only, stock-bond and stock-commodity portfolios as market factors, and find qualitatively similar results.

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are eight markets and three buckets (low, middle and high) for each of the two characteristics: the three-factor model (4.12) yields the lowest absolute pricing error (as measured by the average of the coefficients  $|c_i|$ ), and the highest  $R^2$  (70.7%) in the regression of historical average returns on model-implied expected returns. Furthermore, the model still performs well with respect to the two criteria when it is tested with size-value and size-momentum portfolios of stocks: this property is all the more remarkable as no size factor is explicitly introduced in the right-hand side of (4.12).

AMP13 also measure the VAL and MOM premia, and compare them to the premia of other candidate pricing factors: liquidity, GDP growth, long-run consumption growth, TERM, DEF and market. Measuring these premia enables to identify the factors that contribute to explain the cross section of expected returns (see Section 2.4.2). It turns out that when VAL and MOM are introduced jointly with the other candidate factors, they are the two variables with the most statistically significant (and positive) prices. In particular, their inclusion strongly reduces the significance of the funding liquidity factor (a measure of how easily agents can obtain funding). Moreover, the three-factor model explains on average a substantial 68% of the time series variance of portfolios sorted on value and momentum, 70% of the variance of size-value or size-momentum stock portfolios and a respectable 41% of the variance of 13 hedge fund indices. Overall, the three-factor model appears to be a good description for the cross section of expected returns on various asset classes.

### *Carry Strategies*

Carry strategies have not been used to define equity, bond or commodity pricing

factors, but a recent study by Kojien et al. (2013) (hereafter, KMPHV13) shows that they are profitable in many asset classes.

Carry investing is a classical investment paradigm in the currency class. Carry strategies consist in taking a long position in high-yielding currencies and a short position in low-yielding ones. Evidence of profitability can be found for instance in Ilmanen (2011) Chap. 13.2. He shows that purchasing three of ten currencies and selling three others each week delivered an annualised excess return of 6.1% over the 1983–2009 period. Sharpe ratios also appear high, being above 0.5 most of the time.

KMPHV13 argue that carry is a general concept, which can be applied to any asset class since it can be related to expected returns in a model-free way. The authors show that if the contract matures at date  $t + 1$ , the realised excess return between dates  $t$  and  $t + 1$  on a fully collateralised futures position can be decomposed as:

$$r_{t+1} = C_t + \mathbb{E}_t \left[ \frac{S_{t+1} - S_t}{F_t} \right] + u_{t+1}.$$

In this equation,  $S_t$  and  $F_t$  denote the spot and futures prices,  $u_{t+1}$  is the unexpected return, which has zero expectation, and  $C_t$  is the carry, defined as:

$$C_t = \frac{S_t - F_t}{F_t} \quad (4.13)$$

In particular, the expected return on the futures strategy is the sum of the carry plus the expected spot price change divided by the futures price. Hence, the carry can be informally seen as the expected return subject to the condition that the spot price stays constant.

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KMPHV13 specialise the general expression (4.13) in various asset classes. We review here the expressions that they obtain for currencies, equities, commodities and bonds. In the currency class, they show that:

$$C_t = \frac{r_{dt,f} - r_{dt,d}}{1 + r_{dt,d}},$$

where  $r_{dt,d}$  and  $r_{dt,f}$  are respectively the domestic and the foreign exchange rates. Thus, purchasing high carry currencies and selling low carry ones amounts to going long currencies with high interest rate ( $r_{dt,f}$ ) and short the other ones. In other words, this strategy is equivalent to the usual currency carry strategy.

In the equity class, KMPHV13 show that the carry is given by:

$$C_t = \left( \mathbb{E}_t^{\mathbb{Q}} \left[ \frac{\delta_{t+1}}{S_t} \right] - r_{dt} \right) \frac{S_t}{F_t}, \quad (4.14)$$

where  $\delta_{t+1}$  is the dividend paid at date  $t+1$  and  $\mathbb{Q}$  denotes the risk-neutral probability measure. Thus, the carry is related to the expected dividend yield. This, of course, is reminiscent of the use of the dividend yield as a predictor in the empirical literature (see e.g. Fama and French (1988a), Goetzmann and Jorion (1993) and Welch and Goyal (2008) among many other references), but the quantity of interest here is the risk-neutral expectation of the future ratio, not the current ratio. Since the carry can be computed from observed futures and spot prices, Equation (4.14) enables to recover the expected dividend yield from observable quantities.

In the commodity class, a specific difficulty arises, which is due to the imperfect observation of the spot price. KMPHV13 propose to compute the carry by replacing the spot price by the nearest futures price

and the futures price by the second nearest futures price in (4.13):<sup>22</sup>

$$C_t \approx \frac{F_{1t} - F_{2t}}{F_{2t}}. \quad (4.15)$$

This is exactly the definition of the roll yield (see (4.10)). This indicator also coincides with the measure that Feldman and Till (2006) introduce to assess the degree of backwardation of a contract. Overall, the carry is related to the slope of the term structure, so that a commodity carry strategy, which goes long high carry contracts and short low carry ones, is similar to a strategy that purchases backwardated contracts and sells contangoed ones. Through the identity (4.11), one can also express the carry as a function of the convenience yield net of storage costs:

$$C_t = \frac{(T-t)[y_{t,T}^a + W_t - d_t]}{1 + (T-t)[y_{t,T}^a + W_t - d_t]},$$

where notations are defined in Section 4.3.1. By using the approximation (4.15), one can estimate the net convenience yield,  $d_t - W_t$ .

Finally, in the bond class, KMPHV13 identify several possible notions of carry and propose to define carry as the bond return if the term structure of interest rates remains unchanged. Consider for instance a bond maturing at date  $T$  with fixed coupon payments and a yield to maturity  $\theta_{t,T}$ . KMPHV13 show that:

$$C_t \approx \theta_{t,T} - r_{dt} - D_t(\theta_{t,T-1} - \theta_{t,T}),$$

where  $D_t$  is the (modified) duration. The first term in the right-hand side is the yield spread, which is closely related to the forward-spot spread that Fama and Bliss (1987) propose as a predictor of bond returns (see Section 4.2.1). Thus, bond carry

22 - The viability of the approximation is confirmed by a comparison with the carry as computed from (4.13) in asset classes where spot prices are observable. KMPHV13 find that the two measures are "nearly identical" to each other.

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is related to standard predictors of bond returns. The second term in the right-hand side is in general negative, at least to the extent that the term structure is increasing.

KMPHV13 construct portfolios where securities are weighted according to their carry rank. Specifically, the weighting scheme is the same as in Asness et al. (2013), and is such that the portfolio is long the high carry items and short the low carry ones, and that the weights sum up to zero. The sample period depends on the asset class: for US Treasuries, the data covers 1971-2012; for commodity futures, the data starts between 1980 and 2002; and for global equity indices, the starting date is between 1988 and 2005. In these three classes, as well as others (currencies, global fixed income, US credit and equity index options), the long-short portfolios earn significant positive returns over the sample period. Moreover, significant alphas subsist in each class after the value and momentum factors of the class are taken into account, which suggests that these two factors could usefully be complemented with several single-class or one multi-class "carry factor(s)".

### 4.4.3 Examples of Underlying Risk Factors

As argued above, it is reasonable to think that interesting candidate pricing factors can be found among macroeconomic variables, given that aggregate shocks such as output or monetary shocks are likely to impact future cash flows and the discount rates applied to these cash flows. In this perspective, Ilmanen (2011) proposes four factors: growth, inflation, liquidity and tail risk.

#### *Growth*

Ilmanen (2011) Fig. 16.6 shows that over the period 1990-2009, most asset

classes and strategies tend to be positively correlated with revisions in the consensus forecast of next year US GDP growth: this is true for developed and emerging equities, commodities, a strategy long credit bonds and short Treasuries. Notable exceptions include momentum strategies in equities and commodities and a strategy that purchases long-duration bonds and sells short-duration ones. Nevertheless, a closer look at the equity markets in the US, Europe and emerging regions reveals no clear relationship between the long-term GDP growth and equity market return (see Ilmanen (2011) Fig. 16.1).

#### *Inflation*

The risk of increasing prices is a major concern for many investors. Nominal bonds are obviously exposed to this risk, so that if investors are ready to pay a premium for insuring against inflation risk, bond prices should be lower. On the other hand, commodities should offer a protection against inflation to the extent that inflation shocks are driven by increasing commodity prices (see Gorton and Rouwenhorst (2006)). If the inflation premium is positive, this should lower expected commodity returns. Ilmanen (2011) Chap. 17 estimates this premium and finds that it has shrunk since the early 1980s, due to the decreasing trend in inflation uncertainty.

The relation between inflation and stock returns is less clear, because, as noted by Ilmanen (2011) Chap. 17.1, stocks are "real assets". On the one hand, higher inflation leads to higher nominal interest rates if a central bank attempts to control inflation expectations, which raises discount rates and thus lower the value of future dividends. On the other hand, dividends themselves may be good hedges for inflation risk over the long run (see Boudoukh and Richardson (1993) and Schotman and Schweitzer

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(2000)). In fact, the empirical evidence is in favour of a pronounced horizon effect in the stock-inflation correlation: both a continuous-time model of the nominal and real term structures (see Martellini and Milhau (2013)) and an error correction model (see Amenc and Martellini (2009)) predict that stocks are better inflation hedgers in the long run than in the short run.

### *Liquidity*

The existence of a liquidity premium is validated in the data both for the stock (Amihud and Mendelson, 1986) and the bond markets (Amihud and Mendelson, 1991). The concept can refer either to liquidity as a characteristic or to sensitivity to liquidity conditions at the market level. Intuitively, both the characteristic and the exposure should warrant a positive premium: less liquid assets are less desirable, which lowers their prices, and assets more exposed to aggregate liquidity risk are more likely to lose value when liquidity dries up (Pastor and Stambaugh, 2003), which is a characteristic of bad times. Following Brunnermeier and Pedersen (2009), AMP13 make a distinction between market liquidity (which measures the ability to trade easily) and funding liquidity (which measures the ability to obtain funding). Their measures of liquidity have the advantage that they do not require microeconomic data on individual trades. In line with the intuition, they find that aggregate funding liquidity is positively priced across asset classes (stocks, bonds, commodities and currencies). But the introduction of the "value and momentum everywhere" factors in the model makes the liquidity premium lose its significance, which suggests that a liquidity factor is not needed to explain the cross section of expected returns when the value and momentum are present.

### *Tail Risk*

Ilmanen (2011) retains volatility, correlation and skewness as three examples of tail risk. Volatility effects in the stock market have been analysed in Section 4.1.6. The first is the negative pricing of aggregate volatility risk, where aggregate volatility is defined as the volatility of a broad market index or the VIX: stocks that tend to perform well (poorly) when volatility is high (low) earn lower returns than those that have the opposite behaviour. The second effect is the low volatility anomaly initially reported by Ang et al. (2006): even after taking into account aggregate volatility exposure, stocks with high low idiosyncratic volatility have higher average returns than the others. It is only recently that low risk anomalies have been studied in the bond and the commodity classes. For bonds, de Carvalho et al. (2014) document the existence of low risk anomalies for various risk measures based on the duration. In the commodity class, Miffre et al. (2014b) find that long-short portfolios of futures contracts sorted on idiosyncratic volatility do have positive excess returns, but the returns have low statistical significance when specific volatility is measured with respect to traditional commodity risk factors, such as term structure, hedging pressure and momentum. Overall, low risk factors have not yet become standard factors outside the equity class.

### **4.5 Conclusion: The Choice of a Set of Factors**

After reviewing the traditional factors identified in various asset classes, we must now specify the set of the factors that will be taken as constituents in the factor allocation exercises that we conduct in Section 5.

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Following Section 3.2, an ideal set of factors should consist of pricing factors, since this leads to a zero loss of efficiency in the two-step process. In other words, we would like to have a comprehensive set of factors pricing equities, bonds and commodities. Note that this does not require the factors to be priced, i.e. to have a positive premium. On the other hand, the empirical literature surveyed in Section 4 provides a variety of patterns encountered in the average returns of stocks, bonds and commodities. Some of these patterns have given rise to the definition of long-short portfolios whose returns mimic underlying risk factors. These returns can be regarded as pricing factors to the extent that they are needed to explain patterns that would otherwise appear as anomalies. For instance, the equity size and value factors are pricing factors because the size and value effects cannot be explained with the market factor alone. Similarly, equity momentum is not explained by the three Fama-French factors, so that equity momentum is yet another pricing factor. In contrast, Fama and French (1996) show that the long-term reversal effect in equity returns is explained by the three-factor model, so that it does not require the introduction of a new factor.

It is beyond the scope of this paper to propose new factors, so we restrict to those that have been introduced in the literature. Moreover, we do not aim to choose a comprehensive set of pricing factors, which would by definition explain all patterns in expected returns, either documented or still to be discovered. Our goal is less ambitious: from the analysis of the literature, we select variables that act as pricing factors in the sense that they explain notorious patterns. Since other pricing factors may be omitted, the loss of efficiency in the two-step process is non zero

(see Proposition 4), but the purpose of the empirical analysis in Section 5 is to show that it is lower than if other methods are employed for grouping securities. We make a distinction between two approaches to selecting factors. The first is to pick factors from the equity, bond and commodity classes, and the second is to use multi-class factors.

For the first approach, we restrict to factors that satisfy at least one of the following conditions:

- The historical performance of the factor should be documented by several academic studies using different time periods and geographic areas, in order to ensure the time and spatial robustness of the performance;
- The performance of the factor should rely on plausible economic justification, whether the explanation relies on systematic risk factors that matter to investors or persistent behavioural biases. This is to ensure that a positive past performance is not simply a property of a dataset and will continue in the future, or conversely, that a lack of historical performance does not mean that the factor should not be considered for future investment.

In addition, we require that the existence of the factor should be widely accepted by practitioners, even though it can be debated.

It is clear that the three conditions call for the inclusion of equity size and value as well as commodity term structure. On the other hand, some of the conditions lead us to exclude some factors that would be potentially relevant. For instance, the first and the third conditions rule out factors that have only recently emerged in the literature, such as bond value and momentum, and commodity value. Nevertheless, we will test factors that capture these patterns by using

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multi-class value and momentum factors (this is our second approach to choosing factors, which we will introduce below). For the same reason, we do not include factors designed to reflect low volatility effects in bonds and commodities. Thus, we retain momentum only for the equity and the commodity classes, and volatility only for the equity class. We do not include equity investment and profitability, because the literature on these factors is very recent. However, we acknowledge that the theoretical and empirical evidence in favour of the existence and the robustness of investment and profitability patterns is strong.

We invoke the second condition to retain term and credit factors in bonds. The theoretical arguments and the historical evidence summarised in Section 4.2.1 suggest that the term premium can change sign over time and is not a monotonic increasing function of the maturity. Similarly, the historical performance of corporate bonds over Government bonds is rather disappointing, and in any case well below the observed credit spreads. But there are theoretical arguments that suggest that these factors are at least *pricing* factors even if they are not *priced* factors, following the distinction made in Section 2.2.2. First, the term and the credit factors explain about 80% or more of the returns to Government and corporate bonds rated Baa or above (Fama and French, 1993). This suggests that they are pricing factors in the sense of the APT. Second, the term factor is related to the term spread, which is a predictor of bond returns (see Section 4.2.1). Hence, an ICAPM argument suggests that this factor is important for pricing.

The second approach to choosing factors is to take a multi-class perspective and to use factors that capture similar patterns

across asset classes. The literature on this topic is emerging and, to this date, is mainly represented by the recent contribution of AMP13, who introduce two multi-class value and momentum factors to account for the value and momentum effects in equities, bonds, commodities and currencies (see Section 4.4). These factors are recent and have not yet become standard among academics or practitioners, but we include them in our study precisely because we have a multi-class focus and we are thus interested in testing these innovations.

The following list gives the patterns that are taken into account by our set of factors, and it briefly summarises the empirical evidence surveyed above. Section 5.2 explains in detail how the factors are computed:

- Value (equities, bonds and commodities) Evidence of the profitability of value strategies in equities, Government bonds, commodities and currencies is provided by AMP13, but the value effect in the equity market has of course been documented in a number of preceding papers. In the stock market, the book-to-market ratio (BE/ME) is a popular measure of value, especially since the three-factor model of Fama and French (1993), where the HML factor is defined as the excess return of high BE/ME stocks over low BE/ME ones. In asset classes where there is no notion of book value, Israel and Moskowitz (2013) and AMP13 propose the five-year past return as a criterion to identify "cheap" and "expensive" securities, thereby recalling that in the equity class, the BE/ME effect has been connected to the long-term reversal effect documented by DeBondt and Thaler (1985) (see Fama and French (1996));

- Size (equities)

This style is specific to the equity class, and consists to purchase small firms and sell large ones. Although the small-minus-big factor is often used to explain returns in the

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context of the Fama-French three-factor model, the robustness of the size effect has been questioned by Fama and French (2012), both in the US and other developed stock markets;

- Momentum (equities, bonds and commodities)

A momentum effect is clearly observed in almost all equity markets around the world (see Jegadeesh and Titman (1993) and Israel and Moskowitz (2013)), except Japan. Evidence of the profitability of momentum strategies is also unambiguous in commodities (Miffre and Rallis, 2007). The situation is more contrasted in the bond market, where one has to make a distinction between Government or investment-grade corporate bonds and non investment-grade corporates: it is only in the third class that a significantly positive momentum effect appears (Jostova et al., 2013);

- Term Structure (commodities)

As shown by (Feldman and Till, 2006) and Erb and Harvey (2006), the long-term returns to passive strategies rolling over commodity futures contracts are mainly driven by the roll returns. Investors can expect positive returns by going long backwardated contracts and short contangoed contracts, to the extent that they are able to identify backwardation and contango, which is by no means easy to do ex-ante (see Kolb (1992), Erb and Harvey (2006), Feldman and Till (2006)). The role of hedging pressure in the explanation of returns on passive strategies has been empirically established by De Roon et al. (2000). The term structure and the hedging pressure effect are somewhat intertwined with each other and with the momentum effect: the hedging pressure theory argues that the degree of backwardation depends on the hedging behaviour of producers and consumers, and Miffre and Rallis (2007) show that long-short momentum portfolios tend to purchase backwardated contracts and sell

contangoed ones. Nevertheless, Miffre et al. (2014a) argue that momentum is not subsumed in the term structure effect, and they use the three styles (term structure, hedging pressure and momentum) as pricing factors in their implementation of the ICAPM.

- Term (bonds)

The level factor is by far the dominant factor in sovereign yield curves (Litterman and Scheinkman, 1991). Since long-term bonds are more exposed to this factor due to their longer durations, they should earn higher average returns than short-term bills, at least to the extent that the level risk premium is negative (not positive – bond prices are decreasing in the level of interest rates). But empirical investigation precisely suggests that this premium is not constant: several studies (Fama (1984b), Fama and Bliss (1987), Ilmanen (2011)) show that the ex-post term premium has changed sign over time and that the historical average bond return is not always increasing in the maturity. Similar conclusions are reached by using estimates for the ex-ante premium, whether they are drawn from the yield curve alone (Fama and Bliss (1987), Cochrane and Piazzesi (2005)) or they incorporate macroeconomic variables (Ludvigson and Ng, 2009): the premium appears to vary with the business cycle. Because positive and negative returns tend to offset each other, a strategy that purchases long-term bonds and sells short-term bills has only small positive average return in the long run (Fama and French, 1993). But this does not imply that the conditional expected return is close to zero in any market conditions;

- Credit (bonds)

This factor is the second one which is traditionally considered in the fixed-income class. Intuitively, credit risk should be positively rewarded because firms are more likely to default in bad times, so investors in defaultable bonds are more likely to lose

## 4. Survey of Empirical Literature

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money in times when they have the highest marginal utility loss from one dollar less. But the empirical evidence in favour of a reward for bearing credit risk is not as clear as the intuition: Ilmanen (2011) reports that over the period 1973-2009, CCC-rated bonds had worse performance than investment grade bonds, and that it is the BB-rated sector that performed the best. As for the term premium, Fama and French (1993) find that the unconditional default premium (measured as the ex-post excess return of long-term corporates over long-term sovereigns) is low;

- Low volatility (equities)

Historically, the first low risk anomaly documented in the equity class is a low beta anomaly: the empirical relationship between stock returns and the market beta is close to flat, or even inverted, in obvious contradiction with the predictions of the CAPM (Black et al. (1972), Haugen and Heins (1975)), which has led to the introduction of the "betting-against-beta" factor by Frazzini and Pedersen (2014). The low volatility anomaly is another example of the low risk anomaly, which was uncovered by Ang et al. (2006) in the US market and Ang et al. (2009) outside the US: stocks with high total or idiosyncratic (with respect to the three-factor Fama-French model) volatility tend to outperform those with low volatility. Ang et al. (2006) argue that this effect can be partly explained by the fact that high volatility stocks have higher exposure to aggregate volatility, which is a negatively priced factor, but they note that this does not fully explain the anomaly.

## 4. Survey of Empirical Literature

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# 5. Empirical Analysis of Factor Investing



## 5. Empirical Analysis of Factor Investing

In this section, we perform a number of empirical comparisons between factor investing and investing in "traditional" forms of indices, such as asset class, sector or country indices. Our tests can be classified in two broad categories: (1) in-sample tests, which compare the in-sample mean-variance efficient frontiers before and after the introduction of factor indices, and (2) out-of-sample tests, which measure the benefits, if any, of replacing traditional indices by factor indices in portfolios subject to realistic implementation constraints. We describe the test methodologies in Section 5.1 and the data in Section 5.2. Next, we separately study the added value of equity pricing factors, bond factors, commodity factors and multi-class factors.

### 5.1 Methodologies for Comparing Investment Universes

In this section, we present the test methodologies for comparing two investment universes  $\mathcal{I}_0$  and  $\mathcal{I}_1$ , containing respectively  $n$  and  $N$  assets. In the empirical applications,  $\mathcal{I}_0$  will contain traditional indices, and  $\mathcal{I}_1$  will contain traditional and/or factor indices.

#### 5.1.1 In-Sample Mean-Variance Spanning Tests

Mean-variance spanning tests were introduced by Huberman and Kandel (1987), and several variants of the test statistics have been proposed since then, which essentially differ through their properties in small samples. Comprehensive surveys of these tests can be found in DeRoos and Nijman (2001) and Kan and Zhou (2012) (henceforth KZ12). The tests that we describe below relate to the long-short

efficient frontier (technical details are relegated to Appendix C.1). If short sales are imposed to the mean-variance efficient portfolios, the test procedures must be adapted (see DeRoos et al. (2001)).

As explained in Section 3.3.2, the null hypothesis that the original efficient frontier spans the extended one, and therefore that the introduction of new assets does not involve any welfare gain, is equivalent to having  $\hat{\sigma}_{GMV,0}^2 = \hat{\sigma}_{GMV,1}^2$  and  $\hat{\zeta}_{MRR,0} = \hat{\zeta}_{MRR,1}$  and these conditions are themselves equivalent to the following restrictions on the regression coefficients in (3.9):

$$\alpha = \mathbf{0}_K, \quad \beta' \mathbf{1}_n = \mathbf{1}_K. \quad (5.1)$$

As a result, there are  $2K$  linear restrictions to test. Standard procedures include the likelihood ratio test, the Wald test and the Lagrange multiplier test. Following KZ12, we use the LR test, which compares the likelihood of the model (3.9) with or without the restrictions (5.1). The test statistics and its small-sample distribution are written in Appendix C.1.

On the other hand, it is well known (see e.g. Merton (1980)) that volatility estimates are much more accurate than expected return estimates. Thus, differences in the variance of global minimum variance (GMV) portfolios can be easy to detect, while differences in risk-return ratios of maximum risk-return (MRR) portfolios must be large to be perceived as statistically significant. As pointed by Kan and Zhou (2012), this makes a case for disentangling the test of variance equality and the test of risk-return ratio equality. Specifically, they first test  $\alpha = \mathbf{0}_K$  (which corresponds to testing whether the maximum Sharpe

## 5. Empirical Analysis of Factor Investing

ratios are equal), and then test  $\beta' \mathbf{1}_n = \mathbf{1}_K$  conditional on the fact that  $\alpha = \mathbf{0}_K$ . They call this two-step test a "stepdown test". The corresponding test statistics and their distributions under the null hypothesis are given in Appendix C.1.

Overall, mean-variance spanning tests compare the ex-ante variances of the GMV portfolios and the ex-ante risk-return ratios of the MRR portfolios, which are a function of the sample moments of the assets. Specifically, we have, in each of the universes  $\mathcal{I}_0$  and  $\mathcal{I}_1$ :

$$\hat{\sigma}_{GMV}^2 = \frac{1}{\mathbf{1}' \hat{\Sigma}^{-1} \mathbf{1}},$$

$$\hat{\zeta}_{MRR}^2 = \hat{\boldsymbol{\mu}}' \hat{\Sigma}^{-1} \hat{\boldsymbol{\mu}},$$

where  $\mathbf{1}$  denotes a vector of ones of suitable size. These tests are called "in-sample" because they use the sample estimates for the GMV and the MRR portfolios, which would be implementable only if the full sample was known in advance.

### 5.1.2 Out-of-Sample Tests

To measure the added value of factor indices, if any, in a more realistic setting, we now consider a framework where:

- Investors construct their portfolios using past data only, without any look-ahead bias;
- They do not necessarily attempt to construct explicit proxies for mean-variance efficient portfolios such as the GMV or the MRR, and may rely instead on "heuristic" diversification schemes presented in Section 3.1;
- They not only focus on volatility and expected return, but also on relative risk and performance measures, such as tracking error and information ratio, and

also have a concern over the turnover of their portfolios, which they should keep relatively low in the presence of transaction costs and other forms of market frictions.

The general principle is to backtest the two portfolios on a historical sample and then compare their expected returns, volatilities and Sharpe ratios. For all weighting schemes that require the covariance matrix, or the relative covariance matrix, we estimate the parameters over a two-year rolling window. Portfolios are rebalanced every quarter, which is a reasonable frequency for practical applications. Given two portfolios 1 and 2, the null hypothesis of the three tests are:

$$\mu_1 = \mu_2, \quad \sigma_1 = \sigma_2, \quad \lambda_1 = \lambda_2.$$

For the Sharpe ratio, a test was introduced by Jobson and Korkie (1981) (and the test was subsequently corrected by Memmel (2003)). However, the formula of Jobson and Korkie is based on the assumption that underlying asset returns are stationary, multivariate normal in the cross section and independent in the time series. The stationarity assumption is a reasonably good approximation of reality, but the normality assumption rules out the possibility of fat tails and the independence assumption is at odds with the observation that squared returns on stocks have significantly positive autocorrelation (see e.g. Campbell et al. (1997) Chap. 12.2).

Ledoit and Wolf (2008) relax the assumptions of normal and independent returns and introduce two variants of another test, which differ in the way the covariance matrix of the sample difference  $\hat{\lambda}_1 - \hat{\lambda}_2$  is estimated. The first test relies on inference robust to heteroscedasticity

## 5. Empirical Analysis of Factor Investing

and autocorrelation (HAC inference), in the spirit of the Newey-West estimator for the covariance matrix (Newey and West, 1987). The second method aims at improving the small-sample properties of the test through bootstrapping. Comparing different test procedures on simulated data, Ledoit and Wolf show that if returns are not Gaussian and independent, the Jobson-Korkie-Memmel test rejects the null of equality too often compared to the confidence level set by the statistician. The same is true for the HAC test, although the probability of Type I error is lower than with the previous test. Only the bootstrap test has a probability of Type I error in line with the chosen confidence level.

Given that our backtests involve a large number of data points, we use the former version, which is also less time-consuming.<sup>23</sup> The equality test for Sharpe ratios is adapted by Ledoit and Wolf (2008) to test for the equality of volatilities. As far as expected returns are concerned, we follow an approach similar to Ledoit and Wolf (2008) by adapting the test for Sharpe ratios. Appendix C.2 provides additional detail on the various tests. It should be noted that the volatility and Sharpe ratio tests can also be implemented in a straightforward manner for returns relative to a benchmark, which enables us to test for the equality of tracking errors and information ratios.

### 5.2 The Data

Our analysis of the added value of factor investing requires two types of series:

- Traditional indices, which are broad asset class indices;

- Factor indices, which correspond to proxies for the factors identified in Section 4.5.

We need such series in the equity, bond and commodity asset classes. In addition, we also need series of multi-asset class value and momentum. Our base case geographical area will be the United States, for which the longest time series are available (since 1970 for equity series and 1973 for bond series).

#### 5.2.1 Traditional Indices

Traditional indices form the original investment universe, which we denote by  $\mathcal{I}_0$ . We consider broad indices, whose construction process does not involve any explicit tilt towards a characteristic. In the equity universe, we use the CRSP database to collect daily data for the period Jun. 1970 - Dec. 2013 on a broad index that consists of the 500 stocks eligible to the US universe of ERI Scientific Beta.<sup>24</sup> Unless otherwise stated and in the base case, stocks are weighted by their capitalisation, prices and market values being obtained from CRSP. We also use the long-term database of ERI Scientific Beta, which covers the period Jun. 1970 - Dec. 2013, to study alternative weighting schemes, known as "smart" weighting schemes. In a nutshell, these schemes aim to capture the risk-return trade-off in a given universe better than capitalisation-weighted portfolios do (see Amenc et al. (2012)). In our analysis, we focus on the following weighting schemes:

- Maximum Deconcentration weighting scheme, which maximises the portfolio effective number in the portfolio (see Section 3.1.1) so as to generate the closest approximation to an equally-weighted

23 - We are grateful to Olivier Ledoit for sharing the Matlab code implementing the test on his webpage.

24 - See <http://www.scientificbeta.com/download/file/grd-universe> for details on the construction of the universe.

## 5. Empirical Analysis of Factor Investing

portfolio consistent with turnover constraints;<sup>25</sup>

- Minimum Volatility weighting scheme, which aims to minimise the ex-ante volatility by using the volatilities and the correlation structure (see Section 3.1.2);<sup>26</sup>
- Multi-Strategy weighting scheme, which is an equally-weighted combination of five alternative weighting schemes. These schemes either require no parameter estimation (maximum deconcentration) or require only the covariance matrix estimates.<sup>27</sup> This index aims to combine the respective advantages of the various schemes, and to reduce the impact of estimation risk. Indeed, with the exception of maximum deconcentration, all of these schemes are subject to such risk, but they are not affected by it in the same way, so that mixing them mitigates the effects of estimation errors (see Amenc et al. (2014a)).

For the fixed-income universe, we use the Barclays US Treasury bond index (with coupons reinvested), which is available in Datastream as of Jan. 1976 at the monthly frequency, and becomes daily after Feb. 1994. The index factsheet indicates that bonds are weighted by their market values. Finally, our broad commodity index is the S&P GSCI total return, also obtained from Datastream. The time-series is daily and starts in Jan. 1970. The factsheet indicates that the weights of the futures contracts are based on the world production for each commodity.

### 5.2.2 Factor Indices

Factor indices form the additional universe,  $\mathcal{I}_a$ . Coming up with the "best" definition of each factor is not the focus of our paper. We instead adopt straightforward definitions for the attributes used in the

definitions of the various factors, such as size, value and momentum. It is well possible that other choices lead to superior performance in the historical sample, so our results provide a conservative assessment of the benefits of factor investing.

### Equity Factors

The equity factors that we consider are size, value, momentum and volatility. Each of them exists in long-only (L/O) and long-short dollar-neutral (L/S) versions. Again, we borrow the series from the long-term ERI Scientific Beta database. For each factor, stocks are sorted on a given criterion and the universe is divided into two equal groups, the "top" and the "bottom" halves. Each L/S factor has a "long leg", which is the top or the bottom group depending on the factor, and a "short leg", which is the other half. The choice of the criterion and the definition of the long leg are of course determined by the consensus from the academic literature (see Section 4). The following list gives, for each factor, the criterion used and the definition chosen for the long leg:

- Size: stocks are sorted on their market capitalisation and the long leg is the bottom size group;
- Value: stocks are sorted on their book-to-market (BE/ME) ratio and the long leg is the top BE/ME group;
- Momentum: stocks are sorted on their past 52-week return, skipping the last month, and the long leg is the past winner group;
- Volatility: stocks are sorted on their past 104-week volatility, and the long leg is the low risk group.

In what follows, we abbreviate the factor names as Siz, Val, Mom and Vol. Within each leg, stocks can be weighted by their

25 - Details on the methodology are available at <http://www.scientificbeta.com/download/file/scientific-beta-max-deconcentration-indices>.

26 - See <http://www.scientificbeta.com/download/file/scientific-beta-efficient-min-volatility-indices>.

27 - In detail, the five weighting schemes are maximum deconcentration, minimum volatility, maximum decorrelation, inverse volatility and efficient maximum Sharpe ratio. The maximum decorrelation is a version of the minimum volatility which assumes identical volatilities for all constituents, and the efficient maximum Sharpe ratio is a proxy for the MSR portfolio, where expected returns estimates are proportional to semi-volatilities of constituents. Details on the methodologies for estimating parameters are given in Amenc et al. (2014a).

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market capitalisation (the default choice), or weighted according to one of the aforementioned three alternative "smart" weighting schemes. As for the broad indices, all of these factor indices are available from the ERI Scientific Beta database over the period 1970-2013 at the daily level. Thus, we will choose this period and this frequency in our experiments on the equity class.

### *Dollar-Neutral Factor Values*

A L/S factor is a dollar-neutral portfolio that goes long the long leg and short the short leg. To compute the total returns on these L/S factors, we make the following assumptions (which are identical to those made by Koijen et al. (2013) to compute the returns on a portfolio of futures contracts):

- At each rebalancing date  $t$ , the margin capital is equal to the face value of the allocation to each leg (i.e. the position is fully collateralised);
- The margin capital is invested at the risk-free rate  $r_{dt}$  for the period  $[t, t + 1]$ .

Consider an initial position  $V_0 = \$1$ , invested at the risk-free rate prevailing at date 0 for the period  $[0, 1]$ . Sell short \$1 of the short leg and use the proceeds to purchase \$1 of the long leg. At date 1, the wealth is:

$$\begin{aligned} V_1 &= V_0 \times (1 + r_{d0}) + V_0 \times (1 + r_{long,1}) - V_0 \times (1 + r_{short,1}) \\ &= V_0 \times [1 + r_{d0} + r_{long,1} - r_{short,1}]. \end{aligned}$$

At date 1, the same operation is performed with a collateral amount equal to  $V_1$ . Hence, the value of the L/S portfolio dynamically evolves as:

$$V_{t+1} = V_t \times [1 + r_{dt} + r_{long,t+1} - r_{short,t+1}].$$

The total return between dates  $t$  and  $t + 1$  is thus  $r_{dt} + r_{long,t+1} - r_{short,t+1}$ , and

the excess return over the risk-free rate is simply the excess return of the long leg over the short leg. For such a portfolio, the natural benchmark is the portfolio strategy consisting of rolling over the initial wealth at the locally risk-free asset. If the long leg outperforms the short one, we expect the L/S portfolio to outperform the risk-free asset.

This procedure for computing L/S factor returns is applied to all factors in this paper. In principle, the risk-free rate should have a horizon equal to the rebalancing period of the collateralised portfolio, but in order to have a uniform interest rate throughout the study, we take the secondary market rate on the US 3-month Treasury Bill, which is available at the daily frequency from Datastream since 1954. Of course, the annual rate is multiplied by a suitable scaling factor to make its horizon coincide with the rebalancing period. For equities, we rebalance the L/S portfolios on a daily basis.

### *Bond Factors*

We consider two bond factors: term and credit. As in the equity class, we define for each factor a long and a short legs as bond portfolios tilted towards a characteristic. For each factor, the legs are defined as follows:

- Term (abbreviated as Ter): the long leg is the Barclays US Treasury Long index and the short leg is the Barclays US Treasury 1 year - 3 years. Both series are available from Datastream as of Jan. 1976;
- Credit (Cre): the long leg is the Barclays US Corporate Investment Grade index and the short leg is the Barclays US Treasury index, which already serves as a broad index. Both series are available from Datastream as of Jan. 1976.

## 5. Empirical Analysis of Factor Investing

In view of the bond data availability, we will conduct our experiments on bonds over the period 1976–2013 with monthly data. Using daily data would be possible (and would have the advantage of consistency with equity experiments), but this would delay the starting date of the study until 1998, which is the year when all bond series become available at this frequency.

Note that unlike for equities, we have a single weighting scheme within each factor. Barclays website indicates that the bond indices are weighted by their market value, which is the equivalent of capitalisation weighting for equities. The factsheet of the US Treasury index also mentions versions of the index weighted by GDP or by fiscal strength, but there does not seem to exist marketed bond indices that would adopt a "smart weighting scheme" similar to what has been developed for equities. The construction of such indices is an interesting topic per se, and it would also be useful to investigate the benefits of factor investing with smart-weighted factors. Deguest et al. (2013) and Deguest et al. (2014) have recently focused on the first part of the problem, namely the design of bond portfolios requiring the estimation of the covariance matrix and possibly the expected returns.

### *Commodity Factors*

Our commodity factors are momentum and term structure, which are constructed from a sample of 27 futures contracts written on physical commodities.<sup>28</sup> The raw futures data is from Datastream, and continuous series of futures returns are obtained by rolling from the nearest contract to the second nearby contract. Each time the investor changes contracts, he captures the roll return, which arises from the difference between the prices of the two contracts (see Section 4.3.1). If the second contract is more

expensive than the nearest contract (contangoed term structure), the roll return is positive; otherwise (backwardated term structure), it is positive. The period of availability of futures returns and roll yields depends on the contract considered. 22 out of the 27 contracts are available over the period Jan. 1985 – Aug. 2011, and data for the others is only available over subperiods, the shortest record being Mar. 2004 – Aug. 2011 for electricity. All observations are daily. We compute the values of fully collateralised portfolios of futures contracts by following exactly the same principles as for the L/S factors: the position is rebalanced every day, and the total return equals the risk-free rate plus the futures return.

To obtain the long and the short legs of the momentum and term structure factors (abbreviated as Mom and TSt in what follows), the following procedure is used:

- At the end of each month, the past 12-month total return and average roll yield are computed for each contract;
- The  $q$  contracts for which a score is available ( $q \leq 27$ ) are sorted on the criterion. Over the next month, the long leg will consist of the 50% contracts with the highest score (past return or average roll yield), and the short leg will be made of the 50% contracts with the lowest score. In case  $q$  is odd, the quantity  $0.5q$  is rounded to the largest smaller integer;
- Within each leg, contracts are equally weighted at the end of month, and the portfolio is left in a buy-and-hold mode for the next month;
- At the end of the next month, the procedure is repeated. This procedure is similar to that of Fuertes et al. (2015) to compute the returns to portfolios sorted on momentum, roll yield or idiosyncratic volatility. There are also a few differences.

28 - We are very grateful to Ana-Maria Fuertes, Joëlle Miffre and Adrian Fernandez-Perez for sharing the commodity futures data used in their paper Fuertes et al. (2015). The commodities are crude oil, heating oil, gasoline, natural gas, electricity, soybean oil, soybean meal, soybeans, corn, cotton #2, sugar #11, oats, wheat, rough rice, coffee, cocoa, orange juice, live cattle, feeder cattle, pork bellies, lean hogs, lumber, platinum, palladium, gold, copper and silver.

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First, they consider the 20% top and bottom contracts, while we divide our universe into two halves in order to ensure consistency with our approach for stocks. The second difference is that they test several ranking periods, ranging from 1 to 12 months, while we adopt a 12-month window. Again, this choice has been made for consistency, since 12 months is the time span used for stock momentum. It would also be possible to consider holding periods longer than one month in the above procedure.<sup>29</sup> Once the total returns to the long legs have been obtained, we compute the returns to the L/S factors by following the same approach as for equities and bonds. The futures returns start in Jan. 1985, but one year of data is needed to compute the first momentum and roll return ranks, so that the first year of data is lost, and the usable sample period for our commodity experiments will be Jan. 1986 - Aug. 2011.

We emphasise that the L/O bond and commodity factors that we compute are not meant to be investable portfolios. Many implementation constraints would have to be taken into account, such as the limitation of the number of constituents, a liquidity screening, a control of turnover and the introduction of capacity constraints. Thus, these factors only satisfy minimal investability conditions in the sense that they are defined as portfolio returns and are L/O. On the other hand, the L/O equity indices computed by ERI Scientific Beta incorporate stricter investability constraints, so that they represent more easily investable portfolios.

### *Multi-Class Factors*

The multi-class factors that we test are "value and momentum everywhere" constructed after the methodology of AMP13. Each

of them is defined as a combination of the single-class value and momentum factors. While the time series of factors are readily available on Lasse Pedersen's website, we do not use them because the single-class factors are long-short and our out-of-sample tests require L/O portfolios.<sup>30</sup> Thus, we borrow another dataset from Lasse Pedersen's library: in their paper, AMP13 compute the returns to equity, bond and commodity portfolios sorted on value or momentum characteristics and use them as test assets in their asset pricing tests.<sup>31</sup> From our perspective, they have the advantage of being long-only, which allows us to construct L/O and L/S factors. In AMP13's dataset, the underlying assets for equity portfolios are individual stocks, those of bond portfolios are Government bond indices of 10 developed countries, and those of commodity portfolios are 27 futures contracts.<sup>32</sup> Hence, the bond portfolios are global portfolios.

In detail, AMP13 provide the following monthly series:

- 24 tertile portfolios of stocks sorted on book-to-market ratio (BE/ME) or momentum score for US, UK, Europe and Japan. Stocks are equally weighted within each portfolio. The period of availability depends on the geographical zone and the attribute. It is Feb. 1972 - Jul. 2011 for US portfolios, Jul. 1981 - Jul. 2011 for non-US portfolios sorted on BE/ME, Jan. 1972 - Jul. 2011 for UK portfolios sorted on momentum score and Feb. 1974 - Jul. 2011 for the corresponding Europe and Japan portfolios;
- 6 tertile portfolios of bonds sorted on value or momentum score. The series are available over the period Jan. 1982 - Jul. 2011;
- 6 tertile portfolios of commodity futures sorted on value or momentum score. The series cover the period Jan. 1972 - Jul. 2011.

29 - We refer to Miffre and Rallis (2007) for a study of the momentum effect in commodity futures as a function of the ranking and holding periods.

30 - Another reason why we had to recompute the value and momentum everywhere factors is that those of AMP13 include currencies, while the only asset classes that we consider in this paper are equities, bonds and commodities.

31 - We are grateful to the authors for sharing their data.

32 - The 10 countries represented in the bond indices are Australia, Canada, Denmark, Germany, Japan, Norway, Sweden, Switzerland, the UK and the US. 20 of the 27 commodities are common with our dataset. Electricity, lumber, oats, orange juice, palladium, pork bellies and rice are replaced by aluminium, nickel, zinc, lead, tin, Brent crude oil and gas oil.

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Thus, for each one of the four attributes (low or high value, low or high momentum), we have a total of six series (4 equity, 1 bond and 1 commodity series). We aggregate them by following a procedure similar to that of AMP13:

- We first aggregate the regional equity portfolios by computing the returns to an inverse volatility-weighted portfolio of the available constituents (i.e. the portfolio return in each period is the weighted sum of returns available in this period). This portfolio is rebalanced every month, and the volatilities are estimated from the past 24 monthly returns. The result is a global growth, value, loser or winner equity portfolio;
- Next, we aggregate this portfolio with the corresponding bond and commodity portfolios, still by weighting the three constituents by their volatilities.

Repeating the operation for each attribute, we obtain four multi-class portfolios. Among them, the high value and the high momentum ones are the long legs of the multi-class value and momentum factors. We denote them as Mul L/O Val and Mul L/O Mom. The opposite tilts serve to compute the returns to the L/S factors, referred to as Mul L/S Val and Mul L/S Mom. Given that the first two years of data are needed to compute the volatilities, our multi-class series cover the period Dec. 1973 - Jul. 2011.

Our L/S portfolios are not identical to those of AMP13 because the authors first compute L/S portfolios at the asset class level and then aggregate them, while we perform aggregation of L/O portfolios before we compute L/S returns. Moreover, our L/S returns are obtained as the differences between inverse volatility weighted returns while the weights of AMP13 L/S portfolios are functions of the constituents'

ranks in the sort. We keep the idea of weighting constituents by the inverses of their volatilities because as explained by AMP13, this ensures that all asset classes have roughly the same contribution to volatility, even though equities and commodities are more volatile than bonds.

### 5.3 Descriptive Statistics

We start with descriptive statistics on the various factors.

#### 5.3.1 Equity Factors

Table 1 presents the descriptive statistics for L/O and L/S equity factors. Panel (a) shows the expected effect: in the long run, each L/O factor outperforms the broad capitalisation-weighted (CW) index, both in terms of average return and Sharpe ratio. As explained by Amenc et al. (2014b), this is the result of the intended tilt towards a rewarded characteristic (note that the weighting scheme is the same for all indices and thus does not contribute to explain the differences). In fact, the broad index is by construction tilted towards large and growth stocks, and it is known empirically that these stocks underperform the small and value ones. All L/O factors have high (greater than 90%) correlations between them and with the broad index. The highest correlation is between the broad index and the L/O Mom factor, which is probably due to the fact that the CW scheme by construction leads to overweight stocks whose prices have recently risen. The range of volatilities is rather narrow, between 15.48% and 17.53%. That the L/O volatility factor has the lowest volatility is not surprising but is not a trivial result, because stocks are selected on the basis of their past volatility, and those with the lowest in-sample volatilities are not necessarily the least volatile ones out

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of sample. It turns out that picking the least risky stocks in the sample delivers low out-of-sample volatility, which is to be taken as an indication that there is some persistence in volatility.

L/S factors have much lower, and often negative, correlations. They also have lower volatilities, because the two legs have

similar risks (except for the Vol factor) and high correlation, so that taking the difference mechanically reduces the volatility. To see this formally, write the volatility of the L/S factor as a function of the volatilities and the correlation of the long and short legs:

$$\sigma_{L/S} = \sqrt{\sigma_L^2 + \sigma_S^2 - 2\rho_{SL}\sigma_S\sigma_L},$$

Table 1: Summary statistics on equity factor indices (1970-2013).

(a) L/O factors.					
	Avg. excess return (%)	Volatility (%)	Sharpe ratio	Tracking error (%)	Information ratio
Equ Broad CW	5.58	17.00	0.33	0.00	-
Equ L/O-Siz CW	8.23	17.40	0.47	5.86	0.45
Equ L/O-Mom CW	6.53	17.14	0.38	3.41	0.28
Equ L/O-Vol CW	5.84	15.48	0.38	4.28	0.06
Equ L/O-Val CW	7.33	17.53	0.42	4.65	0.38

Correlations (%)					
	Equ Broad CW	Equ L/O-Siz CW	Equ L/O-Mom CW	Equ L/O-Vol CW	Equ L/O-Val CW
Equ Broad CW	100.00	94.22	98.01	96.95	96.42
Equ L/O-Siz CW		100.00	91.82	90.94	94.60
Equ L/O-Mom CW			100.00	95.05	93.98
Equ L/O-Vol CW				100.00	93.88
Equ L/O-Val CW					100.00

(b) L/S factors.			
	Avg. excess return (%)	Volatility (%)	Sharpe ratio
Equ L/S-Siz CW	2.65	6.59	0.40
Equ L/S-Mom CW	1.59	8.81	0.18
Equ L/S-Vol CW	-1.09	10.60	-0.10
Equ L/S-Val CW	2.15	7.06	0.30

Correlations (%)					
	Equ Broad CW	Equ L/S-Siz CW	Equ L/S-Mom CW	Equ L/S-Vol CW	Equ L/S-Val CW
Equ Broad CW	100.00	-10.23	-5.07	-52.29	-0.39
Equ L/S-Siz CW		100.00	-16.26	-10.82	40.51
Equ L/S-Mom CW			100.00	3.72	-20.56
Equ L/S-Vol CW				100.00	-1.99
Equ L/S-Val CW					100.00

Statistics are estimated from daily returns over the period Jun. 1970 - Dec. 2013 for a broad equity index of US stocks and four equity portfolios proxying for the size, momentum, volatility and value factors. Average excess return is the difference between the geometric average of the index and that of the risk-free asset (the risk-free rate is the US 3-month Treasury bill rate). Average return and volatility are annualised and the Sharpe ratio is the ratio of the two quantities. Factor indices are capitalisation-weighted (CW). The tracking error and the information ratio are computed with respect to the broad CW equity index.

## 5. Empirical Analysis of Factor Investing

and denote the minimum and the maximum of  $\sigma_S$  and  $\sigma_L$  as  $\sigma_{\min}$  and  $\sigma_{\max}$ . For a positive correlation  $\rho_{SL}$ , we thus have that:

$$\frac{\sigma_{L/S}}{\sigma_{\min}} \leq \sqrt{2 \left[ \left( \frac{\sigma_{\max}}{\sigma_{\min}} \right)^2 - 1 \right]}.$$

If  $\frac{\sigma_{\max}}{\sigma_{\min}} \leq \sqrt{\frac{3}{2}} \approx 1.22$ , which is the case if both legs have close volatilities, then we have  $\sigma_{L/S} \leq \sigma_{\min}$  that is we find that the L/S portfolio is less volatile than both legs. It is for the Vol factor that the reduction is the lowest, precisely because the short leg is by construction highly volatile. The ordering of average returns and Sharpe ratios is the same as for L/O factors. From Section 2.2.2, if a pricing factor is the return to a L/S portfolio, its risk premium equals the expected excess return of the long leg over the short one. Thus, if our equity factor indices are pricing factors, the average excess returns estimate their risk premia. For size, momentum and value, the premium is positive, which is in line with the literature that has documented the size, momentum and value effects. The case of the volatility factor is more surprising since it has a negative premium (−1.09% per year).

We examine the origin of this effect in more detail below.

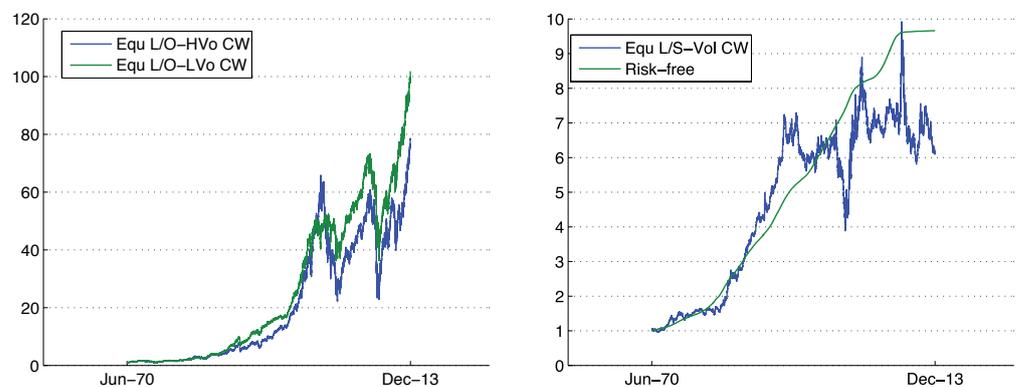
### The Case of the Volatility Factor

The negative premium for the L/S volatility factor means that over the sample period, this L/S portfolio underperforms the risk-free asset, which may be regarded as yet another evidence of the lack of robustness of this effect (see also the references in Section 4.1.6). To better understand the situation, we plot the value of \$1 invested in Jun. 1970 in the L/S portfolio or the bank account in Figure 1. But plotting the values of the long and short legs shows that the low volatility stocks have outperformed the high volatility ones, thereby providing further evidence for the low volatility puzzle of Ang et al. (2006) and Ang et al. (2009). We thus face something of a paradox.

The explanation lies in the difference between the arithmetic and the geometric means. For any portfolio, we have the following approximated relation (which is good for daily returns) between the sample geometric mean, the sample arithmetic mean and the sample variance:

$$\hat{\mu}_g \approx \hat{\mu}_a - \frac{1}{2} \hat{\sigma}^2. \quad (5.2)$$

Figure 1: Value of \$1 invested in the high or low volatility equity portfolio, the L/S volatility factor or the risk-free asset.



The left picture shows the value of \$1 invested in Jun. 1970 in the high or low volatility equity portfolio. Both portfolios are capitalisation-weighted (CW). The right picture shows the value of \$1 invested in the L/S volatility factor or the risk-free asset. The L/S factor is long the low volatility portfolio and short the high volatility one, and the value of the risk-free asset is the compounded US 3-month Treasury bill rate. The unit on the y-axis is the dollar.

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The total return on the L/S volatility factor is the sum of the risk-free rate and the excess return of the long leg over the short one. Because the arithmetic mean is linear, we obtain:

$$\hat{\mu}_{g,L/S} \approx \hat{\mu}_{a,rd} + \hat{\mu}_{a,L} - \hat{\mu}_{a,S} - \frac{1}{2}\hat{\sigma}_{L/S}^2, \quad (5.3)$$

where notations have obvious meanings. The left panel of Figure 1 shows that  $\hat{\mu}_{g,L} > \hat{\mu}_{g,S}$  (i.e. the long leg outperforms the short one). But – and this is the cornerstone of the argument –, the order of arithmetic means is reversed, i.e. we have  $\hat{\mu}_{a,L} < \hat{\mu}_{a,S}$  (see Table 2). By (5.3), this implies that  $\hat{\mu}_{g,L/S} < \hat{\mu}_{a,rd}$ . For the risk-free asset over a long period, the variance of daily returns is very small compared to the cumulative return, so that (5.2) gives  $\hat{\mu}_{a,rd} \approx \hat{\mu}_{g,rd}$  with a very high precision. Thus, we end up with  $\hat{\mu}_{g,L/S} < \hat{\mu}_{g,rd}$ , which is exactly what is seen in the right panel of Figure 1.

To summarise, the apparent inconsistency between the performances of the legs of the volatility factor and that of the L/S factor arises because the geometric mean of the long leg exceeds that of the short leg, while the opposite holds for arithmetic means. This effect does not occur for the other factors: as appears from Table 2, both means are ranked in the same order. What is special about the volatility factor is the built-in large volatility spread between the long and the short legs, which causes the order of means to be flipped (see (5.2)). For the other factors, the long and the short legs have much closer volatilities so that differences between arithmetic and geometric returns are less substantial.

Incidentally, Table 2 shows that in our sample, the existence of the low volatility anomaly depends on the formula applied to compute returns: the high volatility portfolio has lower geometric return than the low volatility one – which reveals

Table 2: Comparison between arithmetic and geometric means for equity factor indices (1970-2013).

Factor	Portfolio	Arithmetic avg. excess return (%)	Geometric avg. excess return (%)	Volatility excess return (%)	Approximated geometric avg. excess return (%)
Siz	High	6.34	5.26	17.07	4.88
	Low	9.05	8.23	17.40	7.53
	L/S	2.70	2.65	6.59	2.49
Mom	High	7.49	6.53	17.14	6.02
	Low	5.60	4.24	18.15	3.96
	L/S	1.89	1.59	8.81	1.50
Vol	High	7.07	5.18	21.28	4.81
	Low	6.60	5.84	15.48	5.40
	L/S	-0.48	-1.09	10.60	-1.04
Val	High	8.27	7.33	17.53	6.73
	Low	6.00	4.88	17.10	4.54
	L/S	2.27	2.15	7.06	2.02

Statistics are estimated from daily returns over the period Jun. 1970 - Dec. 2013 for a broad equity index (the ERI Scientific Beta Long-Term US capitalisation-weighted index) and four equity portfolios proxying for the size, momentum, volatility and value factors. Excess returns are computed with respect to the risk-free asset, whose value is the compounded US 3-month Treasury bill rate. For each factor, three capitalisation-weighted portfolios are defined: the portfolio of high characteristic stocks, the portfolio of low characteristic stocks, and the long-short dollar-neutral portfolio. The first two columns contain the arithmetic and geometric means of excess returns, both expressed in annual terms. The fourth column contains the approximated geometric mean, obtained as the arithmetic mean minus half the variance.

## 5. Empirical Analysis of Factor Investing

an anomaly – but it has higher average arithmetic return. This observation is in line with the findings of Baker et al. (2011) and Van Vliet et al. (2011): unlike us, both papers report that the high volatility portfolio has lower arithmetic mean, but the return spread between the low and the high volatility groups is smaller with arithmetic than with geometric returns.

### 5.3.2 Bond Factors

As appears from Table 3, L/O bond portfolios are less volatile than L/O equity portfolios: the maximum volatility is 10.71% for bonds and the minimum volatility is 15.48% for equities. The L/O term and credit factors outperform an index that consists of lower-duration Treasury bonds: hence, the ex-post

term and credit premia are positive. They have also higher volatilities, which can be explained by their higher sensitivities to common sources of risk. First, the L/O term factor consists of long-duration bonds, so it is more exposed to the risk of changes in interest rates than the Barclays Treasury index. Second, the L/O credit factor is exposed to the risk of changes in credit spreads. To the extent that these changes are not strongly negatively correlated with changes in the risk-free rate, this results in a higher volatility.

L/O bond portfolios also have lower risk premia, around 3% versus more than 5% for equity portfolios. But the reduction in expected return is more than compensated

Table 3: Summary statistics on bond factor indices (1976-2013).

(a) L/O factors.

	Avg. excess return (%)	Volatility (%)	Sharpe ratio	Tracking error (%)	Information ratio
Bnd Broad	2.45	5.41	0.45	0.00	-
Bnd L/O-Cre	3.11	7.06	0.44	3.84	0.17
Bnd L/O-Ter	3.38	10.71	0.32	5.83	0.16

Correlations (%)

	Bnd Broad	Bnd L/O-Cre	Bnd L/O-Ter
Bnd Broad	100.00	84.21	95.01
Bnd L/O-Cre		100.00	80.85
Bnd L/O-Ter			100.00

(b) L/S factors.

	Avg. excess return (%)	Volatility (%)	Sharpe ratio
Bnd L/S-Cre	0.69	3.86	0.18
Bnd L/S-Ter	2.18	8.93	0.24

Correlations (%)

	Bnd Broad	Bnd L/S-Cre	Bnd L/S-Ter
Bnd Broad	100.00	17.26	88.27
Bnd L/S-Cre		100.00	12.00
Bnd L/S-Ter			100.00

Statistics are estimated from monthly returns over the period Jan. 1976 - Dec. 2013 for a broad bond index (the Barclays US Treasury index) and two bond portfolios proxying for the term and credit factors. Each factor exists in long-only and long-short versions. Average excess return is the difference between the geometric average of the index and that of the risk-free asset (the risk-free rate is the US 3-month Treasury bill rate). Average return and volatility are annualised and the Sharpe ratio is the ratio of the two quantities. The tracking error and the information ratio are computed with respect to the broad bond index.

## 5. Empirical Analysis of Factor Investing

by the decrease in volatility, and bond portfolios have relatively high Sharpe ratios: the broad index and the L/O "Cre" factor (fore credit factor) are as good as equity factors according to this criterion. These historical Sharpe ratios are likely to overstate prospective Sharpe ratios, because the sample period was characterised by an unusually long decrease in yields that has brought them close to zero. A standard mean reversion argument suggests that it more likely for rates to grow than to fall down in the future, so that expected future bond returns may currently be lower than what is suggested by historical averages (see Martellini et al. (2015) for a discussion of the implications for risk parity portfolios, which are typically heavily invested in bonds). Finally, in our sample, long-duration bonds and corporate bonds

As in the equity class, L/O bond factors are highly correlated. Moving from L/O to L/S factors leads to a strong decrease in correlations, except for the pair Broad-Ter: the correlation between the L/S Ter factor and the broad index remains as high as 88.27%. Indeed, for this factor, the long leg is much more volatile than the short one, due to the large duration gap. Hence, most of the risk of the L/S factor comes from the long leg, so the correlation with the broad index essentially captures the correlation between the long leg and the broad index.

### 5.3.3 Commodity Factors

As can be seen from Table 4, the broad commodity index has a high volatility (21.08%) over the period Jan. 1986 - Aug. 2011. This is largely due to the drawdown that took place in the second half of 2008 when the S&P GSCI lost about 70% of its value between July 2008 and February 2009. As a result, the index return was only 3% per year higher than that

of the risk-free asset, which is less than what is obtained with a portfolio of long-duration Government bonds (see Table 3). Due to the low average return and the high volatility, the ex-post Sharpe ratio is comparatively low, at 0.14 only. Had the sample period been stopped just before the 2008 downturn, the picture would have been very different. For instance, over the period Jan. 1986 - Jun. 2008, the average excess return is 7.70% and the volatility falls to 19.10%, which implies a Sharpe ratio of 0.40. These values are close to those obtained with L/O equity factors (see Table 1). The two factors experience their worst drawdown approximately over the same period as the broad index, that is, in the second half of 2008, but the losses are more limited (see Panel (a) in Figure 2): the maximum drawdowns of the momentum and term structure factors are respectively 52.6% and 56.4%, which results in a better cumulative return in the sample. A distinctive feature of commodity factors with respect to equity and bond factors is that the L/S portfolios outperform their L/O counterparts, while the opposite is observed in the other two universes. This is explained by the fact that the short legs of the L/S factors have negative cumulative returns over the period: the average excess returns (not reported in the table) on the portfolio of past losers and past contangoed contracts are  $-5.92\%$  and  $-5.17\%$  per year, respectively.

The correlations between L/O portfolios are high, and in any case much higher than those between L/S portfolios, but they are overall lower than within the equity and bond universes. This observation suggests that there is more heterogeneity in the commodity class. To have an informal illustration, consider Panel (b) of Figure 2, where we have plotted the value of \$1 invested in Jan. 1985 in the S&P GSCI or

## 5. Empirical Analysis of Factor Investing

Table 4: Summary statistics on commodity factor indices (1986–2011).

(a) L/O factors.

	Avg. excess return (%)	Volatility (%)	Sharpe ratio	Tracking error (%)	Information ratio
Cmd Broad	3.00	21.08	0.14	0.00	-
Cmd L/O-Mom	4.97	13.86	0.36	15.23	0.13
Cmd L/O-TSt	4.82	13.91	0.35	13.52	0.13

Correlations (%)

	Cmd Broad	Cmd L/O-Mom	Cmd L/O-TSt
Cmd Broad	100.00	69.24	77.61
Cmd L/O-Mom		100.00	82.91
Cmd L/O-TSt			100.00

(b) L/S factors.

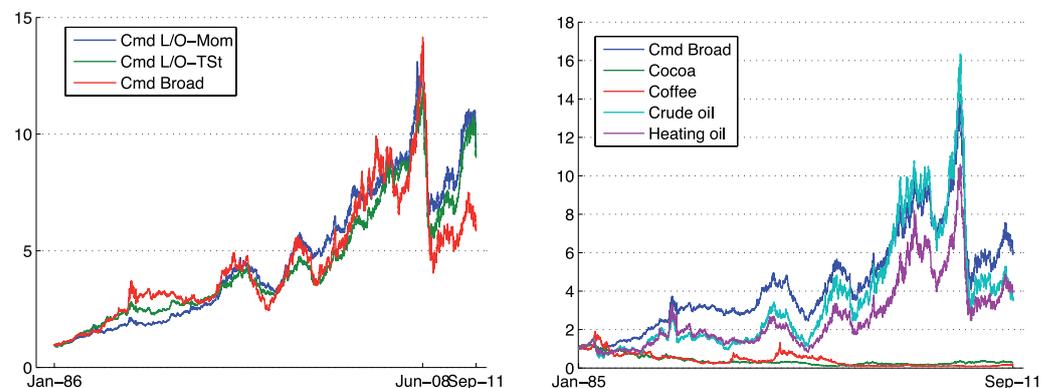
	Avg. excess return (%)	Volatility (%)	Sharpe ratio
Cmd LS-Mom	10.48	14.15	0.74
Cmd LS-TSt	9.62	13.49	0.71

Correlations (%)

	Cmd Broad	Cmd L/S-Mom	Cmd L/S-TSt
Cmd Broad	100.00	13.42	31.72
Cmd L/S-Mom		100.00	33.50
Cmd L/S-TSt			100.00

Statistics are estimated from daily returns over the period Jan. 1986 – Aug. 2011 for a broad commodity index (the S&P GSCI) and two portfolios of commodity futures proxying for the momentum and term structure factors. Each factor exists in long-only and long-short versions. Average excess return is the difference between the geometric average of the index and that of the risk-free asset (the risk-free rate is the US 3-month Treasury bill rate). Average return and volatility are annualised and the Sharpe ratio is the ratio of the two quantities. The tracking error and the information ratio are computed with respect to the broad commodity index.

Figure 2: Value of \$1 invested in the S&P GSCI, in L/O commodity factors or in passive strategies rolling over cocoa, coffee, crude oil or heating oil contracts.



In Panel (a), \$1 is invested at the end of January 1986 in the broad index (the S&P GSCI) or in the long-only portfolios proxying for the term structure and momentum factors. In Panel (b), \$1 is invested at the end of January 1985 in the broad commodity index, or in a strategy that passively rolls over cocoa, coffee, crude oil or heating oil contracts. In each plot, the unit on the y-axis is the dollar.

## 5. Empirical Analysis of Factor Investing

33 - The bond series become daily after 1994 for the Treasury and the Treasury Long indices, 1998 for the 1 year - 3 years index and 1989 for the Corporate Investment Grade index. Thus, an alternative would have been to construct a daily sample for the period 1998-2011, where all the data is available at the daily frequency. We have favoured the sample length over the higher frequency. Moreover, working with monthly data is a common practice in related empirical work (see e.g. AMP13).

a roll-over of cocoa, coffee, crude oil or heating oil futures contracts. As is well known, the S&P GSCI is strongly correlated with oil contracts. On the other hand, the passive strategies rolling over cocoa or coffee contracts have followed a very different pattern, with a tendency to lose money. These patterns are explained in large part by the roll returns: as shown by Erb and Harvey (2006), the long-term returns to passive strategies are strongly related to the roll yields, while being only loosely related to changes in spot prices. As a matter of fact, we verify that in our dataset, over the period Jan. 1985 - Aug. 2011, the daily average roll yield was negative for cocoa and coffee (respectively  $-1.50\%$  and  $-1.58\%$ ) and positive for crude oil and heating oil (respectively  $0.29\%$  and  $0.10\%$ ). Thus, the cross sectional differences between the performances of futures strategies are mainly due to the differences in the shapes of the term structures: in this example, cocoa and coffee have been on average contangoed while the two oil products have been on average backwardated.

### 5.3.4 Equity, Bond and Commodity Factors

For asset allocation purposes, it is useful to take a multi-class perspective and to compare the factors on a common time period. Recall that our equity series are daily and cover the period 1970-2013, our bond series are monthly and cover the period 1976-2013 and our commodity series are daily, start in 1986 and end in 2011. The availability of bond and commodity series determines the sample length and the frequency of the dataset in multi-class experiments: the data will be monthly and the period of study will be 1986-2011.<sup>33</sup>

For brevity, we only report results for long-only factors, which are more relevant than long-short versions for practical applications. Table 5 reports the standard risk and return indicators of the various indices over the sample (we have not included any relative indicator because there is no natural benchmark for a multi-class portfolio). Changing the sample does not materially affect the previous observations. Equity factors have the highest average returns, and bond portfolios still have impressive Sharpe ratios, due to the combination of strong returns and low volatility on the sample period. Starting the sample period in 1986 even reinforces this effect: by removing the impact of negative bond returns until the early 1980s, the overall performance of the bond portfolios is further improved, as can be seen by comparing Tables 3 and 5. This results in extremely high Sharpe ratios of 0.70 for the corporate bond portfolio and 0.67 for the Treasury index.

Correlations between L/O portfolios within an asset class are high, especially in the equity and bond classes. On the other hand, correlations across classes are lower (less than 30%). Interestingly, the equity and commodity momentum factors are positively correlated (17.76%), as are the "global stock momentum" and commodity momentum portfolios of by AMP13 (see their Table II). This is interpreted by AMP13 as the possible sign of the existence of common risk factors that drive the returns to these strategies. In unreported results, we have computed the correlations between L/S factors and have found that the equity and commodity momentum factors are still correlated (11.76%). A difference between the figures of AMP13 and those in Table 5 is that we obtain a positive correlation (17.63%) between the equity value factor and the commodity

## 5. Empirical Analysis of Factor Investing

Table 5: Summary statistics on long-only equity, bond and commodity factor indices (1986-2011).

	Avg. excess return (%)	Volatility (%)	Sharpe ratio								
Broad Indices											
Equ Broad CW	5.17	15.79	0.33								
Bnd Broad	3.24	4.87	0.67								
Cmd Broad	3.01	20.67	0.15								
Equity Factors											
Equ L/O-Siz CW	7.06	17.78	0.40								
Equ L/O-Mom CW	6.41	15.93	0.40								
Equ L/O-Vol CW	5.52	13.91	0.40								
Equ L/O-Val CW	5.49	16.40	0.33								
Bond Factors											
Bnd L/O-Cre	3.85	5.49	0.70								
Bnd L/O-Ter	5.43	9.98	0.54								
Commodity Factors											
Cmd L/O-Mom	4.98	14.58	0.34								
Cmd L/O-TSt	4.83	14.24	0.34								
Correlations (%)											
	Equ Broad CW	Bnd Broad	Cmd Broad	Equ L/O-Siz CW	Equ L/O-Mom CW	Equ L/O-Vol CW	Equ L/O-Val CW	Bnd L/O-Cre	Bnd L/O-Ter	Cmd L/O-Mom	Cmd L/O-TSt
Equ Broad CW	100.00	-0.42	13.02	92.70	96.89	93.96	95.83	28.97	0.13	15.81	18.55
Bnd Broad		100.00	-7.59	-5.10	2.84	6.14	-1.55	74.44	95.43	-3.21	-10.56
Cmd Broad			100.00	14.16	16.62	8.80	16.24	7.29	-10.12	64.99	74.56
Equ L/O-Siz CW				100.00	88.57	88.51	93.26	27.32	-3.38	19.84	21.42
Equ L/O-Mom CW					100.00	90.43	91.92	28.41	2.91	17.76	19.91
Equ L/O-Vol CW						100.00	92.40	29.85	6.25	11.26	13.86
Equ L/O-Val CW							100.00	29.50	-1.08	17.63	21.77
Bnd L/O-Cre								100.00	72.54	14.75	13.55
Bnd L/O-Ter									100.00	-5.43	-12.95
Cmd L/O-Mom										100.00	85.16
Cmd L/O-TSt											100.00

Statistics are estimated from monthly returns over the period Jan. 1986 - Aug. 2011. The broad indices are the ERI Scientific Beta Long-Term US equity index, the Barclays US Treasury index and the S&P GSCI. The four equity portfolios proxy for the size, momentum, volatility and value factors, the two bond portfolios for the term and credit factors and the two commodity portfolios for the momentum and term structure factors. Average excess return is the difference between the geometric average of the index and that of the risk-free asset (the risk-free rate is the US 3-month Treasury bill rate). Average return and volatility are annualised and the Sharpe ratio is the ratio of the two quantities.

momentum factor, while AMP13 report a negative correlation between their "global stock value" and commodity momentum portfolios (-17% in their Table II). The

discrepancy may arise because we use L/O factors, while they consider L/S portfolios. By using our L/S equity value and commodity momentum factors, we recover a negative

## 5. Empirical Analysis of Factor Investing

correlation of  $-6.67\%$ . Another noteworthy empirical finding is that the equity factor to which bond factors are most correlated is the volatility factor. This can be explained by the bond-like nature of low volatility stocks: Coqueret et al. (2014) show that these stocks tend to pay more stable and predictable dividends, which results in a lower tracking errors with respect to bonds.

### 5.3.5 Multi-Class Factors

Table 6 reports descriptive statistics for the multi-class factors and the three broad indices over the same sample period. Two usual effects can be observed here: the short legs (portfolios of past losers and growth constituents) underperform the long legs (portfolios of past winners and value constituents), both in terms of average return and Sharpe ratio, and the L/S portfolios have lower pairwise correlations amongst themselves and with the broad indices compared to the L/O ones.

The striking feature is the magnitude of the Sharpe ratios of the L/O factors, which reach 1.28 for the momentum factor and 1.22 for the value factor. The figures in Panel (a) suggest that these extremely high values are mainly due to unusually high performance: the L/O momentum and value factors earned respectively  $13.11\%$  and  $11.35\%$  per year in excess of the risk-free asset. For comparison purposes, the single-class L/O factor that performed best in our sample is the equity size factor, which earned only  $8.23\%$  per year above the risk-free rate. The high performance of the multi-class factors can be traced back to the single-class portfolios of AMP13. Consider for example the US stock value and momentum portfolios, which are value-weighted combinations of the 30% stocks with the highest BE/ME or the highest past

12-month return. According to Table I in AMP13, the value portfolio has mean excess return and Sharpe ratio of  $13.2\%$  and 0.83 over the period 1972-2011. The figures are respectively  $14.2\%$  and 0.77 for the past winner portfolio. In our dataset, the value and momentum equity portfolios have excess returns of  $7.33\%$  and  $6.53\%$ , respectively, and Sharpe ratios of 0.42 and 0.38, respectively over the period 1970-2013, which is comparable to the sample period of AMP13 (see Table 1). To summarise, our equity L/O factors have much lower excess returns and Sharpe ratios than the L/O portfolios of AMP13. When the various single-class portfolios are mixed to form the multi-class combinations, the (time series) average return is close to the cross-sectional mean of the average returns of the constituents, but the volatility is reduced by the virtue of diversification. Eventually, this generates the very high Sharpe ratios observed in Panel (a).

Two factors may explain the performance gap between the equity portfolios of AMP13 and ours. First, our equity L/O factor indices are formed on the basis of a selection of half of the universe, while in AMP13 one third of the universe is selected for each attribute. The more pronounced tilt towards a rewarded characteristic is likely to improve the performance, even though its impact on Sharpe ratio is less straightforward to predict (selecting fewer stocks mechanically reduces diversification in the naive sense, which could lower the Sharpe ratio). The second explanation is that our equity portfolios are marketed equity indices, which are subject to a number of liquidity and turnover constraints aiming to facilitate their replication. These constraints may adversely impact the gross performances (i.e. performances before transaction costs) that we measure here.

## 5. Empirical Analysis of Factor Investing

Table 6: Summary statistics on multi-class factor indices (1976–2011).

(a) L/O factors.

	Avg. excess return (%)	Volatility (%)	Sharpe ratio
Equ Broad CW	5.18	15.20	0.34
Bnd Broad	2.49	5.50	0.45
Cmd Broad	2.03	19.19	0.11
Mul L/O-Mom	13.11	10.24	1.28
Mul L/O-Val	11.35	9.32	1.22
Mul L/O-Los	6.30	9.19	0.69
Mul L/O-Gro	6.91	10.29	0.67

Correlations (%)

	Equ Broad CW	Bnd Broad	Cmd Broad	Mul L/O-Mom	Mul L/O-Val	Mul L/O-Los	Mul L/O-Gro
Equ Broad CW	100.00	12.48	14.85	61.43	68.59	66.64	66.80
Bnd Broad		100.00	-5.96	17.98	14.96	18.78	18.25
Cmd Broad			100.00	39.30	35.19	32.02	34.92
Mul L/O-Mom				100.00	69.17	72.48	83.76
Mul L/O-Val					100.00	82.99	69.13
Mul L/O-Los						100.00	72.47
Mul L/O-Gro							100.00

(b) L/S factors.

	Avg. excess return (%)	Volatility (%)	Sharpe ratio
Mul L/S-Mom	6.27	7.42	0.85
Mul L/S-Val	3.75	7.92	0.47

Correlations (%)

	Equ Broad CW	Bnd Broad	Cmd Broad	Mul L/S-Mom	Mul L/S-Val
Equ Broad CW	100.00	12.48	14.85	2.49	-5.76
Bnd Broad		100.00	-5.96	3.00	-4.73
Cmd Broad			100.00	14.64	-3.87
Mul L/S-Mom				100.00	-38.14
Mul L/S-Val					100.00

Statistics are estimated from monthly returns over the period Jan. 1976 – Jul. 2011. The broad equity index is the ERI Scientific Beta Long-Term US capitalisation-weighted index, the broad bond index is the Barclays US Treasury index and the broad commodity index is the S&P GSCI. Each of the four L/O multi-class portfolios aggregates four equity portfolios, one bond portfolio and one commodity futures portfolio. Each of the single-class portfolio is tilted towards past winners (L/O-Mom), value constituents (L/O-Val), past losers (L/O-Los) or growth constituents (L/O-Gro). Each L/S multi-class portfolios is constructed by taking the difference between the returns to two L/O portfolios. Average excess return is the difference between the geometric average of the index and that of the risk-free asset (the risk-free rate is the US 3-month Treasury bill rate). Average return and volatility are annualised and the Sharpe ratio is the ratio of the two quantities. The tracking error and the information ratio are computed with respect to the broad commodity index.

Overall, the presence of transaction costs, capacity constraints and other forms of frictions is expected to have in implementation a strong negative impact on the reported performance of the multi-class factor portfolios.

## 5. Empirical Analysis of Factor Investing

### 5.4 Equity Factors in Allocation Decisions

In this section, we measure the value added by equity factor indices.

#### 5.4.1 In-Sample Tests

The in-sample tests are mean-variance spanning tests, which focus on the following question: does the introduction of equity factor indices in a universe that already contains traditional equity indices improve the efficient frontier? The test methodology is described in Section 5.1.1, and we recall that the spanning test is equivalent to a joint test of the equality of minimum variances and maximum risk-return ratios before and after the extension. Thus, we report these statistics before we perform the formal equality test. Note that when the original universe contains only the broad CW index, the mean-variance spanning test is merely a test of efficiency of this portfolio within a broader universe that also contains the factor indices.

#### *Impact of Universe Extension on the Efficient Frontier*

We look separately at L/O and L/S factors in Panels (a) and (b) of Figures 3. The absolute correlation is the canonical correlation given by Proposition 7, and the minimum volatility, the maximum risk-return ratio and the maximum Sharpe ratio are computed from the sample moments of the constituents

as:

$$\begin{aligned}\hat{\sigma}_{GMV,j} &= \frac{h}{\mathbf{1}'\hat{\Sigma}^{-1}\mathbf{1}}, \\ \hat{\zeta}_{MRR,j} &= \sqrt{h}\sqrt{\hat{\boldsymbol{\mu}}'\hat{\Sigma}^{-1}\hat{\boldsymbol{\mu}}}, \\ \hat{\lambda}_{MSR,j} &= \sqrt{h}\sqrt{\hat{\boldsymbol{\mu}}'\hat{\Sigma}^{-1}\hat{\boldsymbol{\mu}}}, \\ j &= 0, 1.\end{aligned}\quad (5.4)$$

In these formulas, the subscript 0 refers to the original universe, which contains only the broad CW, 1 refers to the extended universe, which also contains one or four factor index (indices), and  $\mathbf{1}$  is a conforming vector of ones. Hats denote quantities estimated over the period 1970–2013, and  $h$  is an annualisation factor, equal to the number of observations in a year. Following KZ12, we use monthly returns to estimate the covariance matrix and the expected returns in the mean-variance spanning tests, so  $h$  equals 12. As a consequence, the correlations are not strictly identical to those shown in Table 1. Nevertheless, similar features are recovered, including the fact that L/O factors are much more correlated (above 90%) with the broad CW than the L/S ones.

As expected, L/S factors appear to be much more effective than their L/O equivalents to reduce risk: one reduces the minimum volatility from 15.09% (the broad CW volatility) to 13.24% by introducing the four L/O factors, and to less than 4% with the four L/S indices. Even with a single L/S factor, a substantial decrease in volatility is achieved. But the GMV portfolio of L/O factors contains sizable short positions, as can be seen by computing the compositions of the in-sample efficient portfolios:

$$\begin{aligned}\hat{\mathbf{x}}_{GMV,j} &= \frac{\hat{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\Sigma}^{-1}\mathbf{1}}, \\ \hat{\mathbf{x}}_{MRR,j} &= \frac{\hat{\Sigma}^{-1}\hat{\boldsymbol{\mu}}}{\mathbf{1}'\hat{\Sigma}^{-1}\hat{\boldsymbol{\mu}}}, \\ \hat{\mathbf{x}}_{MSR,j} &= \frac{\hat{\Sigma}^{-1}\hat{\boldsymbol{\mu}}}{\mathbf{1}'\hat{\Sigma}^{-1}\hat{\boldsymbol{\mu}}}, \\ j &= 0, 1.\end{aligned}$$

## 5. Empirical Analysis of Factor Investing

In detail, we have, respectively with L/O and L/S factors:

$$\hat{x}_{GMV,1,L/O} = \begin{pmatrix} -0.40 & \text{Siz} \\ 0.17 & \text{Mom} \\ 1.54 & \text{Vol} \\ 0.25 & \text{Val} \\ -0.56 & \text{Broad} \end{pmatrix},$$

$$\hat{x}_{GMV,1,L/S} = \begin{pmatrix} 0.24 & \text{Siz} \\ 0.20 & \text{Mom} \\ 0.24 & \text{Vol} \\ 0.17 & \text{Val} \\ 0.15 & \text{Broad} \end{pmatrix}.$$

The negative weights in the former portfolio are due to the high correlations between the constituents. On the other hand, the GMV generated from L/S factors has only nonnegative weights, and is well diversified from a naive sense, due to the moderate correlation levels. Because the long-short GMV portfolio of L/O factors involves negative weights, the volatility gain reported in Panel (a) is already an overestimation for what could be achieved by an investor subject to long-only constraints. For comparison purposes, Figure 4 shows the various risk and return indicators computed under long-only

constraints in the constituents. To obtain these numbers, we numerically solve the following programs in each universe:

$$\min_{x \geq 0} x' \Sigma x, \quad \max_{x \geq 0} \frac{x' \mu}{\sqrt{x' \Sigma x}},$$

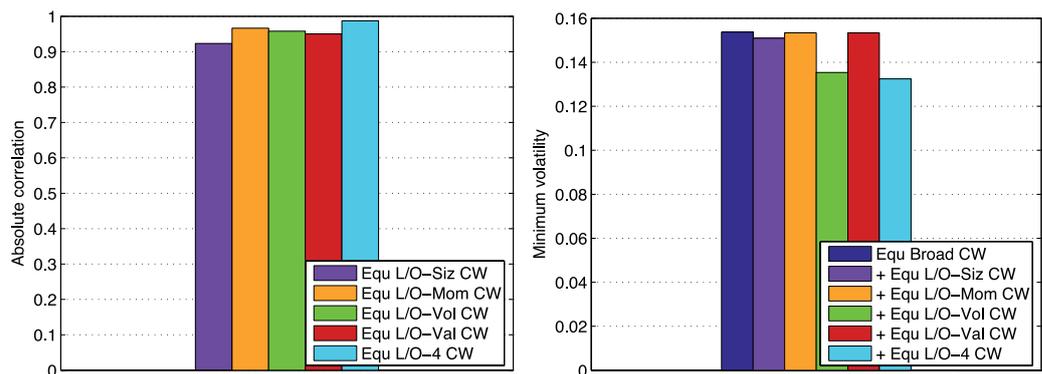
$$\max_{x \geq 0} \frac{x' \tilde{\mu}}{\sqrt{x' \Sigma x}}.$$

Interestingly, imposing long-only constraints does not imply a large increase in risk: the minimum volatility in the largest universe grows from 13.24% to 13.82% (see Figure 4). As a conclusion, the increase in volatility to be expected from the imposition of long-only constraints in the allocation is zero with L/S factors and is very small with L/O factors.

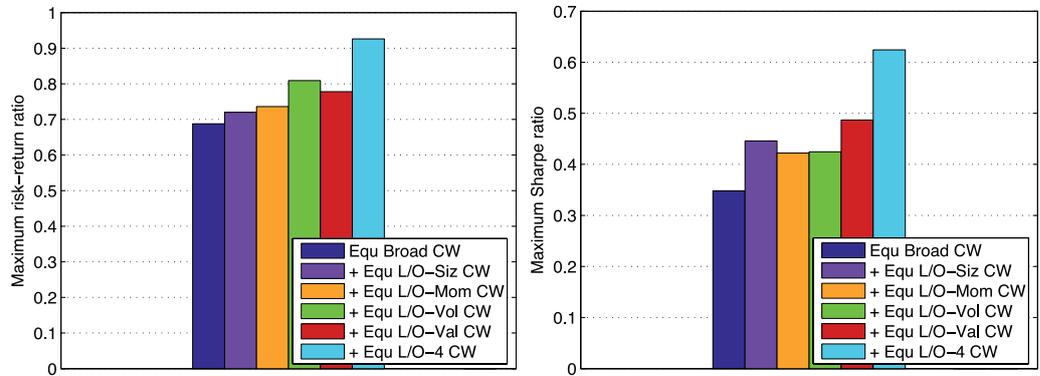
The risk-return benefits of the factors can be assessed through the gain in maximum risk-return ratio or in the maximum Sharpe ratio. The two indicators are different because the average risk-free rate over the sample is 5.35% per year (they would be equal if the risk-free rate had been zero in the sample). As a general rule, the L/S factors lead to higher ratios of either type. With L/O factors, the gains are visible, but they are obtained at the cost of large short positions, and the opportunity costs of

Figure 3: Impact on risk and return indicators of universe extension with equity factors (1970-2013).

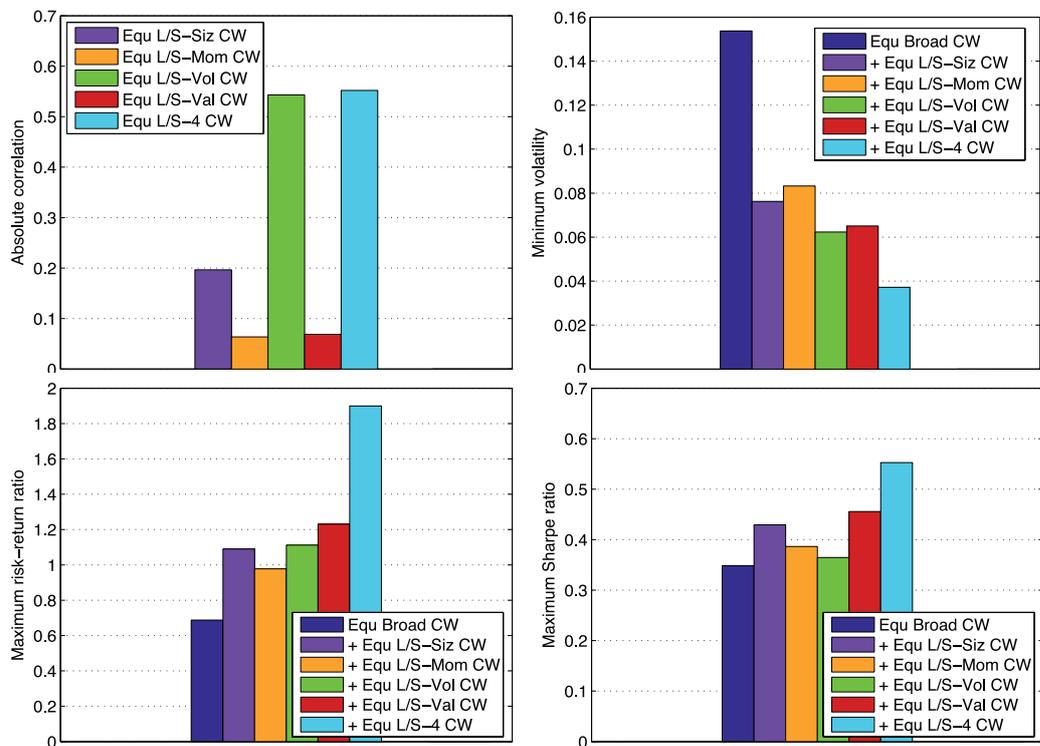
(a) L/O factors.



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(b) L/S factors.



The broad equity index is the ERI Scientific Beta Long-Term US capitalisation-weighted (CW) index and the four equity factor indices proxy for the size, momentum, volatility and value factors. The absolute correlation is the absolute value of the correlation between the broad CW equity index and a factor index, or the maximum absolute value of the correlation between the broad CW and a portfolio of the four factor indices. The minimum volatility, the maximum risk-return ratio and the maximum Sharpe ratio are computed over a universe that contains only the broad CW, or the broad CW plus one factor index, or the broad CW plus the four indices. All long-only factor indices, as well as the long and short legs of the long-short factors, are weighted by capitalisation. The statistics are computed from monthly returns over the whole sample (Jun. 1970 - Dec. 2013) and are annualised.

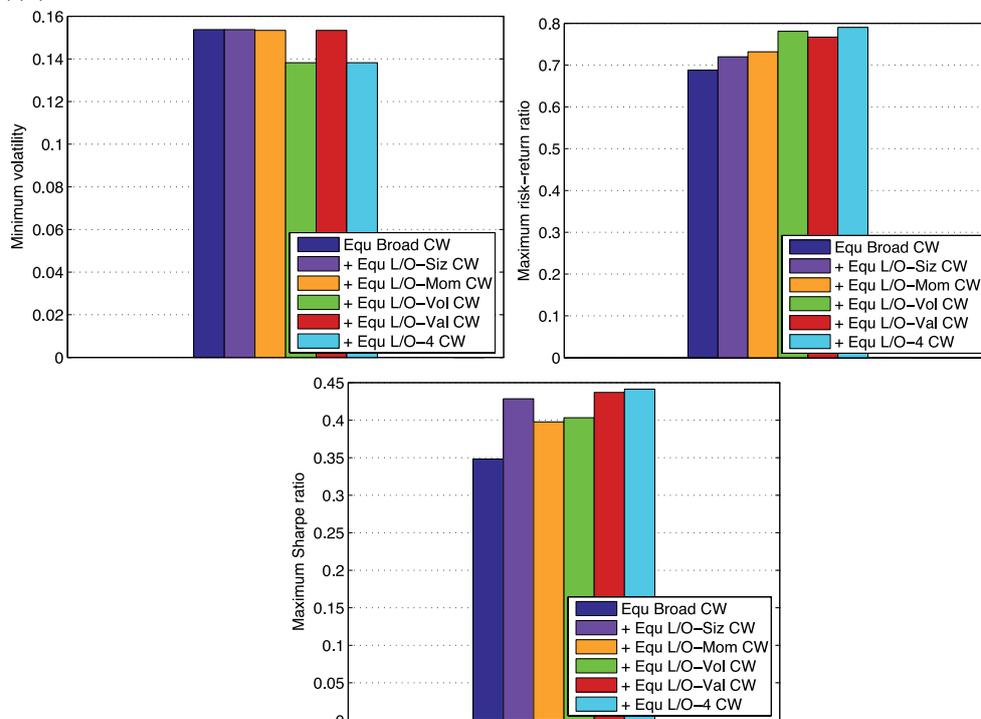
imposing long-only constraints are higher here than for the minimum volatility: the maximum risk-return ratio and Sharpe ratio are 0.93 and 0.62 respectively if short sales are permitted, and 0.79 and 0.44 if they are prohibited. With L/S factors, the lower correlations enable to preserve

long-only positions and to conciliate scientific diversification (i.e. optimisation of a risk or return indicator) and naive diversification (i.e. balanced dollar allocation to constituents). In particular, short sales constraints are not binding in the long-only efficient portfolios, and the statistics

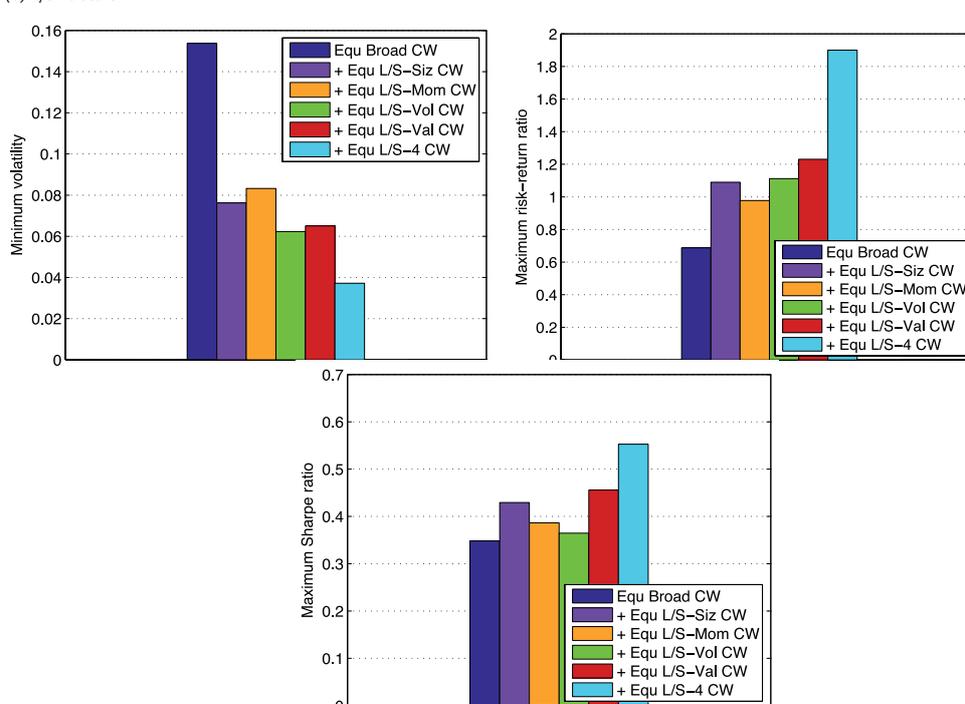
## 5. Empirical Analysis of Factor Investing

Figure 4: Impact on risk and return indicators of universe extension with equity factors when short sales are prohibited (1970-2013).

(a) L/O factors.



(b) L/S factors.



The broad equity index is the ERI Scientific Beta Long-Term US capitalisation-weighted (CW) index and the four equity factor indices proxy for the size, momentum, volatility and value factors. The minimum volatility, the maximum risk-return ratio and the maximum Sharpe ratio are computed over a universe that contains only the broad CW, or the broad CW plus one factor index, or the broad CW plus the four indices. All long-only factor indices, as well as the long and short legs of the long-short factors, are weighted by capitalisation. The minimum variance, maximum risk-return ratio and maximum Sharpe ratio portfolios are subject to long-only constraints. The statistics are computed from monthly returns over the whole sample (Jun. 1970 - Dec. 2013) and are annualised.

## 5. Empirical Analysis of Factor Investing

computed with such constraints are strictly equal to those obtained by relaxing them (Figures 3 and 4).

### Mean-Variance Spanning Tests

Following KZ12, we perform three tests to test for the statistical significance of the changes in the risk and return indicators:

- A joint test of the null hypothesis that  $\hat{\sigma}_{GMV,1} = \hat{\sigma}_{GMV,0}$  and  $\hat{\zeta}_{MRR,1} = \hat{\zeta}_{MRR,0}$ , i.e. a spanning test;
- A test of the null hypothesis that  $\hat{\zeta}_{MRR,1} = \hat{\zeta}_{MRR,0}$ , which is a test of equality of the two MRR portfolios;
- A test of the null hypothesis that  $\hat{\sigma}_{GMV,1} = \hat{\sigma}_{GMV,0}$  given that  $\hat{\zeta}_{MRR,1} = \hat{\zeta}_{MRR,0}$ , which is a test of equality of the two GMVs given that the two MRRs coincide.

The test statistics are reported in Table 7, together with the p-values obtained from the small-sample distributions (see Appendix C for the detailed expressions of the statistics and the distributions).

For L/S factors, the F-statistics are exceedingly large, so that the p-values are infinitesimal and the null hypothesis of spanning is rejected for each set of factor(s) at all confidence levels. The fact that the statistics are so high is largely due to the large sample size: we have  $T = 523$ , which corresponds to a bit more than 43 years of monthly observations. With so many data points and a low number of constituents (5 in the largest universe), the number of degrees of freedom is large and it only takes a small difference between volatilities or risk-return ratios for the test to detect a statistically significant change. It is therefore important to assess the economic significance of the results, e.g. by examining the effects of factor introduction in the context of an allocation exercise as in Section 5.4.2 below.

With L/O factors, the statistics are lower, and they are comparable in magnitude to those reported by KZ12 in the context of a broadly

Table 7. Mean-variance spanning tests with equity factors (1970-2013).

Universe	Spanning tests		Tests on MRR		Tests on GMV	
	F-stat	p-value	F-stat	p-value	F-stat	p-value
L/O Factors						
+ L/O-Siz CW	12.48	0.00	1.88	0.17	23.04	0.00
+ L/O-Mom CW	2.22	0.11	2.86	0.09	1.57	0.21
+ L/O-Vol CW	75.45	0.00	7.56	0.01	141.56	0.00
+ L/O-Val CW	3.39	0.03	5.51	0.02	1.25	0.26
+ L/O-4 CW	22.24	0.00	3.99	0.00	43.07	0.00
L/S Factors						
+ L/S-Siz CW	799.70	0.00	29.83	0.00	1487.30	0.00
+ L/S-Mom CW	629.58	0.00	20.11	0.00	1195.20	0.00
+ L/S-Vol CW	1332.80	0.00	31.83	0.00	2486.70	0.00
+ L/S-Val CW	1195.30	0.00	43.44	0.00	2170.40	0.00
+ L/S-4 CW	407.15	0.00	32.50	0.00	1652.80	0.00

The original investment universe contains the broad CW equity index, and it is extended with one or four equity factor index/indices. The factors are size, momentum, volatility and value. The long-only factors and the long and short legs of the long-short factors are weighted by capitalisation. For each universe extension, three tests are performed: (1) a spanning test, which is a joint test of the equality of the global minimum variance (GMV) and maximum risk-return ratio (MRR) portfolios before and after the extension; (2) a test of equality of the two MRR portfolios; (3) a test of equality of the two GMV portfolios conditional on the equality of the two MRRs. The table reports the F-statistic of each test as well as the p-value derived from the small-sample distribution of the test statistic.

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similar test (the extension of a stock-bond universe with equity indices and a slightly smaller sample (1970-2007)). Because the statistics are lower, it is harder to reject the null hypothesis. For size, volatility and the four-factor universe, the null of spanning is rejected at the 1% confidence level, but the stepdown test shows that for size, rejection is determined only by the benefits in the GMV: the increase in the MRR is so low (see Panel (a) of Figure 3) that it appears as insignificant. In other words, the introduction of the L/O size factor significantly reduces the minimum volatility but it does not have a decisive effect on the MRR. For the value factor, the benefits are dubious in the GMV, but evidence in favour of an improvement in the MRR is more convincing: overall, the null of spanning is rejected at the 5% level, but not at the 1% level. For the momentum factor, it cannot even be rejected at the 10% level, because the benefits of the introduction of this factor on the GMV and the MRR are too uncertain.

As a conclusion, the mean-variance tests unambiguously reject the null of spanning for all universe extensions with L/S factors. With L/O factors, the null is not as easily rejected, and the empirical evidence presented here is inconclusive about the added value of the momentum factor. For the other L/O factors, there are enough benefits in terms of at least one criterion (volatility reduction or improvement in MRR) to ensure rejection at the 5% level.

### 5.4.2 Out-of-Sample Tests

The in-sample tests suggest that equity factors do add value, but their results may be "too good to be true". Indeed, they are based on a comparison of in-sample efficient frontiers, which depend on the full-sample moments and cannot be known ex-ante. Moreover, they assume that short

positions in the constituents are allowed, while many investors are restricted to long-only positions. Thus, the benefits of factors in long-only portfolios without a look-ahead bias remain to be assessed. This is the focus of this section.

### Extension Tests

As in the in-sample tests, an initial universe consisting of the broad CW index is extended with one factor or the four factors. Thus, all universes except the original universe have more than one constituent, so one has to specify how the multiple constituents are weighted. In what follows, we consider four allocation methods to the various indices, which have been presented in detail in Section 3.1: the MDC portfolio (which is an equally-weighted portfolio here since no turnover constraints are imposed), the RP portfolio, the GMV portfolio and the MENCH portfolio. Table 8 reports several performance and risk indicators for these strategies, together with their one-way annual turnover. At each rebalancing date  $t$ , the one-way turnover is computed as:

$$\Theta_t = \frac{1}{2} \sum_{i=1}^p |x_{it} - x_{i,t-}|,$$

where  $x_{it}$  is the weight of constituent  $i$  imposed on date  $t$  and  $x_{i,t-}$  is the effective weight just before the rebalancing. The annual turnover is the average of the  $\Theta_t$  multiplied by 4 to convert it to an annual quantity.

For all weighting schemes and universe extensions, portfolios that contain one or more factors outperform the broad CW. They also have in general lower volatilities, but there are a few exceptions: in MDC and RP portfolios, the introduction of the value factor slightly increases volatility. But even in those cases, the extended portfolios

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Table 8: Extension of universe with capitalisation-weighted equity factor indices (1972–2013).

Universe	Avg. return (%)	Volatility (%)	Sharpe ratio	Tracking error (%)	Information ratio	Ann. 1-way Turnover (%)
Broad CW	10.41	17.26	0.29	0.00	-	0.0
MDC Portfolios						
+ L/O-Siz CW	11.80 ***	17.19	0.37 ***	2.97	0.47	3.1
+ L/O-Mom CW	10.94	17.25	0.32	1.73	0.31	1.6
+ L/O-Vol CW	10.58	16.35 ***	0.32	2.20	0.08	1.5
+ L/O-Val CW	11.51 **	17.38 **	0.35 **	2.35	0.47	2.1
+ L/O-4 CW	11.73 ***	16.78 ***	0.38 ***	2.11	0.63	3.3
GMV Portfolios						
+ L/O-Siz CW	11.60 **	16.90 ***	0.37 **	2.87	0.41	13.3
+ L/O-Mom CW	10.53	17.05 ***	0.30	1.84	0.06	25.6
+ L/O-Vol CW	10.66	15.90 ***	0.33	3.63	0.07	3.7
+ L/O-Val CW	11.60 **	17.00 ***	0.37 ***	2.14	0.56	16.8
+ L/O-4 CW	11.20	15.95 ***	0.37 *	3.22	0.25	26.1
RP Portfolios						
+ L/O-Siz CW	11.79 ***	17.15 **	0.37 ***	2.89	0.48	4.0
+ L/O-Mom CW	10.92	17.23	0.32	1.70	0.30	2.3
+ L/O-Vol CW	10.60	16.31 ***	0.32	2.34	0.08	1.9
+ L/O-Val CW	11.54 ***	17.34	0.36 **	2.31	0.49	2.5
+ L/O-4 CW	11.74 ***	16.68 ***	0.38 ***	2.17	0.61	4.3
MENUM Portfolios						
+ L/O-Siz CW	11.52 **	16.82 ***	0.37 ***	2.56	0.44	16.5
+ L/O-Mom CW	10.53	17.05 ***	0.30	1.66	0.07	25.1
+ L/O-Vol CW	10.52	15.94 ***	0.32	3.30	0.03	13.6
+ L/O-Val CW	11.74 ***	16.91 ***	0.38 ***	2.12	0.63	15.9
+ L/O-4 CW	11.45	15.82 ***	0.39 ***	3.16	0.33	31.2

The benchmark portfolio is the broad equity capitalisation-weighted (CW) index. The other portfolios contain the broad index plus one equity factor only (size, momentum, volatility, value), or the four factors. All factors are long-only and weighted by capitalisation. The constituents are weighted according to the following schemes, rebalanced every quarter: MDC is the maximum deconcentration portfolio; GMV is the global minimum variance subject to bounds on weights (1/12 and 3/4); RP is the risk parity, which equalises the contributions to volatility; MENUM is the maximum ENUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix. The tracking error and the information ratio are computed with respect to the broad index. The stars in columns 2 to 4 indicate the significance level in an equality test with respect to the broad CW: no star = not significant at the 10% level; \* = 10% level; \*\* = 5% level; \*\*\* = 1% level.

have higher Sharpe ratios. The volatility reduction is almost always significant at the 1% level, but given the greater uncertainty over expected returns, gains in performance and Sharpe ratio are less easy to distinguish. Nevertheless, most of them are significant at the 5% level at least. Thus, it appears that factor indices still add value in a more realistic out-of-sample long-only context.

Which factor or combination of factors has the most positive impact on the universe depends on the indicator used to compare the portfolios. If the ex-post average return is used, the "best" factor is, unsurprisingly, the one that performs best in the sample, that is the size factor (see Table 1). With respect to ex-post volatility, it is the volatility factor, that brings the largest reduction. This result is natural since Vol is the least volatile factor, and it is in line

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with the in-sample results, where Vol is the factor that most reduces the minimum volatility (see Figure 3). Finally, if the ex-post Sharpe ratio is taken as the reference indicator, the strongest gains are achieved by adding the four equity factors. This finding confirms the usefulness of combining factors, which has been put forward in other recent studies: Amenc et al. (2014b) show that mixing the equity factors presented here enables to smooth performance across market conditions, and Asness et al. (2013) report that a simple 50%-50% combination of value and momentum factors has greater long-term performance than both constituents. The previous results are robust to the choice of the allocation method since they hold for the four tested weighting schemes.

On the other hand, the momentum factor is the one that appears to have the most limited impact on the universe. This can be seen by comparing the average return, the volatility and the Sharpe ratio of the portfolios containing the broad CW and the momentum factor to those of the broad CW, or, more directly, by noting that the former portfolios have the lowest tracking errors. This result is in line with previous observations from Table 1 and Figure 3, where it turns out that Mom is the factor that is most correlated with the broad index.

Comparing the weighting schemes, we note that the MDC and the RP have very close analytics. This is because the L/O constituents have relatively uniform volatilities and pairwise correlations (see Table 1). The property of uniform correlations implies that the RP portfolio coincides with the inverse volatility-weighted portfolio (Maillard et al., 2010). If, in addition, volatilities are equal, then the RP portfolio is equally weighted. The MDC and the RP are also the two schemes

with the lowest turnovers. This is a classical property of the MDC, and the RP inherits it because it is close to the MDC in this particular context. The GMV and the MENUB portfolios have much higher turnovers, that can reach 30%. The exception is the GMV portfolio of the broad CW and the volatility factor, whose average turnover is only 3.7%. Indeed, the volatility factor, which consists of low risk stocks, is less volatile than the broad index, so that it dominates the GMV allocation. As a result, this particular GMV portfolio has a more stable composition than the other GMV portfolios.

### *Substitution Tests*

Unlike the mean-variance spanning tests, which focus by construction on universe extensions, the tests of equality between analytics enable to consider non-nested universes. Thus, we now consider another situation, where factors are used as substitutes for the broad CW, as opposed to being treated as additional building blocks. Table 9 shows the analytics of portfolios containing either one factor only or four factors. In the latter case, we add three "relative" weighting schemes (see Section 3.1 for details): the MTE, relative RP and maximum ENRUB. These schemes are not relevant when the broad CW is part of the universe because the relative covariance matrix is then singular and the minimum tracking error is zero, but they lead to non-trivial allocations here.

The main result is easy to summarise: all factor portfolios outperform the broad CW in terms of average return, volatility and Sharpe ratio. When the 4-factor "substituted portfolios" are compared to the "extended portfolios" containing the 4 factors plus the broad CW, two natural effects stand out. First, removing the broad CW from the universe has a positive impact on performance, which was to be expected

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Table 9: Capitalisation-weighted equity factor indices as substitutes for a broad equity index (1972-2013)

(a) Performance and risk indicators

Universe	Avg. return (%)	Volatility (%)	Sharpe ratio	Tracking error (%)	Information ratio
Broad CW	10.41	17.26	0.29	0.00	-
Single-factor portfolios					
L/O-Siz CW	13.01 **	17.64 ***	0.43 **	5.92	0.44
L/O-Mom CW	11.43	17.41 *	0.35	3.46	0.30
L/O-Vol CW	10.68	15.71 ***	0.34	4.38	0.06
L/O-Val CW	12.53 **	17.82 ***	0.40 **	4.70	0.45
Absolute weighting schemes					
MDC L/O-4 CW	12.05 ***	16.70 ***	0.40 ***	2.63	0.62
GMV L/O-4 CW	11.10	15.88 ***	0.36	3.51	0.20
RP L/O-4 CW	12.04 ***	16.58 ***	0.40 ***	2.67	0.61
MENUB L/O-4 CW	11.61	15.72 ***	0.40 ***	3.55	0.34
Relative weighting schemes					
MTE L/O-4 CW	11.39 **	16.62 ***	0.36 ***	1.95	0.50
RRP L/O-4 CW	11.41 **	16.66 ***	0.36 ***	1.94	0.51
MENRUB L/O-4 CW	11.45 **	16.73 ***	0.36 ***	2.06	0.50

(b) Analytics on weights

Universe	Avg. ENC (%p)	Avg. ENCB (%p)	Avg. ENUB (%p)	Avg. ENRCB (%p)	Avg. ENRUB (%p)	Ann. 1-way Turnover (%)
Absolute weighting schemes						
MDC L/O-4 CW	100.0	99.1	92.7	57.2	68.9	3.7
GMV L/O-4 CW	49.5	52.0	96.5	30.2	39.4	28.9
RP L/O-4 CW	99.1	100.0	93.9	59.9	70.7	4.8
MENUB L/O-4 CW	57.0	58.7	99.6	44.0	53.3	26.9
Relative weighting schemes						
MTE L/O-4 CW	69.2	70.5	91.5	78.6	69.3	15.3
RRP L/O 4-CW	79.4	80.5	92.1	100.0	86.0	11.3
MENRUB L/O-4 CW	79.9	80.7	91.9	75.6	99.6	18.0

The benchmark portfolio is the broad equity capitalisation-weighted (CW) index. The other portfolios contain either one equity factor only (size, momentum, volatility, value), or the four factors. All factors are long-only and weighted by capitalisation. In 4-factor universes, the constituents are weighted according to the following schemes, rebalanced every quarter: MDC is the maximum deconcentration portfolio; GMV is the global minimum variance subject to bounds on weights (1/12 and 3/4); RP is the risk parity, which equalises the contributions to volatility; MENUB is the maximum ENUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix. The last three weighting schemes focus on relative risk: MTE minimises the tracking error subject to bounds on weights; RRP is the relative risk parity, which equalises the contributions to the tracking error; MENRUB is the maximum ENRUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix of excess returns. In Panel (a), the tracking error and the information ratio are computed with respect to the broad CW. The stars in columns 2 to 4 indicate the significance level in an equality test with respect to the broad index: no star = not significant at the 10% level; \* = 10% level; \*\* = 5% level; \*\*\* = 1% level. In Panel (b), ENC is the effective number of constituents; ENCB is the effective number of correlated bets (effective number of constituent contributions to volatility); ENUB is the effective number of uncorrelated bets; ENRCB is the effective number of relative correlated bets (effective number of constituent contributions to tracking error); ENRUB is the effective number of relative uncorrelated bets (the factors being extracted from the relative covariance matrix). All these indicators are computed at each quarterly rebalancing date and then averaged. They are expressed as percentages of the universe size,  $p = 4$ . The last column reports the annual one-way turnover.

because this index underperforms each of the factors. The impact on the Sharpe ratio is also slightly positive, except for the GMV. Second, tracking errors are higher

without the broad CW. This could have a negative impact on the information ratio, but interestingly, the increase in average return compensates the contribution of

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the tracking error, and overall, the 4-factor substituted portfolios have roughly the same information ratios as their extended counterparts. The relatively high tracking errors of the GMV and the MENUB portfolios, which are above 3.5%, may be a concern for equity managers who are assigned tighter relative risk budgets. The "relative" weighting schemes have an attractive property in this perspective: they have lower tracking errors than their "absolute" counterparts, and they also have higher information ratios, the only exception being the relative RP which has a lower information ratio compared to the RP portfolio in our sample.

The advantages of combining factors are less clear if one focuses on absolute return and risk indicators: the 4-factor portfolios do not have higher Sharpe ratios than the size factor – which is the best from this perspective in the sample – or lower volatility than the volatility factor. But relative indicators give a more positive picture. First, the 4-factor portfolios have lower tracking errors than their constituents. Indeed, each factor is a portfolio of stocks selected from the broad universe, and these selections are non overlapping. Hence, their mixture better spans the universe than each selection taken in isolation. Second, the 4-factor portfolios, except for the GMV and the MENUB, have higher information ratios than the constituents taken in isolation.

Panel (b) displays detailed information on portfolio weights. The first five columns of numbers are effective numbers of constituents or contributions, which allows us to verify that the weighting schemes achieve their respective goals. They are computed at each rebalancing date (i.e. each quarter) and then averaged over the dates, and are expressed as percentages of the nominal number of constituents

or factors ( $p = 4$  here), so that they can be interpreted as deconcentration ratios. The ENC, defined in (3.1), measures diversification in terms of dollars, and is maximal for the MDC portfolio with equal weights. The ENCB is the effective number of correlated bets, as defined in (3.3). By definition, it is maximal for the RP portfolio, in which all assets have the same contribution to volatility. The relative counterpart of this indicator is the ENRCB, where contributions to tracking error are considered. The ENUB and the ENRUB are effective numbers of uncorrelated bets and they measure the effective number of contributions from the orthogonal factors, extracted either from the covariance matrix or the relative covariance matrix. They are maximised by the MENUB and the MENRUB, respectively, but these portfolios do not exactly achieve a deconcentration of 100% in the factors, which can be explained by the presence of long-only constraints on constituent weights. Were short sales allowed, the FRP would be:

$$w_{FRP} = \frac{A\sigma_f^{-1}}{\mathbf{1}'_K A\sigma_f^{-1}}, \quad (5.5)$$

where  $\sigma_f^{-1}$  is the vector of reciprocal of factor volatilities. Then, the vector of factor contributions to portfolio volatility would be  $c_f = \mathbf{1}_K/K$ , that is, all contributions would be equal. But given the short sales constraints, the portfolio (5.5) is not attainable at all dates, and the effective number of uncorrelated bets is sometimes lower than 100%. With respect to most indicators, the GMV turns out to be the most concentrated of all tested portfolios: the effective numbers of dollar and volatility contributions are approximately equal to one half of the nominal number of constituents, and the effective number of contributions to tracking error is only 30.2% of the nominal number. Its relative

## 5. Empirical Analysis of Factor Investing

equivalent, the MTE portfolio, is also concentrated since it has the lowest deconcentration ratios among the three relative allocation schemes. Considering the turnover, we find that the MDC and the RP portfolios have the most stable compositions, while the GMV and MENUB involve additional rebalancing. The relative schemes are located between these two extremes, with turnover levels between 10% and 20%.

### 5.4.3 The Benefits of Smart-Weighted Equity Factors

A factor index is primarily defined by a selection of stocks with a rewarded attribute. Once a set of constituents has been defined, one may wonder whether the use of improved weighting schemes may lead to enhanced risk-adjusted performance. In particular, it has been shown that so-called "smart" weighting schemes (that is weighting schemes aiming at achieving a better diversified portfolio compared to a scheme weighting constituents as a proportion to their market capitalisation) lead to higher Sharpe ratios than capitalisation weighting thanks to a better diversification of non-rewarded risks (Amenc et al., 2014b). From a theoretical perspective, the interest of this substitution is the following: an increase in the Sharpe

ratios of the factor indices is expected to improve the maximum Sharpe ratio that can be achieved by combining these indices, and this results in a lower loss of efficiency in the two-step process. In this section, we test the three weighting schemes introduced in Section 5.2.1.

Note that for a given attribute (size, momentum, volatility or value), the four versions of the factor index combine stocks from the same universe, which is a half the size of the entire universe. They only differ through the weighting scheme employed. Table 10 shows two unambiguous effects. First, for a given stock selection, the use of a smart weighting scheme as opposed to capitalisation weighting leads to an increase in average return and a decrease in volatility, which results in a substantial increase in Sharpe ratio. Since the universe is fixed, the higher performance may be entirely explained by the weighting scheme, as opposed to a different factor exposure. Second, for a given weighting scheme (CW or smart weighting), selecting those stocks that have a rewarded characteristic generates gains in the three indicators with respect to a broad universe. This result extends to smart weighting schemes the observation already made for capitalisation weighting in Table 1. Overall, the two effects

Table 10: Descriptive statistics on smart-weighted equity factor indices (1970-2013).

	Avg. excess return (%)				Volatility (%)				Sharpe ratio			
	CW	MD	mV	MS	CW	MD	mV	MS	CW	MD	mV	MS
Broad	5.58	7.79	7.84	7.90	17.00	17.03	14.34	15.86	0.33	0.46	0.55	0.50
L/O-Siz	8.23	9.15	10.15	9.77	17.40	17.78	14.81	16.33	0.47	0.51	0.69	0.60
L/O-Mom	6.53	8.82	8.84	8.86	17.14	16.88	14.39	15.88	0.38	0.52	0.61	0.56
L/O-Vol	5.84	8.15	8.15	8.22	15.48	14.71	13.19	14.00	0.38	0.55	0.62	0.59
L/O-Val	7.33	9.71	9.14	9.79	17.53	17.31	14.83	16.15	0.42	0.56	0.62	0.61

Statistics are estimated from daily returns over the period Jun. 1970 - Dec. 2013 for a broad equity index (which contains all stocks from the ERI Scientific Beta Long-Term US universe) and four equity portfolios proxying for the size, momentum, volatility and value factors. Four weighting schemes for the stocks in the equity portfolios are considered: capitalisation weighting (CW), equal weighting (MD), volatility minimisation (mV) and multi strategy (MS). The multi strategy scheme is an equally-weighted combination of five weighting schemes: maximum deconcentration, minimum volatility, maximum decorrelation, inverse volatility and maximum Sharpe ratio. Excess returns are computed with respect to the US 3-month Treasury bill secondary market rate.

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show that both the use of suitable stock selection procedures and suitable weighting schemes can improve the risk-return characteristics of an index (see Amenc et al. (2012)). For brevity, we do not show the correlation matrices for the three smart schemes, which are not sensibly different from the matrix obtained with CW factors (see Table 1).

As can be seen from Table 11, all portfolios where factors are smart-weighted display better absolute risk and return indicators than the portfolios where factors are

capitalisation-weighted. This result holds regardless of how the broad CW and the factors are combined. Moreover, all gains in these three indicators are significant at the 5% level, most of them being actually significant at the 1% level. Table 8 already showed that the introduction of CW factors significantly improves the broad index, so the use of smart-weighted factors represents a further improvement. As far as the relative indicators are concerned, the tracking error increases as expected when the CW factors are replaced by their smart-weighted counterparts. The

Table 11: Extension of universe with smart-weighted equity factor indices (1972-2013).

Universe	Avg. return (%)	Volatility (%)	Sharpe ratio	Tracking error (%)	Information ratio	Ann. 1-way Turnover (%)
MDC Portfolios						
+ L/O-4 CW	11.73	16.78	0.38	2.11	0.63	3.3
+ L/O-4 MD	13.35 ***	16.67 ***	0.48 ***	3.96 ***	0.74	3.5
+ L/O-4 mV	13.41 **	14.69 ***	0.55 ***	5.02 ***	0.60	2.9
+ L/O-4 MS	13.47 ***	15.80 ***	0.51 ***	4.14 ***	0.74	3.1
GMV Portfolios						
+ L/O-4 CW	11.20	15.95	0.37	3.22	0.25	26.1
+ L/O-4 MD	12.84 ***	15.78 ***	0.47 ***	4.43 ***	0.55 **	18.3
+ L/O-4 mV	13.33 ***	13.95 ***	0.57 ***	6.22 ***	0.47 *	32.2
+ L/O-4 MS	13.26 ***	14.92 ***	0.53 ***	5.04 ***	0.57 ***	18.9
RP Portfolios						
+ L/O-4 CW	11.74	16.68	0.38	2.17	0.61	4.3
+ L/O-4 MD	13.29 ***	16.55 ***	0.48 ***	3.94 ***	0.73	4.6
+ L/O-4 mV	13.45 **	14.60 ***	0.55 ***	5.17 ***	0.59	4.1
+ L/O-4 MS	13.47 ***	15.71 ***	0.52 ***	4.21 ***	0.73	4.2
MENUMB Portfolios						
+ L/O-4 CW	11.45	15.82	0.39	3.16	0.33	31.2
+ L/O-4 MD	12.56 **	15.56 ***	0.46 **	4.36 ***	0.49	31.0
+ L/O-4 mV	13.41 **	13.72 ***	0.59 ***	6.47 ***	0.46	33.7
+ L/O-4 MS	13.11 ***	14.75 ***	0.53 ***	5.13 ***	0.53 *	33.4

Each universe contains the broad CW equity index plus four portfolios that each proxy for one equity factor (size, momentum, volatility, value). Four weighting schemes for the stocks in the equity portfolios are considered: capitalisation weighting (CW), equal weighting (MD), volatility minimisation (mV) and multi strategy (MS). The MS scheme is an equally-weighted combination of five weighting schemes: maximum deconcentration, minimum volatility, maximum decorrelation, inverse volatility and maximum Sharpe ratio. The broad CW and the four factor indices are weighted according to the following schemes, rebalanced every quarter: MDC is the maximum deconcentration portfolio; GMV is the global minimum variance subject to bounds on weights ( $1/(3p)$  and 1, where  $p$  is the nominal number of constituents); RP is the risk parity, which equalises the contributions to volatility; MENUMB is maximum ENUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix. The tracking error and the information ratio are computed with respect to the broad CW. In each panel of the table, tests of equality are performed by taking the combination of the broad CW and the four factors with the same allocation (first row of the panel) as a reference. The stars in columns 2 to 6 indicate the significance level: no star = not significant at the 10% level; \* = 10% level; \*\* = 5% level; \*\*\* = 1% level.

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gain in outperformance, however, is in general sufficient to balance this effect, and in most cases, the portfolios of smart-weighted factors have higher information ratios compared to the corresponding portfolios of CW factors. The only exceptions to this rule occur when stocks within factors are weighted so as to minimise volatility and the allocation to factors is based according to MDC or RP methodologies. Indeed, minimum volatility factors lead to the highest tracking errors, and the increase in average return is not always large enough to compensate for the increase in relative risk.

Table 12 contains similar results, except that it is now assumed that the factors replace the broad CW. In order to save space, we do not report the results for RP portfolios, which are, as in the previous tables, very similar to the MDC portfolios. On the other hand, we report the risk and return indicators for relative weighting schemes (namely the MTE, the relative RP and the maximum ENRUB weighting schemes),

which are relevant only in a substitution context. The comparison of extension and substitution exercises shows that the effects already mentioned for CW factors also occur with smart-weighted factors: removing the broad CW from the universe increases both the average excess return and the tracking error, but the two effects roughly compensate each other, so that information ratios are almost unchanged.

For all contexts, the ranking of the various allocation methods by turnover is similar for CW and smart-weighted factors. The GMV and the MENUB portfolios have the largest turnovers – between 18.5% and 34.0% per year – and the MDC and the RP portfolios have the lowest – less than 5%. The relative schemes display intermediate values. In particular, for a given four-factor universe, the maximum ENRUB portfolio has a lower turnover than its absolute counterpart. This property is not as well verified by the GMV and the MTE, as it is found that it is only in two universes out of four (CW and minimum volatility factors) that the MTE has a lower turnover.

Table 12: Smart-weighted equity factor indices as substitutes for a broad equity index (1972–2013).

Universe	Avg. return (%)	Volatility (%)	Sharpe ratio	Tracking error (%)	Information ratio	Ann. 1-way Turnover (%)
MDC Portfolios						
L/O-4 CW	12.05	16.70	0.40	2.63	0.62	3.3
L/O-4 MD	14.04 ***	16.66	0.52 ***	4.94 ***	0.74	3.5
L/O-4 mV	14.12 **	14.26 ***	0.61 ***	6.26 ***	0.59	2.9
L/O-4 MS	14.21 ***	15.59 ***	0.57 ***	5.17 ***	0.74	3.1
GMV Portfolios						
L/O-4 CW	11.10	15.88	0.36	3.51	0.20	28.9
L/O-4 MD	13.11 ***	15.60 ***	0.50 ***	5.08 ***	0.53 ***	15.2
L/O-4 mV	13.40 ***	13.75 ***	0.58 ***	6.69 ***	0.45 *	34.0
L/O-4 MS	13.33 ***	14.76 ***	0.54 ***	5.55 ***	0.53 ***	18.5
MENUB Portfolios						
L/O-4 CW	11.61	15.72	0.40	3.55	0.34	31.2
L/O-4 MD	13.07 ***	15.41 ***	0.50 ***	5.06 ***	0.53 *	31.0
L/O-4 mV	13.62 **	13.62 ***	0.61 ***	6.75 ***	0.48	33.7
L/O-4 MS	13.45 ***	14.63 ***	0.55 ***	5.56 ***	0.55 *	33.4

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MTE Portfolios						
L/O-4 CW	11.39	16.62	0.36	1.95	0.50	15.3
L/O-4 MD	13.62 ***	16.11 ***	0.51 ***	4.12 ***	0.78 *	18.6
L/O-4 mV	13.72 ***	14.59 ***	0.57 ***	5.62 ***	0.59	28.0
L/O-4 MS	13.77 ***	15.46 ***	0.54 ***	4.56 ***	0.74	22.4
RRP Portfolios						
L/O-4 CW	11.41	16.66	0.36	1.94	0.51	11.3
L/O-4 MD	13.76 ***	16.34 ***	0.51 ***	4.45 ***	0.75 *	9.3
L/O-4 mV	13.99 ***	14.37 ***	0.60 ***	6.01 ***	0.60	6.9
L/O-4 MS	14.04 ***	15.53 ***	0.56 ***	4.86 ***	0.75	8.0
MENRUB Portfolios						
L/O-4 CW	11.45	16.73	0.36	2.06	0.50	18.0
L/O-4 MD	13.58 ***	16.32 ***	0.50 ***	4.10 ***	0.77 **	16.8
L/O-4 mV	13.68 ***	14.53 ***	0.57 ***	5.54 ***	0.59	22.7
L/O-4 MS	13.73 ***	15.60 ***	0.54 ***	4.47 ***	0.74 *	18.0

Each universe contains four portfolios that each proxy for one equity factor (size, momentum, volatility, value). Four weighting schemes for the stocks in the equity portfolios are considered: capitalisation weighting (CW), equal weighting (MD), volatility minimisation (mV) and multi strategy (MS). The MS scheme is an equally-weighted combination of five weighting schemes: maximum deconcentration, minimum volatility, maximum decorrelation, inverse volatility and maximum Sharpe ratio. The broad CW and the four factor indices are weighted according to the following schemes, rebalanced every quarter: MDC is the maximum deconcentration portfolio; GMV is the global minimum variance subject to bounds on weights (1/12 and 3/4); RP is the risk parity, which equalises the contributions of constituents to volatility; MENUB is maximum ENUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix; MTE is the minimum tracking error; RRP is the relative risk parity, which equalises the contributions of constituents to tracking error; MENRUB is the portfolio that maximises the effective number of contributions of uncorrelated factors to tracking error, the factors being extracted from the relative covariance matrix. The tracking error and the information ratio are computed with respect to the broad CW. In each panel of the table, tests of equality are performed by taking the combination of the broad CW and the four factors with the same allocation (first row of the panel) as a reference. The stars in columns 2 to 6 indicate the significance level: no star = not significant at the 10% level; \* = 10% level; \*\* = 5% level; \*\*\* = 1% level.

### 5.4.4 Comparison with Sector Investing

In the previous comparisons, the original universe contains only one constituent, which is the broad CW equity index. Using one index only is a very crude representation of an asset class, and it is standard in practice to diversify an equity portfolio across several sector indices. In this section, we wish to compare the benefits of sector diversification to the benefits of factor diversification. In particular, this analysis will allow us to check whether the benefits of factor investing are merely due to an increase in the number of constituents in the portfolio, rather than the fact that factor indices happen to be particularly relevant constituents. For this analysis, we consider the ten Datastream US equity

sector indices, which are available at the daily frequency from January 1973 to December 2013: technology, financials, utilities, telecom, consumer services, healthcare, consumer goods, industrials, basic materials and oil and gas.

#### *In-Sample Tests*

Figures 5 and 6 describe the effects of introducing the factors on the usual indicators characterizing the position of the efficient frontier. As in the previous in-sample tests, these statistics are computed over the entire sample period. A comparison with the previous figures shows that the risk reduction ability of L/O factors is more limited when the original universe already contains the ten sector indices. When short sales are

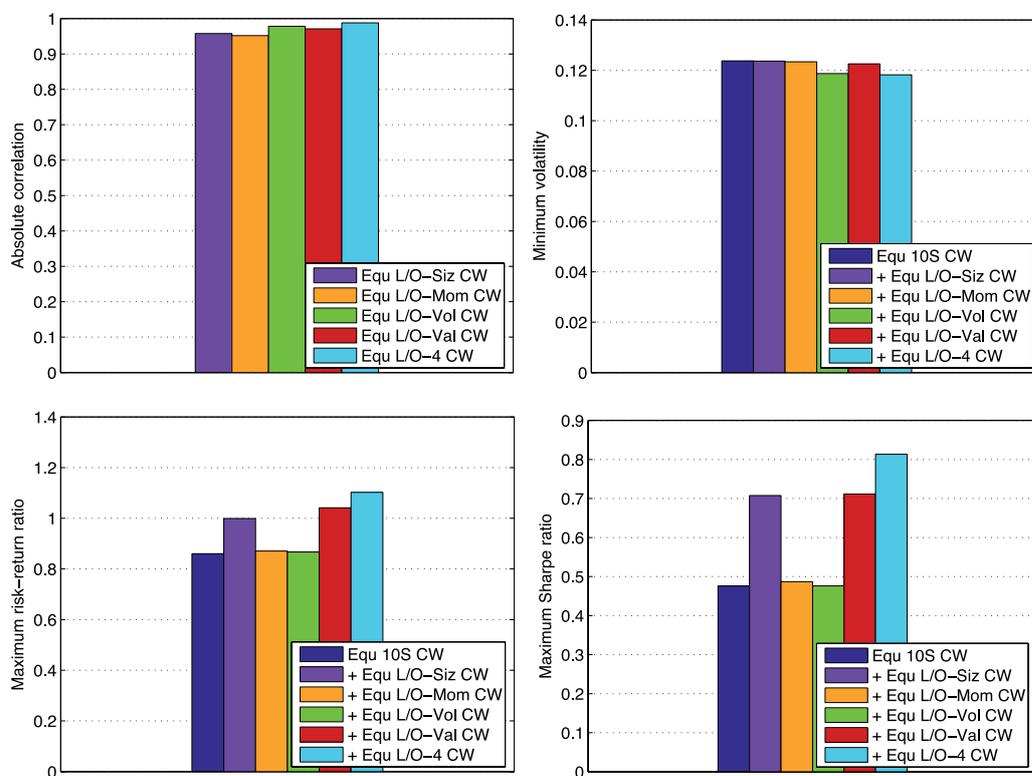
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precluded, the introduction of factors has actually no impact at all on the minimum volatility. The increases in the maximum risk-return ratio and the maximum Sharpe ratio appear to be substantial, except in the presence of short sales constraints. However, the gains observed in Panel (a) of

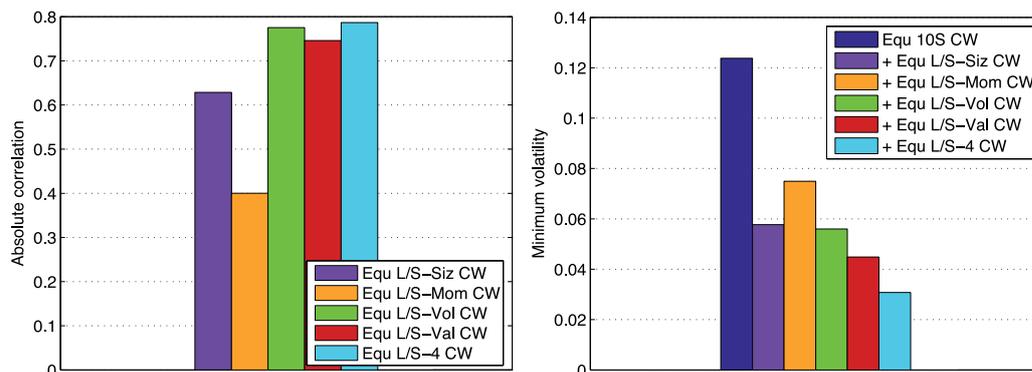
Figure 5 are achieved at the cost of short positions in the MRR and MSR portfolios. It is by investing in L/S factors that all indicators can be improved in an appreciable way. Overall, the picture is similar to that obtained by having only the broad index in the original universe.

Figure 5: Impact on risk and return indicators of universe extension with equity factors, starting from 10 sector indices (1973–2013).

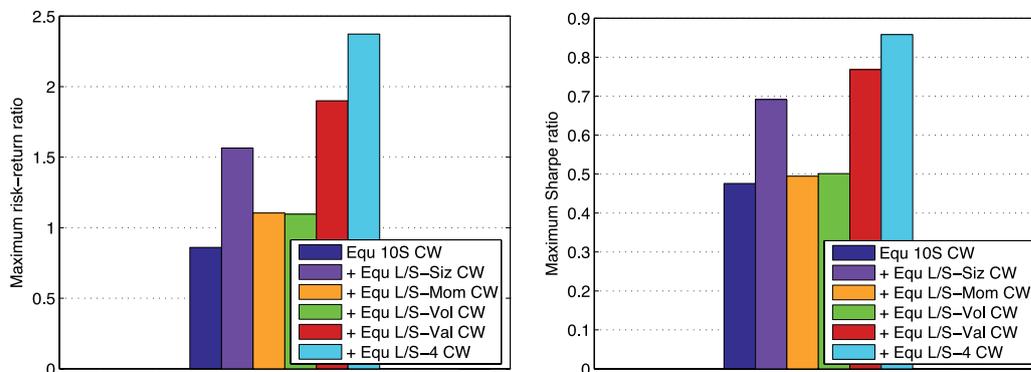
(a) L/O factors.



(b) L/S factors.



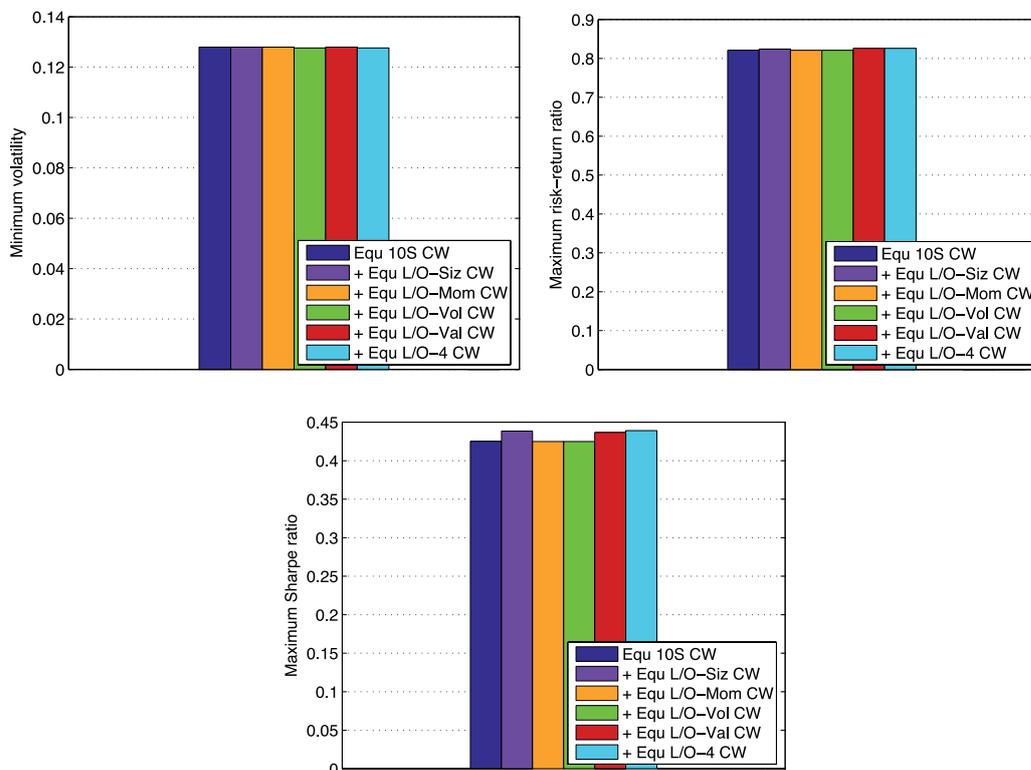
## 5. Empirical Analysis of Factor Investing



The original universe consists of the 10 Datastream equity sector indices, and it is extended with equity factor indices proxying for the size, momentum, volatility and value factors. The absolute correlation is the absolute value of the correlation between a portfolio of the 10 sectors and a factor index, or the maximum absolute value of the correlation between a portfolio of the 10 sectors and a portfolio of the 4 factor indices. The minimum volatility, the maximum risk-return ratio and the maximum Sharpe ratio are computed in each of the original or extended universes. All long-only factor indices, as well as the long and short legs of the long-short factors, are weighted by capitalisation. The statistics are computed from monthly returns over the whole sample (Jan. 1973 - Dec. 2013) and are annualised.

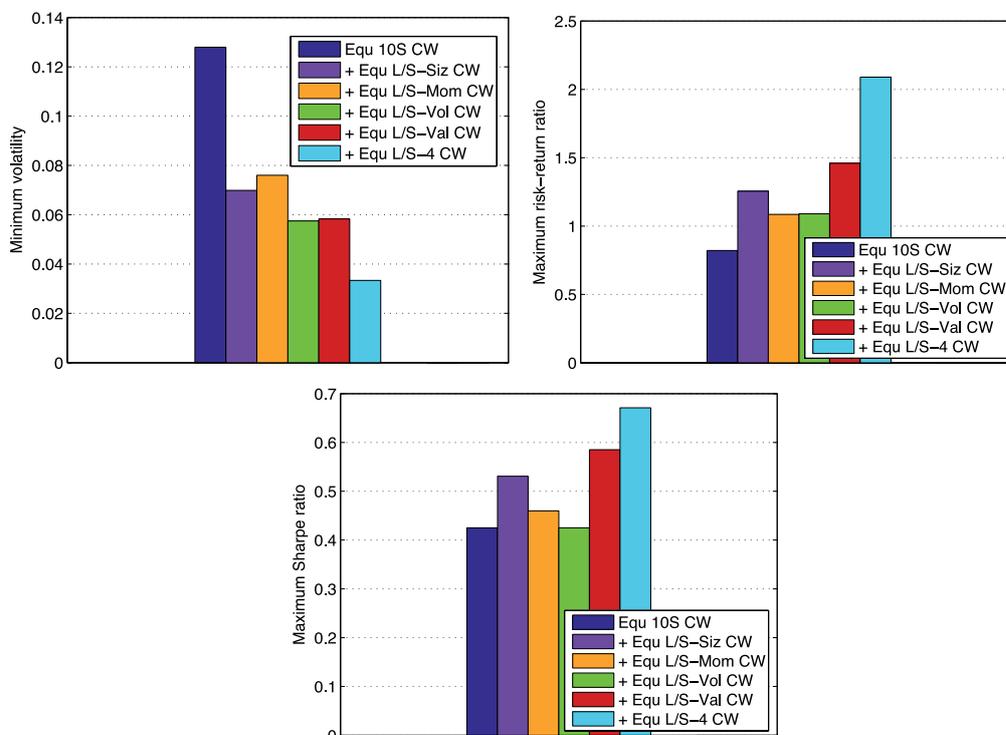
Figure 6: Impact on risk and return indicators of universe extension with equity factors, starting from 10 sector indices and with short sales constraints (1973-2013).

(a) L/O factors.



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(b) L/S factors.



The original universe consists of the 10 Datastream equity sector indices, and it is extended with equity factor indices proxying for the size, momentum, volatility and value factors. The absolute correlation is the absolute value of the correlation between a portfolio of the 10 sectors and a factor index, or the maximum absolute value of the correlation between a portfolio of the 10 sectors and a portfolio of the 4 factor indices. The minimum volatility, the maximum risk-return ratio and the maximum Sharpe ratio are computed in each of the original or extended universes, subject to a no short sales constraint. All long-only factor indices, as well as the long and short legs of the long-short factors, are weighted by capitalisation. The statistics are computed from monthly returns over the whole sample (Jan. 1973 - Dec. 2013) and are annualised.

To check for the statistical significance of the variations, we perform mean-variance spanning tests in Table 13 and decompose the spanning test using the stepdown methodology of KZ12. The results are in line with the visual insights from the figures. For L/S factors, changes in the MRR and the minimum volatility are significant at the 10% level, so that the null of spanning is clearly rejected. For L/O factors, the situation is more contrasted. The test statistics are much lower, and for the momentum factor, the null of spanning cannot be rejected, even at a loose confidence level such as 10%. The results of the stepdown tests indicate that this lack of significance is mainly due to the low increase in MRR. Similarly, the volatility factor, while being

useful in the GMV, does not help increase the MRR. These results are rather unsurprising, given that the momentum and volatility factors have the lowest performance in our sample. When the universe consisted of the broad index only, the null of no change in the MRR with momentum only was also impossible to reject at the 5% level (see Table 7).

### Out-of-Sample Tests

In the next series of tests, we run a horse race between portfolios formed with the ten sector indices and portfolios containing factor indices, either in addition to or in replacement for the sectors, the latter case (substitution as opposed to expansion) being the most relevant for an analysis of

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Table 13: Mean-variance spanning tests with equity factors, starting from 10 sector indices (1973-2013).

Universe	Spanning tests		Tests on MRR		Tests on GMV	
	F-stat	p-value	F-stat	p-value	F-stat	p-value
L/O Factors						
+ L/O-Siz CW	6.06	0.00	9.75	0.00	2.33	0.13
+ L/O-Mom CW	1.38	0.25	0.67	0.41	2.09	0.15
+ L/O-Vol CW	21.09	0.00	0.45	0.50	41.76	0.00
+ L/O-Val CW	8.98	0.00	12.98	0.00	4.86	0.03
+ L/O-4 CW	7.75	0.00	4.45	0.00	11.17	0.00
L/S Factors						
+ L/S-Siz CW	865.19	0.00	64.42	0.00	1471.88	0.00
+ L/S-Mom CW	417.83	0.00	18.11	0.00	789.46	0.00
+ L/S-Vol CW	948.91	0.00	17.41	0.00	1818.40	0.00
+ L/S-Val CW	1588.54	0.00	108.21	0.00	2509.55	0.00
+ L/S-4 CW	366.63	0.00	45.80	0.00	1313.87	0.00

The original investment universe contains the 10 Datastream equity sector indices, and it is extended with one or four equity factor index/indices. The factors are size, momentum, volatility and value. The long-only factors and the long and short legs of the long-short factors are weighted by capitalisation. For each universe extension, three tests are performed: (1) a spanning test, which is a joint test of the equality of the global minimum variance (GMV) and maximum risk-return ratio (MRR) portfolios before and after the extension; (2) a test of equality of the two MRR portfolios; (3) a test of equality of the two GMV portfolios conditional on the equality of the two MRRs. The table reports the F-statistic of each test as well as the p-value derived from the small-sample distribution of the test statistic.

the comparative benefits of factor investing versus sector investing. The results are contained in Tables 14 and 15. As expected, increasing the number of constituents in the original universe has a very positive impact on the Sharpe ratio. All benchmark portfolios have a ratio above 0.40, while that of the broad index merely stands at

0.29. In this context, it is more difficult for factor portfolios to deliver a higher Sharpe ratio. Nevertheless, in most cases, they do have at least as high a Sharpe ratio as the benchmark. Focusing on 4-factor portfolios alone, the only exceptions are the two GMV portfolios and the substituted MENUB portfolio.

Table 14: Extension of universe with equity factor indices, starting from 10 equity sector indices (1975-2013).

Universe	Avg. return (%)	Volatility (%)	Sharpe ratio	Tracking error (%)	Information ratio	Ann. 1-way Turnover (%)
MDC Portfolios						
Equ 10S CW	12.56	16.37	0.45	2.82	0.34	8.9
+ Equ L/O-Siz CW	12.77 **	16.42 ***	0.46 **	2.81	0.42	8.5
+ Equ L/O-Mom CW	12.59	16.41 ***	0.45	2.56	0.39	8.4
+ Equ L/O-Vol CW	12.51	16.25 ***	0.45	2.76	0.33	8.4
+ Equ L/O-Val CW	12.63	16.46 ***	0.45	2.72	0.38	8.4
+ Equ L/O-4 CW	12.76	16.43 ***	0.46	2.48	0.47	7.5
GMV Portfolios						
Equ 10S CW	12.67	14.26	0.52	6.18	0.17	29.1
+ Equ L/O-Siz CW	12.70	14.36 ***	0.52	6.00	0.18	29.4
+ Equ L/O-Mom CW	12.71	14.37 ***	0.52	5.98	0.19	29.3
+ Equ L/O-Vol CW	12.64	14.32 **	0.52	6.00	0.18	31.3
+ Equ L/O-Val CW	12.70	14.39 ***	0.52	5.99	0.19	29.6
+ Equ L/O-4 CW	12.70	14.59 ***	0.51	5.45	0.20	31.9

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RP Portfolios						
Equ 10S CW	12.76	15.60	0.48	3.64	0.32	11.9
+ Equ L/O-Siz CW	12.94 **	15.69 ***	0.49 *	3.56	0.38	11.6
+ Equ L/O-Mom CW	12.79	15.68 ***	0.48	3.42	0.35	11.4
+ Equ L/O-Vol CW	12.68	15.57 ***	0.48	3.58	0.30	11.2
+ Equ L/O-Val CW	12.83	15.73 ***	0.48	3.50	0.35	11.4
+ Equ L/O-4 CW	12.91	15.81 ***	0.49	3.22	0.41	10.2
MENUM Portfolios						
Equ 10S CW	12.76	14.18	0.53	5.58	0.21	34.6
+ Equ L/O-Siz CW	13.02 **	14.19	0.55 *	5.54	0.26	38.2
+ Equ L/O-Mom CW	12.83	14.23 ***	0.53	5.45	0.23	39.0
+ Equ L/O-Vol CW	12.55 **	14.16	0.52 *	5.56	0.17	45.6
+ Equ L/O-Val CW	12.94 *	14.24 ***	0.54	5.50	0.25	39.6
+ Equ L/O-4 CW	12.88	14.28 ***	0.53	5.31	0.24	55.4

For each allocation scheme, the benchmark portfolio contains the 10 sector indices. The other portfolios contain the broad index plus one equity factor only (size, momentum, volatility, value), or the four factors. All factors are long-only and weighted by capitalisation. The constituents are weighted according to the following schemes, rebalanced every quarter: MDC is the maximum deconcentration portfolio; GMV is the global minimum variance subject to bounds on weights  $(1/(3p)$  and  $3/p$ , where  $p$  is the universe size); RP is the risk parity, which equalises the contributions to volatility; MENUM is the maximum ENUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix. The tracking error and the information ratio are computed with respect to the broad CW. The stars in columns 2 to 4 indicate the significance level in an equality test with respect to the 10-sector portfolio: no star = not significant at the 10% level; \* = 10% level; \*\* = 5% level; \*\*\* = 1% level.

The results in Table 15 suggest that factor investing generally generates higher performance compared to sector investing. An even more spectacular improvement can be noted when considering relative risk indicators. In all cases, 4-factor portfolios display lower tracking errors and/or higher

information ratios than the 10-sector portfolios. Finally, it is worth noting that the portfolios containing only the four factors have lower turnovers. This can be explained by the fact that they involve fewer constituents, so that they are less demanding in terms of rebalancing.

Table 15: Capitalisation-weighted equity factor indices as substitutes for equity sector indices (1975-2013).

Universe	Avg. return (%)	Volatility (%)	Sharpe ratio	Tracking error (%)	Information ratio	Ann. 1-way Turnover (%)
MDC Portfolios						
Equ 10S CW	12.56	16.37	0.45	2.82	0.34	8.9
Equ L/O-4 CW	13.23	16.73 ***	0.48	2.59	0.63	3.5
GMV Portfolios						
Equ 10S CW	12.67	14.26	0.52	6.18	0.17	29.1
Equ L/O-4 CW	12.06	15.87 ***	0.43	3.48	0.13	27.2
RP Portfolios						
Equ 10S CW	12.76	15.60	0.48	3.64	0.32	11.9
Equ L/O-4 CW	13.17	16.60 ***	0.48	2.64	0.60	4.6
MENUM Portfolios						
Equ 10S CW	12.76	14.18	0.53	5.58	0.21	34.6
Equ L/O-4 CW	12.55	15.68 ***	0.47	3.59	0.27	26.2

## 5. Empirical Analysis of Factor Investing

MTE Portfolios						
Equ 10S CW	11.98	16.97	0.40	1.47	0.27	14.0
Equ L/O-4 CW	12.59	16.60 ***	0.44	1.99	0.50	15.3
RRP Portfolios						
Equ 10S CW	11.98	16.96	0.40	1.47	0.26	13.2
Equ L/O-4 CW	12.62	16.65 ***	0.44	1.97	0.52	11.2
MENRUB Portfolios						
Equ 10S CW	12.45	16.52	0.44	2.64	0.33	38.1
Equ L/O-4 CW	12.63	16.72 ***	0.44	2.10	0.50	17.5

For each allocation method, the benchmark portfolio contains the 10 equity sector indices. The other portfolios contain four equity factor indices proxying for the size, momentum, volatility and value factors. All factors are long-only and weighted by capitalisation. In 4-factor universes, the constituents are weighted according to the following schemes, rebalanced every quarter: MDC is the maximum deconcentration portfolio; GMV is the global minimum variance subject to bounds on weights ( $1/(3p)$  and  $3/p$ , where  $p$  is the universe size); RP is the risk parity, which equalises the contributions to volatility; MENUB is the maximum ENUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix. The last three weighting schemes focus on relative risk with respect to the broad CW equity index: MTE minimises the tracking error subject to bounds on weights; RRP is the relative risk parity, which equalises the contributions to the tracking error; MENRUB is the maximum ENRUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix of excess returns. The tracking error and the information ratio are computed with respect to the broad CW. The stars in columns 2 to 4 indicate the significance level in an equality test with respect to the broad CW: no star = not significant at the 10% level; \* = 10% level; \*\* = 5% level \*\*\* = 1% level. The last column reports the annual one-way turnover.

It should be reminded that our results provide only a lower bound for the benefits of factor investing, given that our set of factors does not exhaust the set of all pricing factors. Larger benefits could certainly be achieved with a more comprehensive set of factors. Overall, factor investing in the equity space appears to be a more parsimonious way to access to rewarded sources of risk than other standard forms of grouping such as grouping in sectors.

### 5.5 Bond Factors in Allocation Decisions

We now repeat the previous analysis for bond factor indices. The protocol we use is the same as for equities, and is divided in in-sample and out-of-sample tests. One first difference is that there are fewer factor indices, in fact only two here, namely term and credit. The second difference is that our dataset, which does not contain return information for individual bonds, does not allow us to test various weighting schemes for the factors and

all bond portfolios are taken to be capitalisation-weighted.

#### 5.5.1 In-Sample Tests

We first look at the impact of introducing the fixed-income factors in a universe that contains the broad bond index on the efficient frontier.

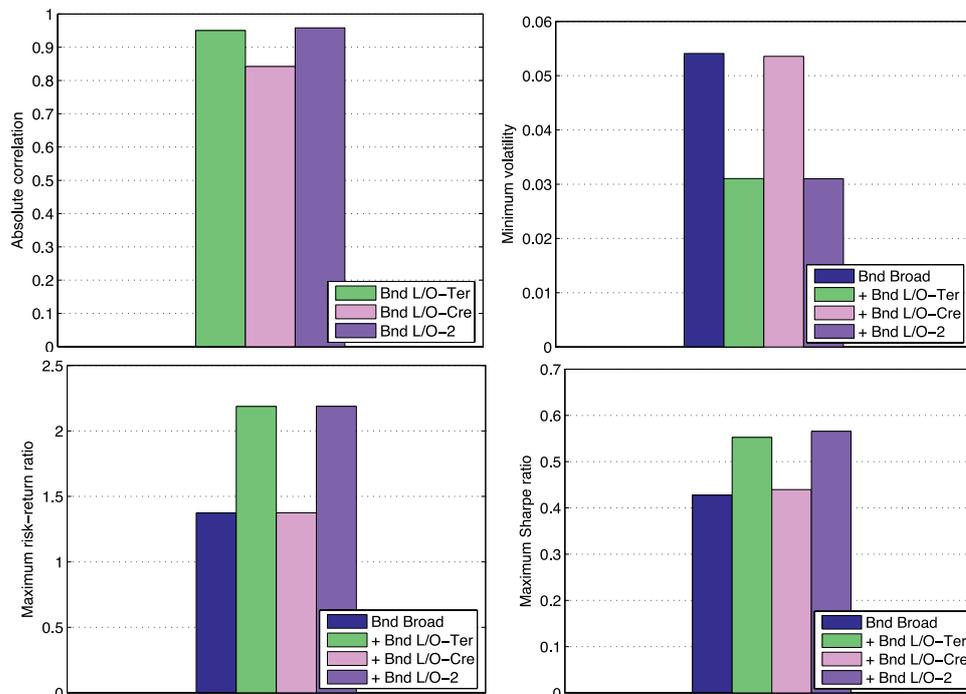
#### *Impact of Universe Extension on the Efficient Frontier*

Figure 7 shows the canonical correlations of bond factors as well as characteristic indicators of the efficient frontier before and after the universe extension. As mentioned earlier, it is only for the credit factor that switching from the L/O to the L/S version substantially reduces the correlation with the broad index. Indeed, both legs are correlated with the broad index and have similar volatilities. For the term factor, the short leg has very low volatility compared to the long one, so that the reduction in correlation is small. But unlike for equity factors, it appears that the introduction of the two L/O factors

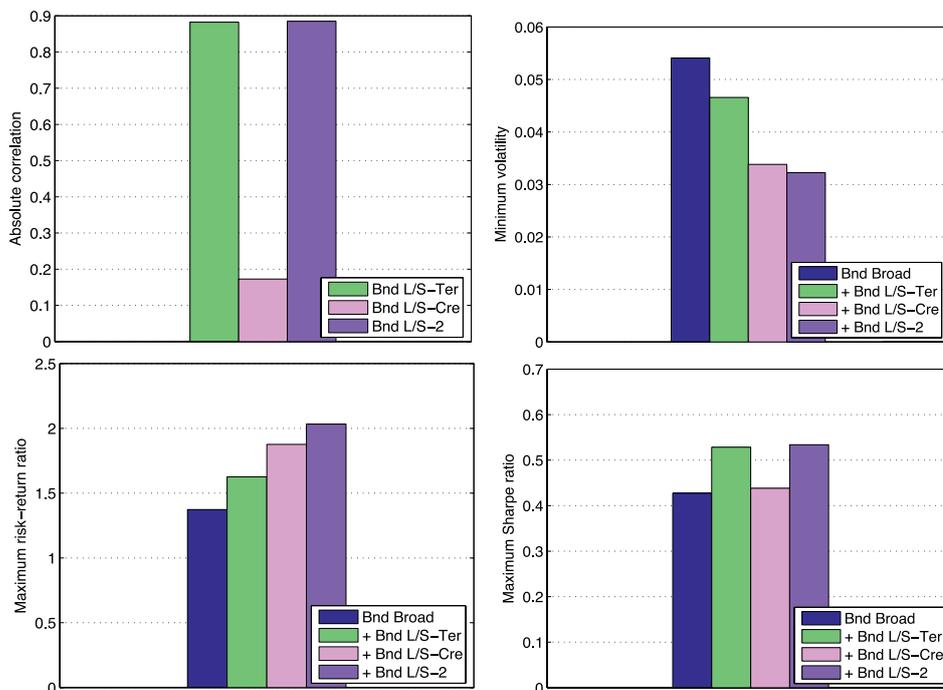
## 5. Empirical Analysis of Factor Investing

Figure 7: Impact on risk and return indicators of universe extension with bond factors (1976-2013).

(a) L/O factors.



(b) L/S factors.



The broad bond index is the Barclays US Treasury index and the two bond factor indices proxy for the term and credit factors. The absolute correlation is the absolute value of the correlation between the broad bond index and a factor index, or the maximum absolute value of the correlation between the broad index and a portfolio of the two factor indices. The minimum volatility, the maximum risk-return ratio and the maximum Sharpe ratio are computed over a universe that contains only the broad index, or the broad index plus one factor index, or the broad index plus the two indices. The statistics are computed from monthly returns over the whole sample (Jan. 1976 - Dec. 2013) and are annualised.

## 5. Empirical Analysis of Factor Investing

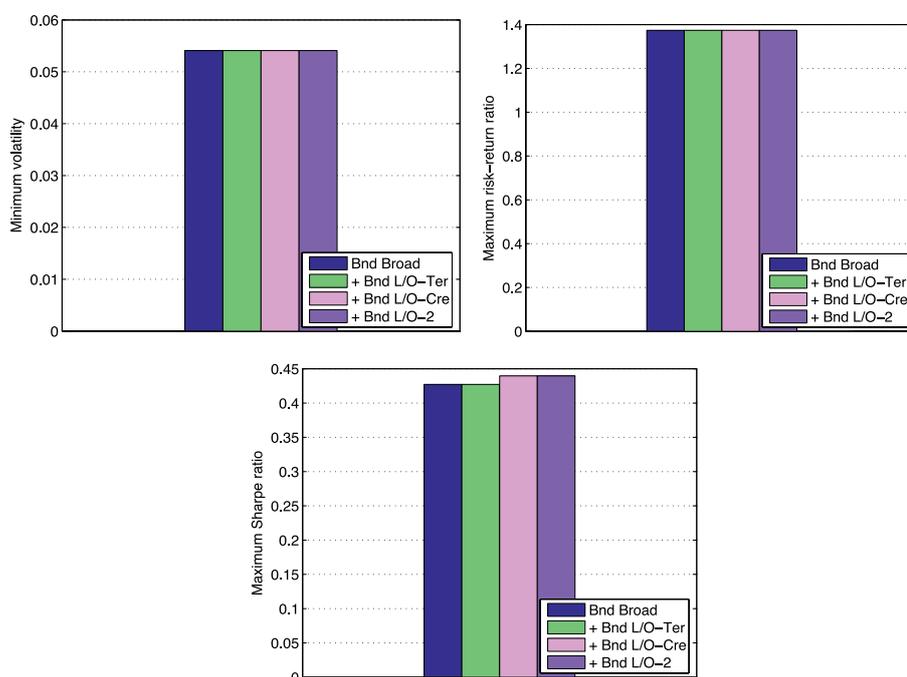
has roughly the same final effect on the risk and return indicators as that of the two L/S factors. With the two L/O factors, the minimum volatility is 3.10% and the maximum risk-return ratio reaches 2.19, and with the two L/S factors, the two indicators are respectively 3.23% and 2.03. Thus, the L/O factors generate an even larger improvement.

However, the gains observed for L/O factors are achieved at the cost of short positions, due to the high correlations between the constituents. Figure 8 describes the impact of L/O or L/S factors when short sales in the GMV, MRR and MSR portfolios are prohibited, and it is clear from Panel (a) that the advantages of L/O factors are almost economically insignificant in terms of volatility reduction or improvement of the risk-return trade-off: the minimum volatility stays at 5.41%, the maximum risk-return ratio at 1.37, and only the Sharpe ratio increases slightly, from 0.43 in the original universe to 0.44 in the universe

augmented with two factors. Thus, an investor constrained to invest in L/O factors and to take only long-only positions will hardly benefit from the introduction of these factors. If he has the opportunity to invest in L/S factors, but still under the constraint of long-only positions in the indices, he can expect more improvements with respect to the various dimensions. However, comparing Panels (b) from Figures 7 and 8 reveals that in the largest universe, the minimum volatility and the maximum ratios are respectively lower and larger when short sales are allowed than when they are prohibited. This indirectly shows that unconstrained efficient portfolios of L/S factors do involve short positions, which can also be directly verified by computing their compositions. The presence of negative weights in portfolios of L/S factors is explained by the high correlation between the term factor and the broad index (see Panel (b) in Figure 7).

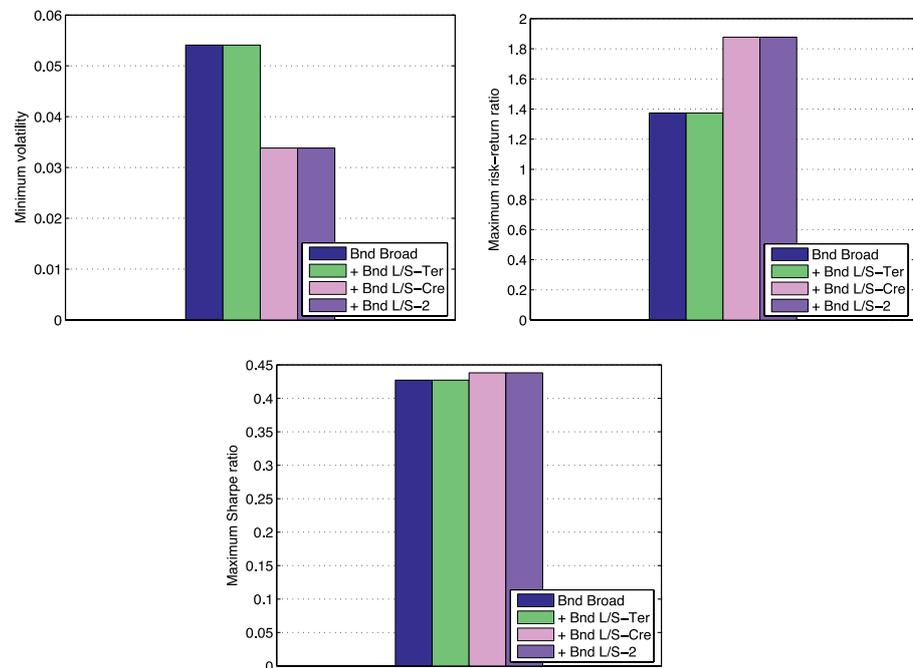
Figure 8: Impact on risk and return indicators of universe extension with bond factors when short sales are prohibited (1976-2013).

(a) L/O factors.



## 5. Empirical Analysis of Factor Investing

(b) L/S factors.



The broad bond index is the Barclays US Treasury index and the two bond factor indices proxy for the term and credit factors. The minimum volatility, the maximum risk-return ratio and the maximum Sharpe ratio are computed over a universe that contains only the broad index, or the broad index plus one factor index, or the broad index plus the two indices. The minimum variance, maximum risk-return ratio and maximum Sharpe ratio portfolios are subject to long-only constraints. The statistics are computed from monthly returns over the whole sample (Jan. 1976 - Dec. 2013) and are annualised.

### Mean-Variance Spanning Tests

The mean-variance spanning tests that we apply compare the long-short efficient frontiers, that is, they take as inputs the performance and risk indicators of Figure 7. As for equity factors, the test statistics in Table 16 are sufficiently large to warrant rejection of the null hypothesis of spanning at the 1% level, both for L/O and L/S factors.

However, the null hypothesis that the maximum risk-return ratio is not improved by the L/O credit factor cannot be rejected, due to a p-value of 87%. The rejection of the spanning hypothesis can be entirely attributed to the reduction in the minimum volatility. But Panel (a) of Figure 7 suggests that the decrease in volatility is very small and is actually close to invisible on the plot: quantitatively, the volatility of the broad index is 5.41% per

year, and with the L/O credit factor, the minimum volatility is 5.36%. It is because the sample is large (38 years of monthly data, which corresponds to 456 data points) that even such an economically insignificant variation achieves statistical significance. On the other hand, the advantages of the L/O credit factor appear to be small, regardless of whether they are measured in terms of volatility reduction or increase in the risk-return ratio (see Figure 7). The tension between the smallness of the effect and its statistical significance stresses the need for further tests which will give an assessment of the economic significance. This is what we turn to in the next section.

## 5. Empirical Analysis of Factor Investing

Table 16: Mean-variance spanning tests with bond factors (1976-2013).

Universe	Spanning tests		Tests on MRR		Tests on GMV	
	F-stat	p-value	F-stat	p-value	F-stat	p-value
L/O Factors						
+ L/O-Ter	462.79	0	94.48	0	689.18	0
+ L/O-Cre	4.95	0.01	0.03	0.87	9.88	0
+ L/O-2	168.88	0	47.19	0	345.54	0
L/S Factors						
+ L/S-Ter	78.95	0	24.68	0	126.62	0
+ L/S-Cre	356.20	0	53.33	0	590.96	0
+ L/S-2	154.64	0	36.57	0	326.53	0

The original investment universe contains the broad bond index (the Barclays US Treasury index), and it is extended with one or two bond factor index/indices (term and credit). For each universe extension, three tests are performed: (1) a spanning test, which is a joint test of the equality of the global minimum variance (GMV) and maximum risk-return ratio (MRR) portfolios before and after the extension; (2) a test of equality of the two MRR portfolios; (3) a test of equality of the two GMV portfolios conditional on the equality of the two MRRs. The table reports the F-statistic of each test as well as the p-value derived from the small-sample distribution of the test statistic.

### 5.5.2 Out-of-Sample Tests

Bond factors can be used in addition to, or in substitution for, the broad index. We now compare portfolios containing one or two bond factor index (indices) to the broad benchmark, using the allocation schemes presented in Section 3.1. The covariance matrix is still estimated over a two-year rolling window, but given data availability, the sampling frequency is monthly.

#### *Extension Tests*

Table 17 focuses on universe extensions. Because both bond factors outperform the broad index, the extended portfolios have higher average returns than the benchmark index. The increase is perhaps less important than in the equity class: in the best case (the MDC portfolio with two factors), the gain in average return is 59 bps per year, while many extended equity portfolios in Table 8 achieved a gain of more than 100 bps with respect to the broad equity index. The most remarkable difference with respect to the equity class is that all extended portfolios have higher volatilities than the benchmark index. Panels (a) of Figures 7 and 8 help to understand why: all L/O bond portfolios are highly positively

correlated, and by prohibiting short sales, it is impossible to reduce the minimum volatility in the sample. Out of sample, the zero change in volatility becomes a positive change, especially for portfolios that do not explicitly attempt at reducing risk (MDC, RP and MENUUB). The combination of an increase in average return and an increase in volatility does not always have the same effect on the Sharpe ratio: some of the extended portfolios display Sharpe ratios equal to or greater than that of the broad index (e.g. MENUUB portfolios); others are behind the benchmark (e.g. most MDC, GMV and RP portfolios). This ambiguous effect on the Sharpe ratio can be related to the very low increase in maximum Sharpe ratio that is observed in the full sample with L/O factors and L/O allocations (see Panel (a) in Figure 8). In terms of turnover, the various allocation schemes rank as in the equity class: as a general rule, MDC and RP portfolios imply lower amounts of rebalancing than GMV and MENUUB portfolios. The GMV and MENUUB of the broad index and the term factor are exceptions on which we comment below.

## 5. Empirical Analysis of Factor Investing

It should be noted that due to the low number of constituents (at most three) and their high correlations, some of the portfolios studied here have trivial compositions. The most obvious example is the MENCH portfolio based on the broad index and the term factor. The zero tracking error indicates that it is fully invested in the broad index. A qualitative explanation of this phenomenon is as follows. This portfolio is the closest approximation to a factor risk parity portfolio that respects the long-only constraints in the constituents. Factor risk parity would be achieved if the two factors had weights inversely proportional to their volatilities. But because the two constituents are highly correlated, any orthogonalisation process introduces a substantial distortion of the initial assets (even though the MLT approach

minimises the distance between the original returns and the uncorrelated factors), so that the factor risk parity portfolio is not feasible without violating the long-only constraints. Thus, the MENCH portfolio will merely overweight the least volatile of the two implicit factors in the limits imposed by the constraints. This factor is the one that corresponds to the broad index, because the other implicit factor has, by definition, the same volatility as the term factor, and is therefore more risky. The larger weight assigned to the first implicit factor results in a larger weight in the corresponding constituent, that is the broad index. The maximum overweighting permitted by the no short-sales constraints is a 100% allocation to the broad index, so that the MENCH portfolio is actually fully invested in the broad index.

Table 17: Extension of universe with bond factor indices (1978-2013).

Universe	Avg. return (%)	Volatility (%)	Sharpe ratio	Tracking error (%)	Information ratio	Ann. 1-way Turnover (%)
Broad	7.77	5.53	0.46	0.00	-	0.0
MDC Portfolios						
+ L/O-Ter	8.32	8.16 ***	0.38 **	2.98	0.18	2.3
+ L/O-Cre	8.08	6.11 ***	0.47	1.97	0.16	1.4
+ L/O-2	8.36	7.55 ***	0.41	2.53	0.23	2.4
GMV Portfolios						
+ L/O-Ter	7.97	6.37 ***	0.43 **	1.00	0.20	1.3
+ L/O-Cre	7.92	5.74 **	0.47	1.56	0.10	24.7
+ L/O-2	8.08	6.24 ***	0.46	1.70	0.18	26.2
RP Portfolios						
+ L/O-Ter	8.17	7.26 ***	0.40 **	1.97	0.20	3.5
+ L/O-Cre	8.02	5.97 ***	0.47	1.73	0.14	5.3
+ L/O-2	8.26	6.92 ***	0.44	2.06	0.24	6.1
MENCH Portfolios						
+ L/O-Ter	7.77	5.53	0.46	0.00	-	0.0
+ L/O-Cre	8.04	5.53	0.51	1.48	0.18	21.9
+ L/O-2	8.27	5.70	0.53	2.10	0.24	19.1

The benchmark portfolio is the broad bond index (the Barclays US Treasury index). The other portfolios contain the broad index plus one bond factor only (term or credit), or the two factors. The constituents are weighted according to the following schemes, rebalanced every quarter: MDC is the maximum deconcentration portfolio; GMV is the global minimum variance subject to bounds on weights ( $1/(3p)$  and 1, where  $p$  is the nominal number of constituents); RP is the risk parity, which equalises the contributions to volatility; MENCH is maximum ENUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix. The tracking error and the information ratio are computed with respect to the broad CW. The stars in columns 2 to 4 indicate the significance level in an equality test with respect to the broad CW: no star = not significant at the 10% level; \* = 10% level; \*\* = 5% level; \*\*\* = 1% level.

## 5. Empirical Analysis of Factor Investing

Another example is the GMV portfolio of the broad index and the term factor. It has an annual turnover of 1.3% only, which is unusually low for this weighting scheme (compare with equity figures and with the turnovers of the other GMV bond portfolios). Further examination of the weights shows that this portfolio is actually a fixed-mix of the two constituents, in which the term factor has the lowest possible weight given the lower bound (16.7%; see (3.2)). Indeed, the correlation between the two constituents is so high that a long-only GMV will be invested only in the least volatile one. The introduction of a minimum weight constraint here precludes a zero weight, so that the portfolio is not fully invested in one of the two assets, but it is still a fixed-mix portfolio.

More generally, all GMV portfolios tend to overweight the broad bond index with respect to the other factor(s) included, because this constituent is the least volatile and the long-only constraints prevent from taking advantage of the high correlations by shorting some of the constituents. As a result, for a given universe, the GMV

is in general not the weighting scheme that leads to the highest tracking error. For two universes out of three (with the term factor and with the two factors), it has even the lowest deviation with respect to the benchmark. This stands in contrast with the equity universe, where the GMV has often higher tracking errors than the other schemes.

### Substitution Tests

In substitution tests, the broad bond index is replaced by the bond factors. As appears from Table 18, the MDC, GMV, RP and MENUB portfolios of two factors have higher average returns and tracking errors than their counterparts based on the factors plus the broad index. These effects are the same as in the equity class. The net impact on the information ratio is close to zero for the MDC and the RP, positive for the GMV and negative for the MENUB. Moreover, as for equities, the two-factor portfolios have lower tracking errors than their constituents, and, except for the MENUB, they have also higher information ratios. Thus, combining factors appears to improve the relative risk and

Table 18: Bond factor indices as substitutes for a broad bond index (1978-2013).

(a) Performance and risk indicators.

Universe	Avg. return (%)	Volatility (%)	Sharpe ratio	Tracking error (%)	Information ratio
Broad	10.41	17.26	0.29	0.00	-
Single-factor portfolios					
L/O-Ter	8.76	10.96 ***	0.32 **	5.95	0.17
L/O-Cre	8.33	7.21 ***	0.43	3.93	0.14
Absolute weighting schemes					
MDC L/O-2	8.62	8.66 ***	0.39	3.79	0.22
GMV L/O-2	8.57	7.63 ***	0.44	3.55	0.23
RP L/O-2	8.60	8.22 ***	0.41	3.58	0.23
MENUB L/O-2	8.29	7.37 ***	0.42	3.49	0.15
Relative weighting schemes					
MTE L/O-2	8.55	7.94 ***	0.42	3.37	0.23
RRP L/O-2	8.54	8.17 ***	0.41	3.48	0.22
MENRUB L/O-2	8.36	7.91 ***	0.40	3.35	0.18

## 5. Empirical Analysis of Factor Investing

(b) Analytics on weights.

Universe	Avg. ENC (%op)	Avg. ENCB (%op)	Avg. ENUB (%op)	Avg. ENRCB (%op)	Avg. ENRUB (%op)	Ann. 1-way Turnover (%)
Absolute weighting schemes						
MDC L/O-2	100.0	91.4	72.7	76.0	72.7 2.5	
GMV L/O-2	69.9	79.5	86.1	72.6	74.2	3.9
RP L/O-2	93.0	100.0	79.9	84.7	77.6	5.7
MENUB L/O-2	57.5	60.2	92.3	52.3	69.9	9.1
Relative weighting schemes						
MTE L/O-2	76.9	84.0	81.8	86.6	85.7	11.2
RRP L/O-2	87.5	93.5	78.6	100.0	90.8	10.1
MENRUB L/O-2	78.1	81.1	80.0	84.4	99.5	15.5

The broad bond index is the Barclays US Treasury index. The other portfolios contain either one bond factor only (term or credit), or the two factors. In 2-factor universes, the constituents are weighted according to the following schemes, rebalanced every quarter: MDC is the maximum deconcentration portfolio; GMV is the global minimum variance subject to bounds on weights (1/6 and 1); RP is the risk parity, which equalises the contributions to volatility; MENUB is the maximum ENUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix. The last three weighting schemes focus on relative risk: MTE minimises the tracking error subject to bounds on weights; RRP is the relative risk parity, which equalises the contributions to the tracking error; MENRUB is the maximum ENRUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix of excess returns. In Panel (a), the tracking error and the information ratio are computed with respect to the broad index. The stars in columns 2 to 4 indicate the significance level in an equality test with respect to the broad index: no star = not significant at the 10% level; \* = 10% level; \*\* = 5% level; \*\*\* = 1% level. In Panel (b), ENC is the effective number of constituents; ENCB is the effective number of correlated bets (effective number of constituent contributions to volatility); ENUB is the effective number of uncorrelated bets; ENRCB is the effective number of relative correlated bets (effective number of constituent contributions to tracking error); ENRUB is the effective number of relative uncorrelated bets (the factors being extracted from the relative covariance matrix). All these indicators are computed at each quarterly rebalancing date and then averaged. They are expressed as percentages of the universe size,  $p = 2$ . The last column reports the annual one-way turnover.

performance indicators. It can be verified in Panel (b) that the MDC, RP, MENUB, RRP and MENRB achieve their respective objectives in terms of diversification of contributions of constituents or factors. The GMV and the MTE portfolios display relatively high concentration levels, in the sense that they lag behind at least one another weighting scheme with respect to all diversification measures. Interestingly, with an effective number of bets equal to 92.3% of the nominal number of constituents, the MENUB is further away from factor risk parity, than its relative counterpart: the MENRUB has an effective number of relative bets of 99.5%. This can be attributed to the lower levels of correlations of relative returns compared to absolute returns. Since the orthogonalisation process distorts less the relative returns, the relative FRP, which would equalise the contributions of factors to the tracking

error, corresponds to a portfolio which is almost feasible, i.e. involves only small short positions. Thus, with short sales constraints, the MENRUB is close to a relative FRP.

### 5.6 Commodity Factors in Allocation Decisions

After equity and bond factors, we conclude the series of single-class experiments by measuring the value added by commodity factors. The protocol is the same as for the other two classes and the results are broadly similar, but there are a few differences to be emphasised.

#### 5.6.1 In-Sample Tests

##### *Impact of Universe Extension on the Efficient Frontier*

Figures 9 and 10 show the impact of introduction of commodity factors on the

## 5. Empirical Analysis of Factor Investing

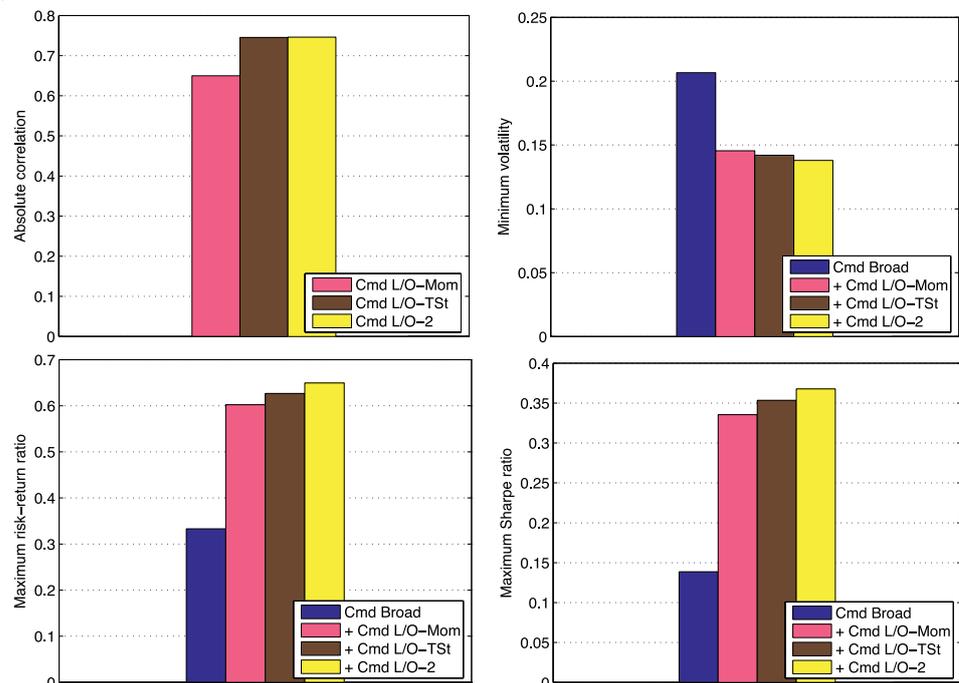
minimum volatility, maximum risk-return ratio and maximum Sharpe ratio, both in the absence and in the presence of short sales constraints. Because correlations between L/S factors and the broad index are moderate, the constraints are not binding in the GMV and the MRR portfolios. They are binding in the MSR portfolio, but the opportunity cost from prohibiting short sales is limited, since the difference between the two MSRs is only 41 bps. L/O factors are more correlated between them and with the broad index, but as noted above, the correlations are lower than in the equity and bond universes: for instance, the L/O term structure and momentum factors have correlations 74.60% and 65.00% with the broad index, while all equity and bond factors have correlations greater than 90% with their benchmark. As a result, the opportunity cost from imposing long-only constraints in the allocation is lower here: the minimum volatility and the maximum Sharpe ratio in the largest universe are

respectively 13.80% and 0.37 if short sales are permitted, and respectively 13.86% and 0.33 otherwise.

The effects of the factors on the minimum volatility and the maximum ratios are substantial. Let us consider the case where they are the most limited, i.e. with L/O factors and long-only constraints. The minimum volatility decreases from 20.67% to 13.86% with the two factors, and the maximum Sharpe ratio increases from 0.14 to 0.33. This should not be a surprise, given that Table 4 shows that in the sample, both factors are less volatile than the broad index and have Sharpe ratios approximately 2.5 times as high as that of this benchmark. But an investor having the opportunity to invest in L/S factors would enjoy larger benefits: the minimum volatility and the maximum Sharpe ratio are 10.97% and 0.81, the difference between the long-only and the long-short allocations being invisible at this level of precision.

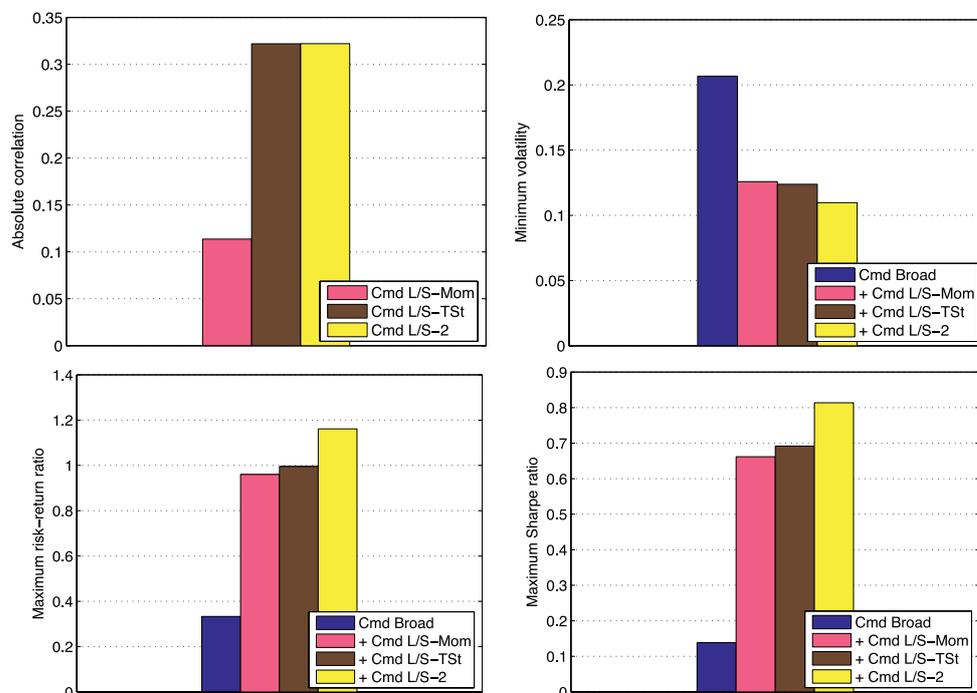
Figure 9: Impact on risk and return indicators of universe extension with commodity factors (1986–2011).

(a) L/O factors.



## 5. Empirical Analysis of Factor Investing

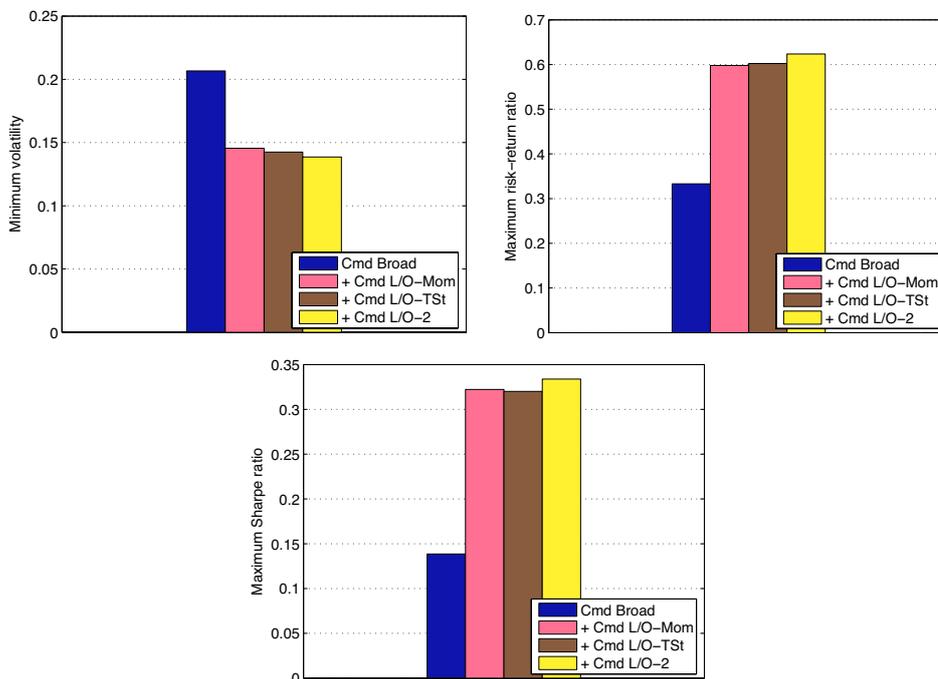
(b) L/S factors.



The broad commodity index is the S&P GSCI and the commodity factor indices proxy for the term structure and momentum factors. The absolute correlation is the absolute value of the correlation between the broad commodity index (the S&P GSCI) and a factor index, or the maximum absolute value of the correlation between the broad index and a portfolio of the two factor indices. The minimum volatility, the maximum risk-return ratio and the maximum Sharpe ratio are computed over a universe that contains only the broad index, or the broad index plus one factor index, or the broad index plus the two indices. The statistics are computed from monthly returns over the whole sample (Jan. 1986 - Sep. 2011) and are annualised.

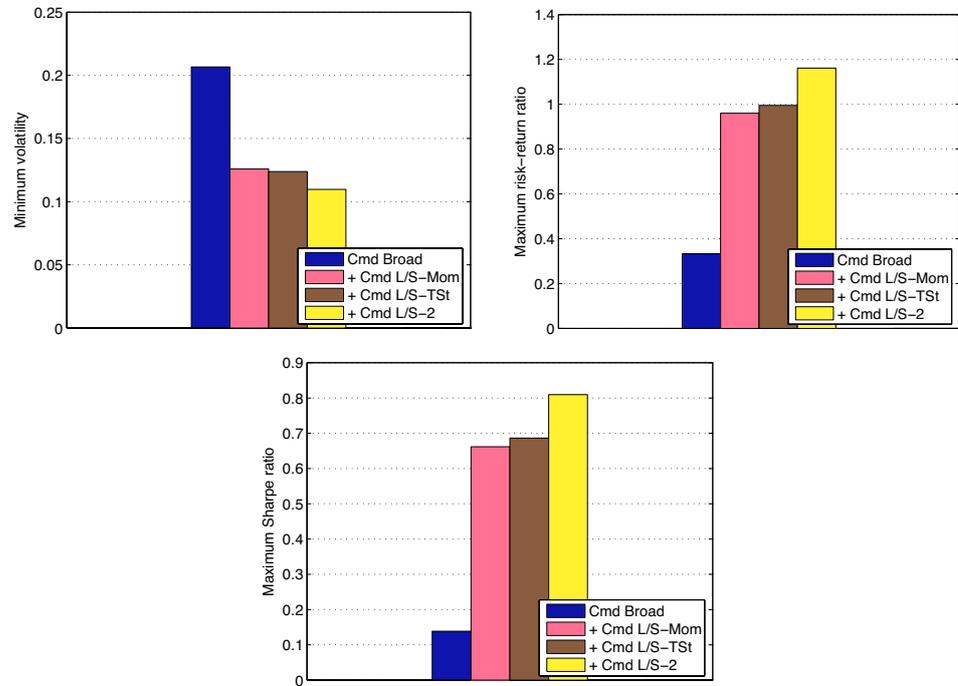
Figure 10: Impact on risk and return indicators of universe extension with commodity factors when short sales are prohibited (1986-2011).

(a) L/O factors.



## 5. Empirical Analysis of Factor Investing

(b) L/S factors.



The broad commodity index is the S&P GSCI and the commodity factor indices proxy for the term structure and momentum factors. The minimum volatility, the maximum risk-return ratio and the maximum Sharpe ratio are computed over a universe that contains only the broad index, or the broad index plus one factor index, or the broad index plus the two indices. The statistics are computed from monthly returns over the whole sample (Jan. 1986 - Sep. 2011) and are annualised.

### Mean-Variance Spanning Tests

Table 19 contains the results of mean-variance spanning tests. In all tests, the null hypothesis that the factors bring no improvement is rejected at least at the 5% level, and in fact at the 1% level when L/S factors are used. For a given set of factors, the F-statistics are lower with L/O

than with L/S factors, which is because the changes in the risk and return statistics are smaller, but the differences are sufficiently large with respect to the sample length to warrant rejection. Hence, the comparison of in-sample long-short efficient frontiers leads to the clear conclusion that commodity factors add value.

Table 19: Mean-variance spanning tests with commodity factors (1986-2011).

Universe	Spanning tests		Tests on MRR		Tests on GMV	
	F-stat	p-value	F-stat	p-value	F-stat	p-value
L/O Factors						
+ L/O-Mom	156.28	0	6.36	0.01	300.95	0
+ L/O-TSt	171.71	0	7.12	0.01	329.72	0
+ L/O-2	76.07	0	3.91	0.02	182.12	0
L/S Factors						
+ L/S-Mom	262.37	0	20.49	0	474.14	0
+ L/S-TSt	276.58	0	22.22	0	496.61	0
+ L/S-2	135.97	0	15.58	0	343.69	0

The original investment universe contains the broad commodity index (the S&P GSCI), and it is extended with one or two commodity factor index/indices (term structure and momentum). For each universe extension, three tests are performed: (1) a spanning test, which is a joint test of the equality of the global minimum variance (GMV) and maximum risk-return ratio (MRR) portfolios before and after the extension; (2) a test of equality of the two MRR portfolios; (3) a test of equality of the two GMV portfolios conditional on the equality of the two MRRs. The table reports the F-statistic of each test as well as the p-value derived from the small-sample distribution of the test statistic.

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### 5.6.2 Out-of-Sample Tests

The next series of tests aim to quantify the economic significance of the results of spanning tests by considering a more realistic context, where portfolios are long-only and do not suffer from the look-ahead bias inherent to in-sample portfolios. The covariance matrix used to form the GMV, RP and MENCH portfolios is estimated from weekly data over a two-year rolling window.

Portfolio analytics are reported in Tables 20 and 21, with a distinction between extensions and substitutions. As for the equity and bond classes, all portfolios containing factors outperform the broad index, whether this index is left or not in the universe. Again, this is a straightforward consequence of the outperformance of

factor indices with respect to the broad benchmark. The benefits in terms of volatility reduction are also spectacular: over the period 1988-2011, the volatility of the S&P GSCI is 21.61%, and all factor portfolios are below 17%, and even 14% for those that give up the broad index. All this implies a large increase in Sharpe ratio. As explained above, the historical Sharpe ratio of the broad index is affected by the large 2008 drawdown, and is only 0.10. This value is at least doubled, and in many cases tripled, by the inclusion of factors. However, due to the large variability in returns, the gains in performance and Sharpe ratio are not statistically significant. Overall, while these results support the idea that factors add value for commodity investors, their magnitude should be

Table 20: Extension of universe with commodity factor indices (1988-2011).

Universe	Avg. return (%)	Volatility (%)	Sharpe ratio	Tracking error (%)	Information ratio	Ann. 1-way Turnover (%)
Broad	5.96	21.61	0.10	0.00	-	0.0
MDC Portfolios						
+ L/O-Mom	7.58	16.47 ***	0.22	7.82	0.21	6.3
+ L/O-TSt	7.06	16.81 ***	0.19	6.94	0.16	5.6
+ L/O-2	7.65	15.20 ***	0.25	9.48	0.18	5.8
GMV Portfolios						
+ L/O-Mom	8.85	14.72 ***	0.34	12.15	0.24	12.3
+ L/O-TSt	7.46	14.62 ***	0.24	11.41	0.13	6.2
+ L/O-2	7.67	14.05 ***	0.27	12.42	0.14	29.5
RP Portfolios						
+ L/O-Mom	7.92	15.76 ***	0.26	9.25	0.21	8.3
+ L/O-TSt	7.19	16.06 ***	0.20	8.29	0.15	6.7
+ L/O-2	7.79	14.71 ***	0.26	10.49	0.17	8.1
MENCH Portfolios						
+ L/O-Mom	8.63	14.83 ***	0.32	11.87	0.23	16.0
+ L/O-TSt	7.88	14.70 ***	0.27	11.41	0.17	14.5
+ L/O-2	8.17	13.82 ***	0.31	12.94	0.17	21.7

The benchmark portfolio is the broad commodity index (the S&P GSCI). The other portfolios contain the broad index plus one or two commodity factors (term structure and momentum). The constituents are weighted according to the following schemes, rebalanced every quarter: MDC is the maximum deconcentration portfolio; GMV is the global minimum variance subject to bounds on weights (1/6 and 1); RP is the risk parity, which equalises the contributions to volatility; MENCH is the maximum MENCH, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix. The tracking error and the information ratio are computed with respect to the broad index. The stars in columns 2 to 4 indicate the significance level in an equality test with respect to the broad index: no star = not significant at the 10% level; \* = 10% level; \*\* = 5% level; \*\*\* = 1% level.

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taken cautiously. Indeed, the broad commodity index displays rather poor performances over the sample period, so that it is an easy-to-beat benchmark. Nevertheless, this should not alter the qualitative message from these tables,

as the profitability of term structure and momentum strategies has been shown to be robust across all empirical studies.

Tracking errors are larger than in the other two classes: they range between

Table 21: Commodity factor indices as substitutes for a broad commodity index (1988-2011).

(a) Performance and risk indicators.

Universe	Avg. return (%)	Volatility (%)	Sharpe ratio	Tracking error (%)	Information ratio
Broad	5.96	21.61	0.10	0.00	-
Single-factor portfolios					
L/O-Mom	8.36	14.03 ***	0.32	15.50	0.15
L/O-TSt	7.57	13.94 ***	0.26	13.80	0.12
Absolute weighting schemes					
MDC L/O-2	8.05	13.43 ***	0.31	14.16	0.15
GMV L/O-2	7.62	13.56 ***	0.27	14.13	0.12
RP L/O-2	7.99	13.43 ***	0.30	14.14	0.14
MENUB L/O-2	7.70	13.47 ***	0.28	14.12	0.12
Relative weighting schemes					
MTE L/O-2	8.46	13.64 ***	0.33	13.94	0.18
RRP L/O-2	8.17	13.44 ***	0.32	14.10	0.16
MENRUB L/O-2	8.51	13.53 ***	0.34	13.96	0.18

(b) Analytics on weights.

Universe	Avg. ENC (%p)	Avg. ENCB (%p)	Avg. ENUB (%p)	Avg. ENRCB (%p)	Avg. ENRUB (%p)	Ann. 1-way Turnover (%)
Absolute weighting schemes						
MDC L/O-2	100.0	99.1	95.7	98.6	93.9	3.0
GMV L/O-2	77.3	78.0	96.9	75.5	88.1	24.6
RP L/O-2	99.3	100.0	97.6	98.3	94.1	5.0
MENUB L/O-2	90.6	92.7	100.0	89.1	91.8	17.9
Relative weighting schemes						
MTE L/O-2	76.6	74.7	87.7	78.3	98.5	20.5
RRP L/O-2	98.9	98.2	95.9	100.0	96.2	6.1
MENRUB L/O-2	85.8	84.2	90.7	88.2	100.0	20.9

The broad commodity index is the S&P GSCI. The other portfolios contain either one commodity factor only (term structure or momentum), or the two factors. In two-factor universes, the constituents are weighted according to the following schemes, rebalanced every quarter: MDC is the maximum deconcentration portfolio; GMV is the global minimum variance subject to bounds on weights (1/6 and 1); RP is the risk parity, which equalises the contributions to volatility; MENUB is the maximum ENUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix. The last three weighting schemes focus on relative risk: MTE minimises the tracking error subject to bounds on weights; RRP is the relative risk parity, which equalises the contributions to the tracking error; MENRUB is the maximum ENRUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix of excess returns. In Panel (a), the tracking error and the information ratio are computed with respect to the broad index. The stars in columns 2 to 4 indicate the significance level in an equality test with respect to the broad index: no star = not significant at the 10% level; \* = 10% level; \*\* = 5% level; \*\*\* = 1% level. In Panel (b), ENC is the effective number of constituents; ENCB is the effective number of correlated bets (effective number of constituent contributions to volatility); ENUB is the effective number of uncorrelated bets; ENRCB is the effective number of relative correlated bets (effective number of constituent contributions to tracking error); ENRUB is the effective number of relative uncorrelated bets (the factors being extracted from the relative covariance matrix). All these indicators are computed at each quarterly rebalancing date and then averaged. They are expressed as percentages of the universe size,  $p = 2$ . The last column reports the annual one-way turnover.

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13% and 16% in substitution portfolios of two factors, and between 10% and 13% in two-factor portfolios that also contain the broad index. This is due to the relatively lower correlations the benchmark with the factors, but also to the high volatility of the benchmark, which is about 1.5 times as high as the volatilities of the two factors (see Table 4). The dispersion of volatilities among equity portfolios was much lower, but a similar effect explains the high tracking error (5.83%) of the term factor with respect to the benchmark index in the bond universe: in this case, it is the factor that is more volatile (about twice as much) than the benchmark, which mechanically increases the tracking error. The difference between the two situations is that in the case of commodities, the correlation between the factor and the benchmark is lower, so that the volatility gap is not compensated. Overall, the high tracking errors suggest that the commodity class is more heterogeneous than the equity and bond classes.

### 5.7 Multi-Class Comparisons

The previous sections have looked into the added value of factors in each asset class considered separately. In this section, we take a multi-class perspective and we analyse the benefits of factor investing for investors who allocate funds to multiple classes. We focus on L/O factors, which are arguably more easily investable in practice, but the analysis could be repeated for L/S factors without difficulty.

#### 5.7.1 In-Sample Tests

The original universe now consists of the three broad equity, bond and commodity indices. We consider extended universes that contain, in addition to these three constituents, the factors of one, two or three class(es). Figure 11 compares risk

and return indicators that characterise the efficient frontiers in these various universes, with or without short-sales constraints. First, the canonical correlations are used to assess how close can be a portfolio of the broad indices and a portfolio of factors. It appears that the equity factors are almost redundant with the broad indices. This is due to the presence of the broad equity index in the original universe: the inclusion of the other two indices is not required to generate such a high correlation (see Panel (a) in Figure 3). Next, the figure enables to evaluate the risk reduction potential of each set of single-class factors in different ways. One can look at the effect on minimum volatility of augmenting the original universe with these factors, but one can also measure the marginal benefits of this set of factors when other factors are already present in the universe. With these criteria, equity and commodity factors appear to have very limited ability to reduce risk: they hardly reduce the minimum volatility of the original universe, and when they are introduced in a universe that already contains the factors of one or two class(es), the marginal decrease in volatility is extremely small. The gains are even more limited when short sales are prohibited. This result contrasts with the single-class comparisons, where equity and commodity factors did show risk reduction abilities with respect to their respective asset class benchmarks (see Panels (b) in Figures 3 and 9). The difference here is that the original universe includes an asset with much lower volatility, which is the bond index: as can be seen from Table 5, its volatility is only 4.87%, while commodity and bond factors have volatilities ranging between 13.91% and 17.78%. Thus, the minimum volatility in the original universe is by definition lower than 4.87%, and even though the introduction of new assets will reduce it, the gain can only be

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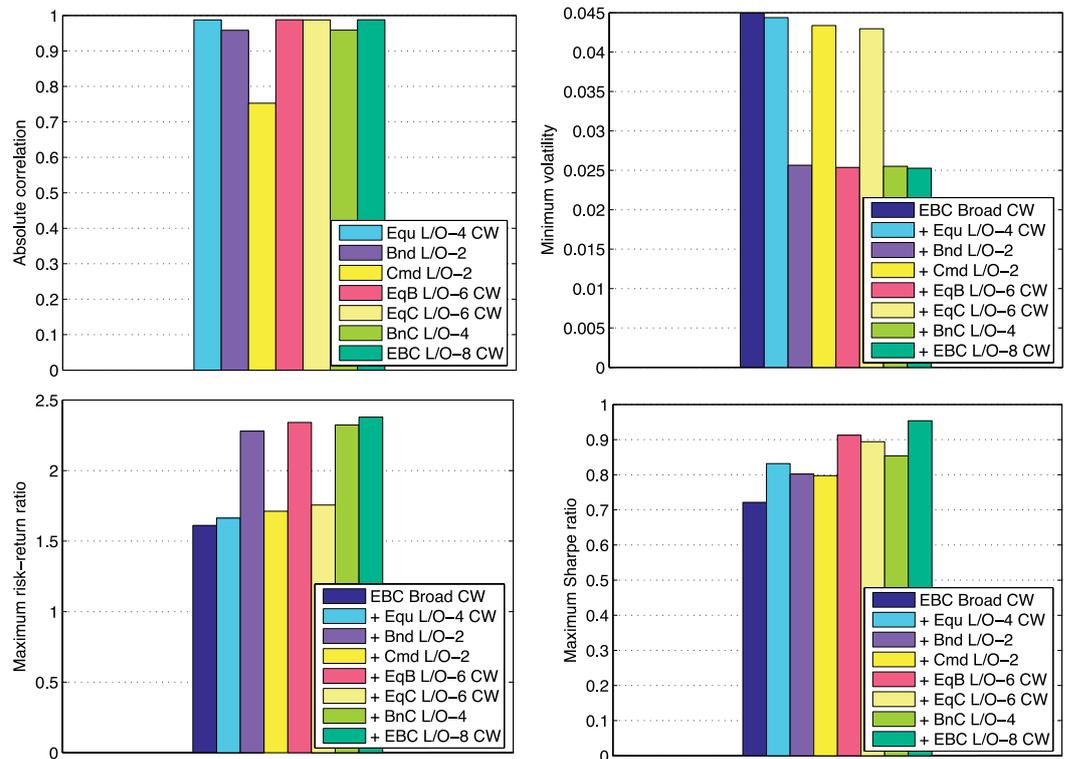
small if the new constituents have large volatilities.

The fact that bonds have both low volatility and relatively high returns in the sample also accounts for the high risk-return ratios and Sharpe ratios in the original universe: the maximum Sharpe ratio of a portfolio of the three asset class indices is

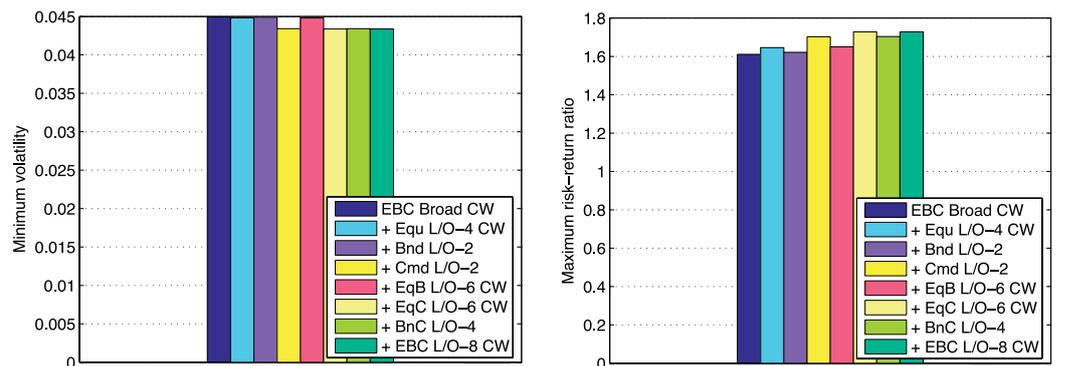
0.72 (whether short sales constraints are imposed or not, since these constraints are not binding here). But introducing factors can improve this value, even if short sales are prohibited: the maximum Sharpe ratio of the 11 constituents (3 asset class indices plus 8 factors) is 0.81 under short sales constraints, and 0.95 without these constraints.

Figure 11: Impact on risk and return indicators of universe extension with long-only equity, bond and commodity factors (1986-2011).

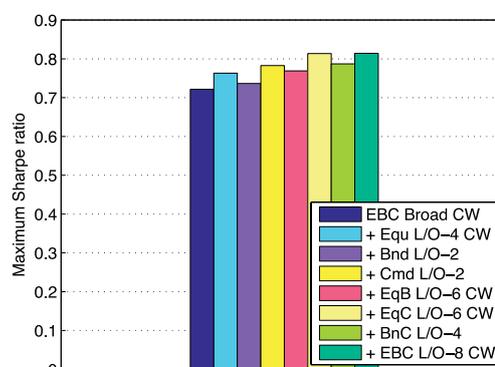
(a) Without short-sales constraints.



(b) With short-sales constraints.



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The absolute correlation in Panel (a) is the absolute value of the correlation between a portfolio of the broad equity, bond and commodity indices, and a portfolio of equity, bond and commodity factors. The four equity factors are size, momentum, volatility and value, the two bond factors are term and credit and the two commodity factors are momentum and term structure. The minimum volatility, the maximum risk-return ratio and the maximum Sharpe ratio are computed over a universe that contains only the three broad indices plus a selection of the eight factors. All factors are long-only, and the equity factors are weighted by capitalisation. The statistics are computed from monthly returns over the whole sample (Jan. 1986 - Dec. 2011) and are annualised.

To assess the statistical significance of all these changes, we conduct mean-variance spanning tests in Table 22. The null of spanning is rejected at the 1% confidence level for all extensions, except for the case where only equity factors are introduced. As mentioned previously, the low reduction in the minimum volatility contributes to this result: it is small (from 4.49% to 4.44%), so that the equality of GMVs can only be rejected at the 6% level. But the main explanation for the high p-value in the spanning test is the lack of evidence against the null of equality between the two MRRs: the F-statistic is small (0.89),

which is insufficient to reject the null at usual confidence levels. On Figure 11, the MRR increases from 1.61 to 1.66, which is small but does not seem to be negligible. However, the sample on which these results are based is shorter than the sample used in the equity tests (309 dates instead of 523), so that it would take a larger variation to achieve statistical significance. On the other hand, the sample is large enough to qualify as significant the small reductions in minimum volatility achieved by introducing commodity factors only (from 4.49% to 4.34%) or equity and commodity factors (from 4.49% to 4.30%). Indeed, for a given

Table 22. Mean-variance spanning tests with equity, bond and commodity factors (1986-2011).

Universe	Spanning tests		Tests on MRR		Tests on GMV	
	F-stat	p-value	F-stat	p-value	F-stat	p-value
+ Equ L/O-4	1.58	0.13	0.89	0.47	2.29	0.06
+ Bnd L/O-2	115.94	0.00	27.02	0.00	249.97	0.00
+ Cmd L/O-2	5.61	0.00	3.48	0.03	7.79	0.00
+ EqB L/O-6	39.74	0.00	9.85	0.00	84.88	0.00
+ EqC L/O-6	2.76	0.00	1.66	0.13	3.89	0.00
+ BnC L/O-4	58.52	0.00	14.45	0.00	124.65	0.00
+ EBC L/O-8	30.02	0.00	7.79	0.00	63.46	0.00

The original investment universe contains three broad asset class indices, equity (the ERI Scientific Beta Long-Term US capitalisation-weighted index), bond (the Barclays US Treasury index) and commodity (the S&P GSCI). It is extended with factor indices from the three asset classes: four equity factors (size, momentum, volatility and value), two bond factors (term and credit) and two commodity factors (term structure and momentum). Equity factors are weighted by capitalisation (CW). For each universe extension, three tests are performed: (1) a spanning test, which is a joint test of the equality of the global minimum variance (GMV) and maximum risk-return ratio (MRR) portfolios before and after the extension; (2) a test of equality of the two MRR portfolios; (3) a test of equality of the two GMV portfolios conditional on the equality of the two MRRs. The table reports the F-statistic of each test as well as the p-value derived from the small-sample distribution of the test statistic.

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sample size, volatility estimates are more accurate than expected return estimates, so that variations in volatility appear more easily to be significant.

### 5.7.2 Out-of-Sample Tests

In the out-of-sample tests, the original universe consists of the three asset class indices, and the equity, bond and commodity factors are introduced separately or simultaneously. As usual, we make a distinction between extension tests, where the original indices are maintained within the universe (Table 23) and substitution tests, where the new universe only contains factors (Table 24).

A first way of reading these tables is to compare portfolios containing factors to portfolios of traditional indices in terms of risk and return analytics. The clearest results relate to average returns: whichever allocation method is adopted, using equity factors always improves this indicator with respect to the benchmark portfolio of the three indices. This result is straightforward, given that the four equity factors outperform all other portfolios included in this study (see Table 5). Bond factors also lead to increases in average return in most cases (the RP and the MENUB portfolios in substitution tests being the two exceptions). For commodities,

Table 23: Extension of universe with equity, bond and commodity factors (1988-2011).

Universe	Avg. return (%)	Volatility (%)	Sharpe ratio	Ann. 1-way Turnover (%)
MDC Portfolios				
EBC Broad	8.50	9.18	0.50	10.3
+ Equ L/O-4	9.56	11.14 ***	0.51	7.5
+ Bnd L/O-2	8.77	6.62 ***	0.73 *	8.4
+ Cmd L/O-2	8.49	10.12 ***	0.45	8.9
+ EqB L/O-6	9.57	9.00	0.63	8.2
+ EqC L/O-6	9.45	10.30 ***	0.54	8.8
+ BnC L/O-4	8.78	7.69 ***	0.63	8.9
+ EBC L/O-8	9.48	8.66 *	0.64	9.0
GMV Portfolios				
EBC Broad	8.18	4.71	0.91	10.8
+ Equ L/O-4	8.47	7.16 ***	0.64	19.1
+ Bnd L/O-2	8.33	4.78	0.92	29.0
+ Cmd L/O-2	8.21	5.39 ***	0.80	17.2
+ EqB L/O-6	8.77	5.41 ***	0.90	18.6
+ EqC L/O-6	8.65	7.48 ***	0.63	34.1
+ BnC L/O-4	8.74	5.00	0.96	27.4
+ EBC L/O-8	8.93	5.40 **	0.93	25.7
RP Portfolios				
EBC Broad	8.65	5.25	0.90	12.7
+ Equ L/O-4	9.50	6.17 ***	0.90	17.6
+ Bnd L/O-2	8.89	5.38	0.92	14.5
+ Cmd L/O-2	8.89	5.49	0.90	18.0
+ EqB L/O-6	9.58 *	5.95 ***	0.95	17.7
+ EqC L/O-6	9.37	6.02 ***	0.91	21.5
+ BnC L/O-4	9.15	5.45	0.96	18.2
+ EBC L/O-8	9.54	5.82 **	0.97	20.8

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MENUB Portfolios				
EBC Broad	8.25	5.24	0.83	15.8
+ Equ L/O-4	8.65	5.91 ***	0.80	41.9
+ Bnd L/O-2	8.48	5.14	0.89	31.5
+ Cmd L/O-2	8.69	5.64 *	0.85	29.3
+ EqB L/O-6	9.02	5.67 **	0.90	72.0
+ EqC L/O-6	8.89	6.00 ***	0.83	51.9
+ BnC L/O-4	8.78	5.38	0.91	50.7
+ EBC L/O-8	9.08	5.65	0.91	77.9

The original investment universe contains three asset class indices (equity, bond and commodity), and it is extended with equity, bond and commodity factor indices. The four equity factors are size, momentum, volatility and value, the two bond factors are term and credit and the two commodity factors are term structure and momentum. The constituents are weighted according to the following schemes, rebalanced every quarter: MDC is the maximum deconcentration portfolio; GMV is the global minimum variance subject to bounds on weights ( $1/(3p)$  and 1, where  $p$  is the nominal number of constituents); RP is the risk parity, which equalises the contributions to volatility; MENUB is maximum ENUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix. The stars in columns 2 to 4 indicate the significance level in an equality test with respect to the portfolio of asset class indices: no star = not significant at the 10% level; \* = 10% level; \*\* = 5% level; \*\*\* = 1% level.

the situation is more contrasted: it is only in extensions that they show ability to generate outperformance in the multi-class portfolio, and only with the GMV, RP and MENUB allocations. This is likely due to the underperformance of commodity factors with respect to the broad equity index in the sample: it is difficult for a portfolio of commodities only to outperform a portfolio containing equities. For volatility, the situation is reversed: most combinations of factors increase the volatility with respect to the benchmark portfolio. Indeed, with the exceptions of the commodity class and the low volatility factor in the equity class, factors tend to be more volatile than the corresponding asset class index. Finally, the impact on Sharpe ratio strongly depends on the context: if the asset class indices are kept in the universe, most combinations of factors improve the Sharpe ratio, while very few portfolios of factors only do better than the portfolios of traditional indices. This may be due to the very high Sharpe ratio of the broad bond index in the sample (0.67): all equity and commodity factors have lower Sharpe ratios (see Table 5). Thus, the advantages of factors in terms of this indicator may only be visible if the bond index is left in the universe.

A second way of looking at the results is to search which factor portfolios rank first in terms of performance, volatility or Sharpe ratio. It turns out that those containing equity factors often outperform the others. In fact, for each allocation method, the two universes that lead to the highest average return contain equity factors. Similarly, for each allocation method, the two universes that imply the lowest volatility contain bond factors, and these are actually the same: they consist respectively of bond factors only and of bond and commodity factors. The inclusion of bond factors also appears as a necessary condition for the maximisation of the ex-post Sharpe ratio: given a weighting scheme, the four universes with the highest Sharpe ratios contain these factors.

Third, the tables can be used to assess the marginal gains from introducing new factors in a universe that already contains factors from another class or from other classes. Most of the time, adding equity factors improves the average return. The only exception is the MENUB weighting scheme in substitution tests, where the portfolio of equity, bond and commodity factors has slightly lower ex-post return

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than the bond-commodity portfolio (9.06% versus 9.07%). Similarly, introducing bond factors reduces volatility and also improves the Sharpe ratio. In fact, for most choices of weighting scheme and factors, combining factors from two classes leads to higher a Sharpe ratio than the ratios of the portfolios that select factors from a single class. Moreover, in most cases, combining factors from the three classes generates a further increase in Sharpe ratio. Interestingly, the exceptions are mainly observed for the MDC weighting scheme: for instance, in substitutions, the MDC portfolios that mix bond factors with equity and/or commodity factors have lower Sharpe ratios than the portfolio that contains only bond factors, and a similar result holds for extensions. This suggests that the MDC scheme does not enable to take the most of the diversification properties of factors with respect to each other. Indeed, this allocation method focuses on diversification of the dollar contributions of constituents, regardless

of their volatilities and correlations. Hence, the MDC scheme treats all factors as if they were indistinguishable in terms of risk and correlations. Within a class, volatilities have the same order of magnitude and correlations are high, but across classes, correlations are lower and volatilities can vary substantially (see Table 5). This makes the assumption of uniformity hardly appropriate for a multi-class allocation exercise.<sup>34</sup>

Table 25 contains analytics computed from weights. For each universe, the two weighting schemes with the highest concentration of dollar positions are the GMV and the MENUB. But the MENUB compensates this concentration by a more balanced allocation to the uncorrelated factors extracted from the covariance matrix. In universes mixing portfolios from multiple classes, the MDC portfolio has an effective number of contributions to volatility comprised between 70% and

34 - The variation in volatility across asset classes is also the reason why AMP13 weight each single-class factor by the reciprocal of its volatility to construct the multi-class factors: this procedure ensures that all asset classes have approximately the same contribution to the risk of the multi-class portfolio.

Table 24: Equity, bond and commodity factors as substitutes for broad asset class indices – Performance and risk statistics (1988-2011).

Universe	Avg. return (%)	Volatility (%)	Sharpe ratio
MDC Portfolios			
EBC Broad	8.50	9.18	0.50
Equ L/O-4	9.98	14.39 ***	0.42
Bnd L/O-2	8.65	6.92 ***	0.68
Cmd L/O-2	8.09	14.01 ***	0.30
EqB L/O-6	9.94	10.09 *	0.60
EqC L/O-6	9.81	11.54 ***	0.51
BnC L/O-4	8.87	7.88 ***	0.63
EBC L/O-8	9.78	8.97	0.65
GMV Portfolios			
EBC Broad	8.18	4.71	0.91
Equ L/O-4	9.20	13.51 ***	0.39 *
Bnd L/O-2	8.25	5.88 ***	0.74
Cmd L/O-2	7.90	14.25 ***	0.28 **
EqB L/O-6	9.61	6.54 ***	0.87
EqC L/O-6	8.96	10.86 ***	0.46
BnC L/O-4	8.86	5.83 ***	0.85
EBC L/O-8	9.28	6.09 ***	0.88

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RP Portfolios			
EBC Broad	8.65	5.25	0.90
Equ L/O-4	9.86	14.24 ***	0.42 *
Bnd L/O-2	8.38	6.42 ***	0.70
Cmd L/O-2	8.13	14.03 ***	0.30 **
EqB L/O-6	10.22	7.50 ***	0.84
EqC L/O-6	9.41	11.13 ***	0.49
BnC L/O-4	9.58	6.35 ***	0.89
EBC L/O-8	10.02	6.81 ***	0.90
MENUB Portfolios			
EBC Broad	8.25	5.24	0.83
Equ L/O-4	9.24	13.37 ***	0.40 *
Bnd L/O-2	7.84	5.61	0.70
Cmd L/O-2	8.08	14.13 ***	0.29 **
EqB L/O-6	8.82	6.49 ***	0.76
EqC L/O-6	9.10	10.61 ***	0.49
BnC L/O-4	9.07	5.87	0.88
EBC L/O-8	9.06	6.21 **	0.83

*The original investment universe contains three broad asset class indices (equity, bond and commodity). The other portfolios contain factors from one, two or three asset class(es). The four equity factors are size, momentum, volatility and value, the two bond factors are term and credit and the two commodity factors are term structure and momentum. The constituents are weighted according to the following schemes, rebalanced every quarter: MDC is the maximum deconcentration portfolio; GMV is the global minimum variance subject to bounds on weights ( $1/(3p)$  and  $1$ , where  $p$  is the nominal number of constituents); RP is the risk parity, which equalises the contributions to volatility; MENUB is maximum ENUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix. The stars in columns 2 to 4 indicate the significance level in an equality test with respect to the portfolio of asset class indices: no star = not significant at the 10% level; \* = 10% level; \*\* = 5% level; \*\*\* = 1% level.*

35 - One difference between Deguest et al. (2013) and our paper is that they extract the uncorrelated factors by principal component analysis, while we use minimum linear torsion.

85%, and conversely, the RP portfolio has an effective number of constituents between 75% and 91%. Hence, minimizing the dispersion of risk contributions and minimizing the dispersion of dollar contributions are two objectives that cannot be simultaneously achieved here. This is due to the heterogeneous volatilities and correlations of the constituents from different asset classes.

Finally, the ranking of the four weighting schemes by turnover shows some regularities. For all universes, the MDC portfolio has the lowest turnover, while for most of them, the MENUB has the largest. The turnover of the MENUB sometimes reaches extremely large values, such as 77.9% per year for the largest universe (consisting of the three broad indices and

the eight factors). Such large numbers are not unusual for this strategy: Deguest et al. (2013) report turnovers greater than 80% for multi-class portfolios designed to maximise the effective number of uncorrelated bets.<sup>35</sup> The GMV and the RP rank between the MDC and the MENUB, with a tendency for the GMV to have higher turnovers than the RP.

### 5.7.3 Policy-Neutral Portfolios

In the absence of a consensual multi-class benchmark, many institutional investors will define a "policy portfolio" against which they measure the performance of a multi-class strategy. The policy portfolio is very often defined as a fixed-mix allocation to various asset classes, so we give ourselves a fixed-mix allocation consisting of 60% stocks, 30% bonds and 10% commodities.

## 5. Empirical Analysis of Factor Investing

Table 25: Equity, bond and commodity factors as substitutes for broad asset class indices – Analytics on weights (1988-2011).

Universe	Avg. ENC (%p)	Avg. ENCB (%p)	Avg. ENUB (%p)	Ann. 1-way Turnover (%)
MDC Portfolios				
EBC Broad	100.0	62.7	62.0	10.3
Equ L/O-4	100.0	98.3	88.6	3.6
Bnd L/O-2	100.0	91.3	69.0	2.7
Cmd L/O-2	100.0	98.6	94.9	2.9
EqB L/O-6	100.0	76.9	67.3	6.6
EqC L/O-6	100.0	85.0	78.0	7.5
BnC L/O-4	100.0	75.8	69.1	7.4
EBC L/O-8	100.0	71.4	63.9	8.3
GMV Portfolios				
EBC Broad	58.8	79.1	81.1	10.8
Equ L/O-4	49.4	52.4	94.1	30.0
Bnd L/O-2	70.2	80.5	82.4	5.3
Cmd L/O-2	77.5	78.5	97.2	32.4
EqB L/O-6	50.4	72.2	71.3	18.5
EqC L/O-6	54.8	61.1	81.7	44.9
BnC L/O-4	50.9	61.0	72.2	20.3
EBC L/O-8	51.1	68.2	67.9	29.7
RP Portfolios				
EBC Broad	71.9	100.0	92.3	12.7
Equ L/O-4	98.4	100.0	90.8	6.2
Bnd L/O-2	92.6	100.0	75.9	6.1
Cmd L/O-2	99.0	100.0	97.4	7.0
EqB L/O-6	75.6	100.0	79.6	16.2
EqC L/O-6	90.4	100.0	91.1	16.1
BnC L/O-4	85.3	100.0	81.6	15.5
EBC L/O-8	76.3	100.0	81.7	21.5
MENUM Portfolios				
EBC Broad	68.1	90.2	100.0	15.8
Equ L/O-4	55.7	57.6	98.6	39.1
Bnd L/O-2	57.6	59.7	89.8	8.5
Cmd L/O-2	89.8	92.0	100.0	25.0
EqB L/O-6	38.2	49.5	93.3	36.3
EqC L/O-6	59.9	60.9	98.6	45.9
BnC L/O-4	51.6	67.2	94.9	25.2
EBC L/O-8	40.4	52.8	94.5	46.7

The original investment universe contains three broad asset class indices (equity, bond and commodity). The other portfolios contain factors from one, two or three asset class(es). The four equity factors are size, momentum, volatility and value, the two bond factors are term and credit and the two commodity factors are term structure and momentum. The constituents are weighted according to the following schemes, rebalanced every quarter: MDC is the maximum deconcentration portfolio; GMV is the global minimum variance subject to bounds on weights ( $1/(3p)$  and 1, where  $p$  is the nominal number of constituents); RP is the risk parity, which equalises the contributions to volatility; MENUM is maximum ENUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix. ENC is the effective number of constituents; ENCB is the effective number of contributions of assets to volatility; ENUB is the effective number of contributions of uncorrelated factors extracted from the covariance matrix. These indicators are computed at each rebalancing dates and then averaged over the dates. They are reported as percentages of the nominal number of constituents,  $p$ . The last column contains the annual one-way turnover.

## 5. Empirical Analysis of Factor Investing

These weights do not aim to achieve any form of optimality, and they are chosen because standard policy portfolios are dominated by equities, with the rest being allocated to fixed income and real assets. In this section, we consider replacing the index of each asset class by a combination of factors from the same class, and we look at the impact of the substitution on the performance. In other words, we compute portfolios of the eight factors that respect a "policy neutrality constraint". In mathematical notations, the optimisation programs written in Section 3.1 are solved subject to the following constraints on weights:

$$\begin{cases} \mathbf{x} \geq \mathbf{0}_K \\ \sum_{i \in \mathcal{F}_S} x_i = 0.6 \\ \sum_{i \in \mathcal{F}_B} x_i = 0.3 \\ \sum_{i \in \mathcal{F}_C} x_i = 0.1 \end{cases},$$

where  $\mathcal{F}_S$ ,  $\mathcal{F}_B$  and  $\mathcal{F}_C$  denote respectively the sets of stock factors, bond factors and commodity factors.

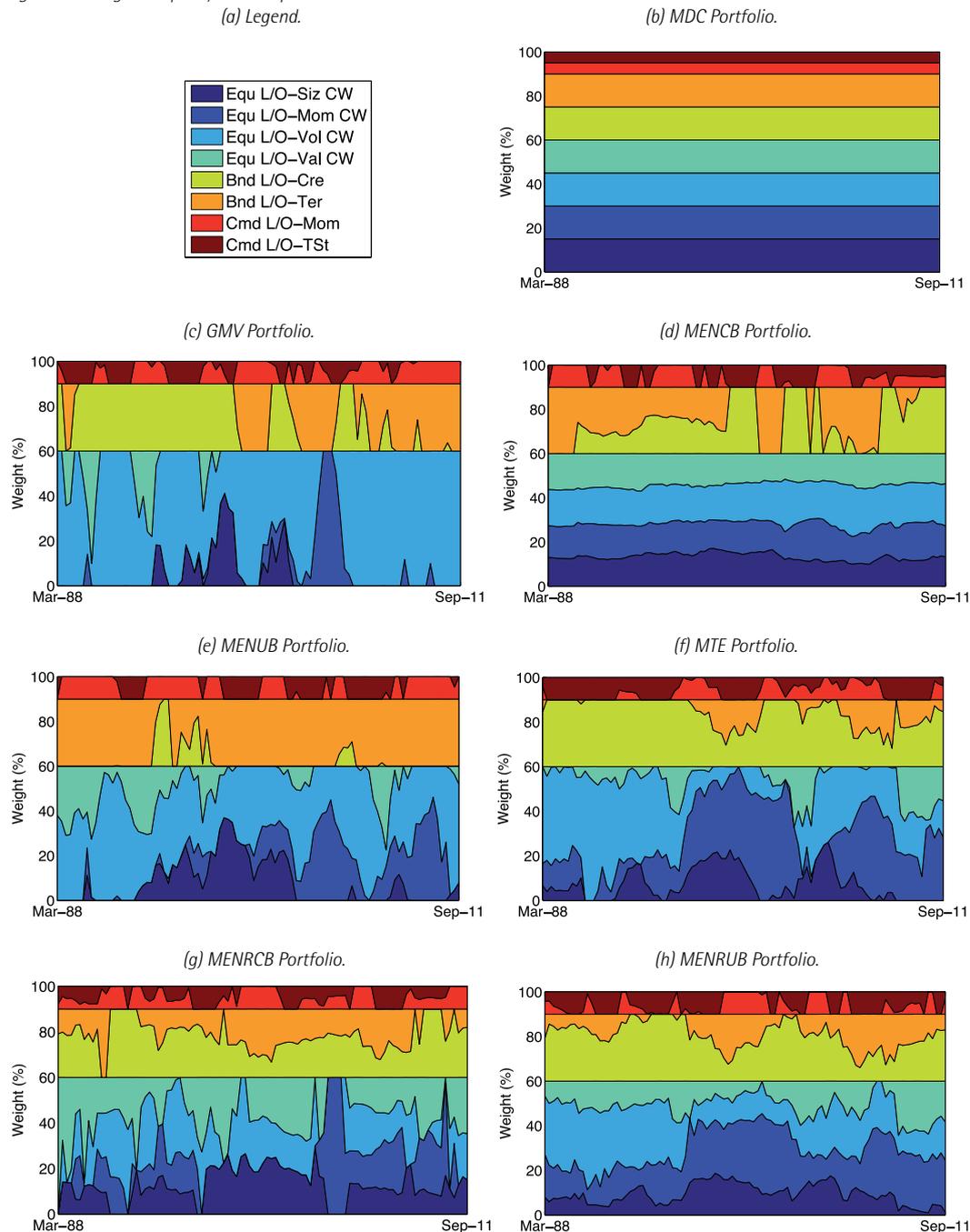
The simplest weighting scheme is the MDC: it is straightforward to verify that the MDC portfolio subject to the neutrality constraint assigns equal weights to factors within each asset class in such a way as to respect the neutrality constraint, that is 15% to each equity or bond factor and 5% to each commodity factor. The next three weighting schemes focus on volatility (GMV, MENCB and MENUB) and the last three focus on relative risk, measured with respect to the policy portfolio (MTE, MENRCB and MENRUB). We recall that the MENCB and the MENUB are portfolios that aim to be as close as possible to risk parity, by minimizing either the dispersion of contributions to volatility of correlated constituents (MENCB) or of uncorrelated factors (MENUB), subject to the neutrality

constraints. Their relative equivalents, in which risk is measured by the tracking error, are the MENRCB and the MENRUB. Figure 12 shows the resulting compositions of the factor portfolios.

As can be seen from Table 26, the substitution of the asset class indices by factors in the policy portfolio has a clear positive effect on performance, and also on the Sharpe ratio, even though the factor portfolios have higher volatilities. These gains are obtained at the cost of deviations from the benchmark, but the tracking errors are not huge. All of them are less than 4%, and the relative schemes, namely the MTE, MENRCB and MENRUB portfolios, display the lowest figures, less than 3%. Each of them has also a higher information ratio than its absolute counterpart. The bottom panel in the table displays analytics on weights. The effective numbers of constituents or contributions to volatility and tracking error provide various measures of diversification. The ENC, ENCB and ENUB are maximised respectively by the MDC, the MENCB and the MENUB portfolios. Without the policy constraints, the maximum deconcentration ratios were equal or close to 100% (see Table 25), but they are lower here, especially for the MENCB and the MENUB: these portfolios achieve ratios of about 65% only. This indicates that the neutrality constraints sometimes conflict with the objective of maximising diversification in a certain sense. In particular, they prevent from achieving perfect risk parity. Finally, the turnover figures give a sense of the relative magnitude of transaction costs across strategies. As in the previous experiments, the GMV and the MENUB have the highest turnovers.

## 5. Empirical Analysis of Factor Investing

Figure 12: Weights of policy-neutral portfolios.



All portfolios are computed subject to a policy neutrality constraint: the total allocations to equity, bond and commodity factors must be respectively 60%, 30% and 10%. The equity factors are size, momentum, volatility and value, the bond factors are term and credit and the commodity factors are momentum and term structure. MDC is the maximum deconcentration portfolio; GMV is the global minimum variance subject to bounds on weights (1/24 and 3/8); RP is the risk parity, which equalises the contributions to volatility; MENUB is the maximum ENUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix. The last three weighting schemes focus on relative risk with respect to the policy portfolio, which has weights 60%, 30% and 10% in the broad equity, bond and commodity indices: MTE minimises the tracking error subject to bounds on weights; RRP is the relative risk parity, which equalises the contributions to the tracking error; MENRUB is the maximum ENRUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix of excess returns.

## 5. Empirical Analysis of Factor Investing

Table 26: Substitution of asset class indices by factor indices in policy portfolio (1988–2011).

(a) Performance and risk indicators.

Universe	Avg. return (%)	Volatility (%)	Sharpe ratio	Tracking error (%)	Information ratio
Policy portfolio	8.82	9.44	0.52	0.00	–
Absolute weighting schemes					
MDC L/O-8	9.90 *	9.46	0.63 *	2.57	0.42
GMV L/O-8	9.54	8.78 ***	0.64	3.22	0.22
MENCB L/O-8	9.37	9.56	0.57	2.68	0.21
MENUB L/O-8	9.76	8.82 ***	0.66	3.55	0.26
Relative weighting schemes					
MTE L/O-8	9.74 *	9.24	0.63 *	2.23	0.41
MENRCB L/O-8	9.56	9.57	0.59	2.90	0.25
MENRUB L/O-8	9.67 *	9.36	0.62 *	2.25	0.38

(b) Analytics on weights.

Universe	Avg. ENC (% $\rho$ )	Avg. ENCB (% $\rho$ )	Avg. ENUB (% $\rho$ )	Avg. ENRCB (% $\rho$ )	Avg. ENRUB (% $\rho$ )	Ann. 1-way Turnover (%)
Policy portfolio	72.5	40.9	43.3	–	–	7.5
Absolute weighting schemes						
MDC L/O-8	89.3	61.1	54.8	44.9	57.7	7.4
GMV L/O-8	54.2	38.0	60.1	25.5	45.0	35.3
MENCB L/O-8	74.4	64.8	57.3	40.2	57.9	29.2
MENUB L/O-8	47.0	36.3	65.2	21.9	41.0	49.4
Relative weighting schemes						
MTE L/O-8	60.6	40.4	53.5	42.5	62.7	29.0
MENRCB L/O-8	64.3	45.1	55.3	63.2	63.5	68.2
MENRUB L/O-8	69.1	48.7	54.8	50.3	86.3	38.6

The benchmark portfolio is a policy portfolio made of 60% equities, 30% bonds and 10% commodities. Each asset class is represented by a broad index: a broad US capitalisation-weighted equity index, the Barclays US Treasury index and the S&P GSCI. The other portfolios combine eight factors: the four equity factors are size, momentum, volatility and value; the two bond factors are term and credit; the two commodity factors are term structure and momentum. All these portfolios are subject to a policy neutrality constraint, i.e. they have the same allocation to each asset class as the policy portfolio. MDC is the maximum deconcentration portfolio; GMV is the global minimum variance subject to bounds on weights (1/24 and 3/8); RP is the risk parity, which equalises the contributions to volatility; MENUB is the maximum ENUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix. The last three weighting schemes focus on relative risk with respect to the policy portfolio, which has weights 60%, 30% and 10% in the broad equity, bond and commodity indices: MTE minimises the tracking error subject to bounds on weights; RRP is the relative risk parity, which equalises the contributions to the tracking error; MENRUB is the maximum ENRUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix of excess returns. In Panel (a), the tracking error and the information ratio are computed with respect to the policy portfolio. The stars in columns 2 to 4 indicate the significance level in an equality test with respect to the portfolio of asset class indices: no star = not significant at the 10% level; \* = 10% level; \*\* = 5% level; \*\*\* = 1% level. In Panel (b), ENC is the effective number of constituents; ENCB is the effective number of contributions of constituents to volatility; ENUB is the effective number contributions of uncorrelated factors extracted from the covariance matrix; ENRCB is the effective number of contributions of constituents to tracking error; ENRUB is the effective number of uncorrelated bets to tracking error. All these indicators are computed at each quarterly rebalancing date and then averaged. They are expressed as percentages of the universe size ( $p = 3$  for the policy portfolio,  $p = 8$  for the others). The last column reports the annual one-way turnover.

### 5.8 Multi-Class Factors in Allocation Decisions

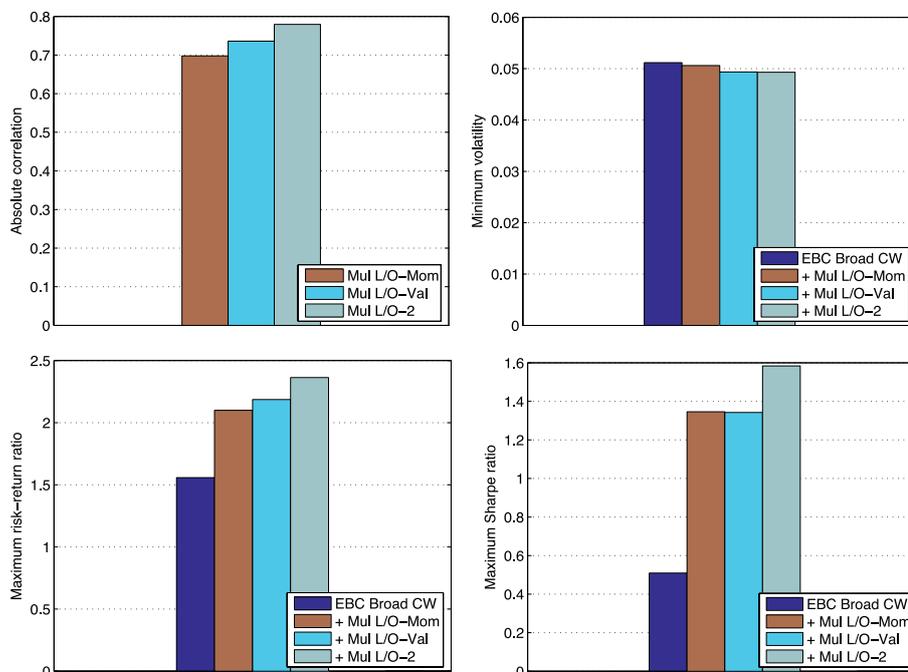
In the previous experiments, factors were formed with securities from one asset class at a time. Instead of combining single-class

factors, one may consider using multi-class factors, which attempt to capture common rewarded sources of risk in various classes. In this section, we briefly study the added value of multi-class factors inspired by

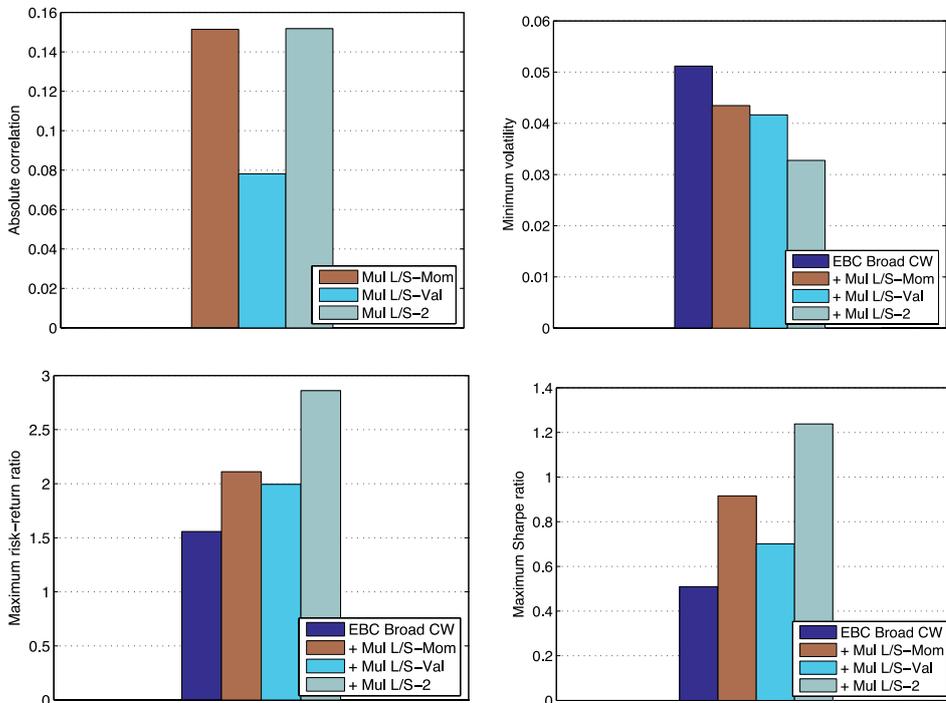
## 5. Empirical Analysis of Factor Investing

Figure 13: Impact on risk and return indicators of universe extension with multi-class factors (1976-2011).

(a) L/O factors.



(b) L/S factors.

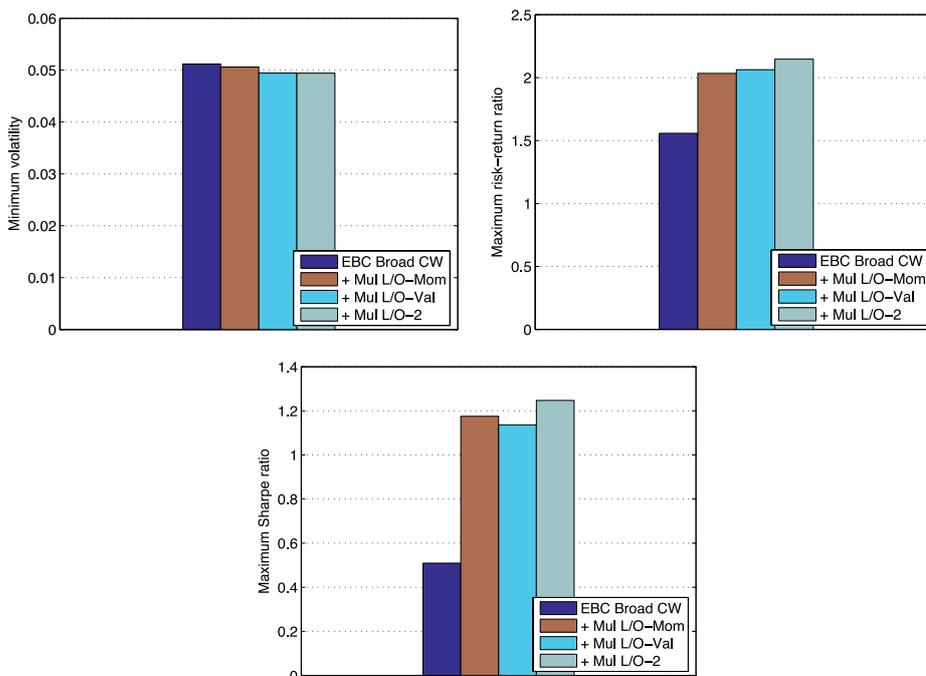


The original universe consists of the three broad asset class indices (equity, bond and commodity), and it is extended with multi-class momentum and value factors. These factors are computed as combinations of single-class value and momentum factors from each asset class, the methodology being adapted from Asness et al. (2013). The absolute correlation in Panel (a) is the absolute value of the correlation between a portfolio of the broad equity, bond and commodity indices, and a portfolio of multi-class factors. The minimum volatility, the maximum risk-return ratio and the maximum Sharpe ratio are computed over a universe that contains the three broad indices plus one or two factor(s). The statistics are computed from monthly returns over the whole sample (Jan. 1976 - Dec. 2011) and are annualised.

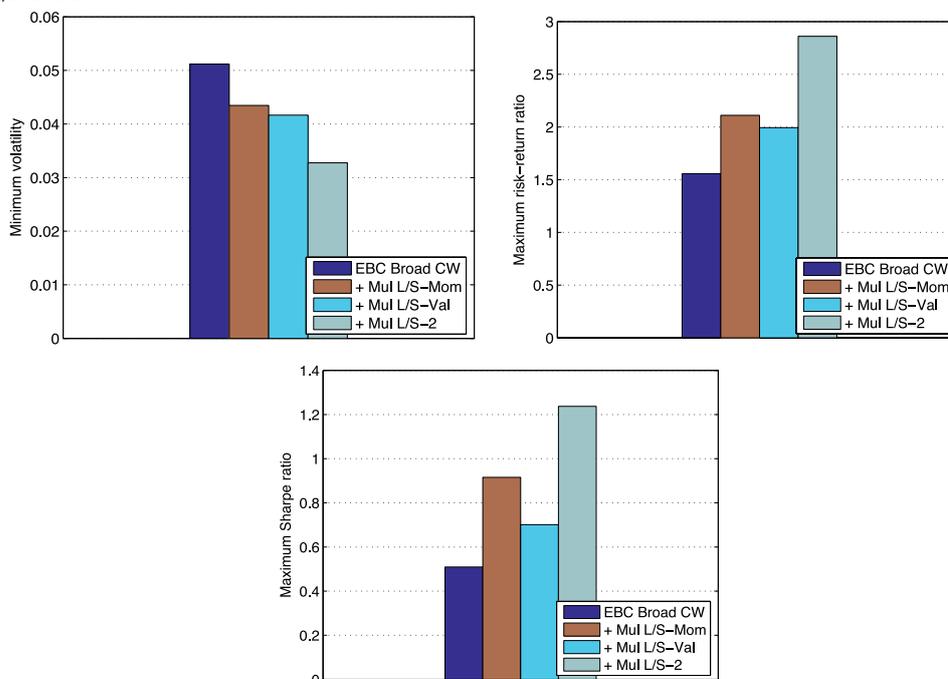
## 5. Empirical Analysis of Factor Investing

Figure 14: Impact on risk and return indicators of universe extension with multi-class factors in the presence of short sales constraints (1976-2011).

(a) L/O factors.



(b) L/S factors.



The original universe consists of the three broad asset class indices (equity, bond and commodity), and it is extended with multi-class momentum and value factors. These factors are computed as combinations of single-class value and momentum factors from each asset class, the methodology being adapted from Asness et al. (2013). The absolute correlation in Panel (a) is the absolute value of the correlation between a portfolio of the broad equity, bond and commodity indices, and a portfolio of multi-class factors. The minimum volatility, the maximum risk-return ratio and the maximum Sharpe ratio are computed over a universe that contains the three broad indices plus one or two factor(s), subject to a constraint of long-only positions in the constituents. The statistics are computed from monthly returns over the whole sample (Jan. 1976 - Dec. 2011) and are annualised.

## 5. Empirical Analysis of Factor Investing

the value and momentum everywhere factors of AMP13. The original universe is the same as before, and still consists of the three asset class indices.

### 5.8.1 In-Sample Tests

Figures 13 and 14 show the impact of these additional factors on the risk and return indicators. It appears that the L/O factors are of little help in terms of volatility reduction: the minimum volatility decreases from 5.12% to 4.93%, a value that is almost unchanged (4.94%) when short sales constraints are imposed. The inclusion of the two L/S factors leads to a more substantial decrease in risk, since the minimum volatility falls to 3.28%, whether short sales are prohibited or not. In the mean-variance spanning tests (see Table 27), the benefits of L/O factors in the GMV appear to be less significant than those of L/S factors. The volatility reduction achieved with the L/O momentum factor alone is even so small that the null hypothesis of a zero gain cannot be rejected. If the null hypothesis is indeed true, this means that the GMV portfolio of the three broad indices and the L/O momentum factor has zero weight in the factor.

The ability of multi-class factors to increase the maximum risk-return ratio and Sharpe ratio is more apparent in the figures. In particular, the gains in Sharpe ratio achieved with one factor only are impressive: while the maximum Sharpe ratio of a long-only or long-short portfolio of the three asset class indices is 0.51, the inclusion of any of the two factors raises this indicator above 1. But these large values are due to the unusually high Sharpe ratios of the factors and, to a lesser extent, to the high in-sample Sharpe ratios of bond portfolios (see Table 6). For instance, the L/O MSR portfolio of the three indices and the momentum factor has weights of 27.4% and 72.6% respectively in the bond index and the factor (composition not shown in the figure). The two constituents have respective Sharpe ratios of 0.45 and 1.28, which accounts for the high Sharpe ratio of the combination.<sup>36</sup> As a result, the formal tests have no difficulty rejecting the null hypothesis of no change in the MRR, and this leads to clear rejection of the spanning hypothesis.

36 - Note that the Sharpe ratios of the constituents extracted from Table 6 are not directly comparable to that of the portfolio as it can be read from Figure 14, because different formulas are used to estimate these ratios. In the tables, the Sharpe ratio is computed as the difference between the annualised geometric returns of the portfolio and the risk-free asset, and in the figure, the MSR is obtained by applying (5.4). Nevertheless, the three Sharpe ratios remain high, whichever convention is adopted.

Table 27: Mean-variance spanning tests with multi-class factors (1976-2011).

Universe	Spanning tests		Tests on MRR		Tests on GMV	
	F-stat	p-value	F-stat	p-value	F-stat	p-value
L/O Factors						
+ L/O-Mom	28.97	0.00	57.93	0.00	0.00	0.99
+ L/O-Val	37.42	0.00	68.85	0.00	5.17	0.02
+ L/O-2	23.68	0.00	46.06	0.00	3.25	0.04
L/S Factors						
+ L/S-Mom	85.23	0.00	59.38	0.00	97.61	0.00
+ L/S-Val	107.95	0.00	45.32	0.00	154.41	0.00
+ L/S-2	120.56	0.00	83.95	0.00	162.11	0.00

The original investment universe contains the three asset class indices (equity, bond and commodity), and it is extended with one or two multi-class factor index/indices, proxying respectively for the value and momentum factors. For each universe extension, three tests are performed: (1) a spanning test, which is a joint test of the equality of the global minimum variance (GMV) and maximum risk-return ratio (MRR) portfolios before and after the extension; (2) a test of equality of the two MRR portfolios; (3) a test of equality of the two GMV portfolios conditional on the equality of the two MRRs. The table reports the F-statistic of each test as well as the p-value derived from the small-sample distribution of the test statistic.

## 5. Empirical Analysis of Factor Investing

### 5.8.2 Out-of-Sample Tests

In Tables 28 and 29, the factor portfolios outperform a combination of the three asset class indices, both in extensions and substitutions, and regardless of the weighting scheme. Some average returns are impressively high (more than 17% per year for all substituted portfolios). Given that all portfolios have moderate volatilities (between 5% and 10%), the gains with respect to a portfolio of broad indices is statistically significant at the 1% level in all cases. Similarly, all increases in Sharpe ratios are significant at this level, but as pointed previously, the very high ratios, often greater than 1, should not be taken at face value. Indeed, the multi-

class factors are unlikely to satisfy realistic implementation constraints, so that they are more theoretical than investable portfolios.

Nevertheless, the results presented here provide encouraging preliminary evidence that indices designed to capture common patterns across asset classes can improve the opportunity set. The design of investable forms of such indices is an interesting topic for future applied research, since combining multi-class factors is a more parsimonious investment approach than mixing single-class factors. As an example, each of the two multi-class factors used in this paper replaces three single-class factors.

Table 28: Extension of universe with multi-class factors (1978-2011).

Universe	Avg. return (%)	Volatility (%)	Sharpe ratio	Ann. 1-way Turnover (%)
MDC Portfolios				
EBC Broad	10.11	9.09	0.49	9.5
+ L/O-Mom	12.12 ***	8.75 **	0.74 ***	8.3
+ L/O-Val	11.97 ***	8.50 ***	0.75 ***	8.4
+ L/O-2	13.21 ***	8.41 ***	0.90 ***	7.5
GMV Portfolios				
EBC Broad	9.00	5.37	0.63	10.8
+ L/O-Mom	9.76 ***	5.31	0.78 ***	12.6
+ L/O-Val	9.92 ***	5.21 ***	0.83 ***	15.2
+ L/O-2 1	1.05 ***	5.41	1.00 ***	16.7
RP Portfolios				
EBC Broad	9.50	5.92	0.66	13.1
+ L/O-Mom	11.08 ***	6.09 *	0.90 ***	15.7
+ L/O-Val	11.03 ***	5.85	0.93 ***	14.9
+ L/O-2	12.04 ***	6.06	1.06 ***	16.9
MENUM Portfolios				
EBC Broad	8.99	5.75	0.59	16.0
+ L/O-Mom	10.57 ***	5.79	0.86 ***	23.4
+ L/O-Val	10.75 ***	5.52 *	0.93 ***	21.7
+ L/O-2	11.64 ***	5.64	1.07 ***	27.7

The original investment universe contains three asset class indices (equity, bond and commodity), and it is extended with multi-class factor indices, which proxy respectively for the value and momentum factors. The constituents are weighted according to the following schemes, rebalanced every quarter: MDC is the maximum deconcentration portfolio; GMV is the global minimum variance subject to bounds on weights ( $1/(3p)$  and 1, where  $p$  is the nominal number of constituents); RP is the risk parity, which equalises the contributions to volatility; MENUM is maximum ENUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix. The stars in columns 2 to 4 indicate the significance level in an equality test with respect to the portfolio of asset class indices: no star = not significant at the 10% level; \* = 10% level; \*\* = 5% level; \*\*\* = 1% level.

## 5. Empirical Analysis of Factor Investing

Table 29: Multi-class factors as substitutes for broad asset class indices – Performance and risk statistics (1978-2011).

Universe	Avg. return (%)	Volatility (%)	Sharpe ratio	Ann. 1-way Turnover (%)
MDC Portfolios				
EBC Broad	10.11	9.09	0.49	9.5
L/O-Mom	18.04 ***	10.30 **	1.21 ***	0.0
L/O-Val	17.38 ***	9.12	1.29 ***	0.0
L/O-2	17.78 ***	8.96	1.36 ***	2.4
GMV Portfolios				
EBC Broad	9.00	5.37	0.63	10.8
L/O-Mom	18.04 ***	10.30 ***	1.21 **	0.0
L/O-Val	17.38 ***	9.12 ***	1.29 ***	0.0
L/O-2	18.36 ***	9.09 ***	1.40 ***	29.6
RP Portfolios				
EBC Broad	9.50	5.92	0.66	13.1
L/O-Mom	18.04 ***	10.30 ***	1.21 ***	0.0
L/O-Val	17.38 ***	9.12 ***	1.29 ***	0.0
L/O-2	17.85 ***	8.95 ***	1.37 ***	7.3
MENUM Portfolios				
EBC Broad	8.99	5.75	0.59	16.0
L/O-Mom	18.04 ***	10.30 ***	1.21 ***	0.0
L/O-Val	17.38 ***	9.12 ***	1.29 ***	0.0
L/O-2	18.02 ***	8.98 ***	1.38 ***	24.7

The original investment universe contains three broad asset class indices (equity, bond and commodity), and it is replaced by universes that contain one or two multi-class factor index/indices. These indices proxy respectively for the value and momentum factors. The constituents are weighted according to the following schemes, rebalanced every quarter: MDC is the maximum deconcentration portfolio; GMV is the global minimum variance subject to bounds on weights ( $1/(3p)$  and 1, where  $p$  is the nominal number of constituents); RP is the risk parity, which equalises the contributions to volatility; MENUM is maximum ENUB, which maximises the effective number of contributions of uncorrelated factors extracted from the covariance matrix. The stars in columns 2 to 4 indicate the significance level in an equality test with respect to the portfolio of asset class indices: no star = not significant at the 10% level; \* = 10% level; \*\* = 5% level; \*\*\* = 1% level. The last column contains the annual one-way turnover.

## 5. Empirical Analysis of Factor Investing

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# Conclusion



## Conclusion

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Factor investing and risk allocation principles have recently led to the development of new investment products by asset managers. Our work suggests that the most meaningful way for grouping individual securities is not by forming arbitrary asset class indices, but instead by forming replicating portfolios for a set of suitably-designed asset pricing factor indices. Building on this theoretical result in support for factor investing, our paper examines the relative efficiency of standard forms of practical implementation of the factor investing paradigm based on commonly-used factors in the equity, fixed-income and commodity universes. Given that a test of the relevance of factor investing is a joint test of the relevance of the chosen factors and the chosen allocation methodology, we assess the out-of-sample performance of a number of portfolios constructed from the factor covariance matrix, without the need to use expected return estimates. Overall, we find that the out-of-sample benefits of factor investing are extremely substantial when short-selling constraints apply, and are robust to the introduction of long-only constraints, especially when improved weighting schemes are used in implementation. While a cost-efficient implementation of this approach is relatively straightforward in the equity space, extending the approach to a multi-asset context involves, however, a number of challenges in terms of the proper identification and implementation of factors beyond the equity universe.

Our work can be extended in several dimensions. On the one hand, one may be tempted to assess whether the use of implicit, as opposed to explicit, factors can be used to perform risk allocation, thus alleviating the concern over possible specification errors in the identification of explicit factors. A successful implementation of this approach would require us to address

the question of which method to use to extract meaningful and robust implicit factors. Another particularly important avenue for further research would consist in extending the empirical analysis of the benefits of factor investing to an asset-liability management context, where the focus should be on an efficient allocation to rewarded risk factors, with factor exposure in the investor's liability portfolio taken as a given. We leave these questions for further research.

# Appendices



## Appendices

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### A. Proofs of Propositions

This appendix collects the proofs of the propositions written in the paper.

#### A.1 Proof of Proposition 1

Take an SDF given by  $m = a + \mathbf{b}'\mathbf{f}$  and consider an asset  $i$  with return  $r_i$ . By (2.2), we have:

$$\mathbb{E}[r_i] = \frac{1}{\mathbb{E}[m]} - 1 - \frac{1}{\mathbb{E}[m]} \text{Cov}[m, r_i],$$

hence:

$$\mathbb{E}[r_i] = \frac{1}{\mathbb{E}[m]} - 1 - \frac{1}{\mathbb{E}[m]} \mathbf{b}' \text{Cov}[\mathbf{f}, r_i]. \quad (\text{A.1})$$

Consider the multivariate regression of asset  $i$  on the factors:

$$r_i = \alpha_i + \beta_i' \mathbf{f} + \varepsilon_i.$$

The vector of betas is given by  $\beta_i = \Sigma_f^{-1} \text{Cov}[\mathbf{f}, r_i]$ . Substituting this equality in (A.1), we obtain:

$$\mathbb{E}[r_i] = \frac{1}{\mathbb{E}[m]} - 1 - \frac{1}{\mathbb{E}[m]} \mathbf{b}' \Sigma_f \beta_i,$$

which has the form

$$\mathbb{E}[r_i] = \kappa + \Lambda' \beta_i, \quad (\text{A.2})$$

with  $\kappa = \frac{1}{\mathbb{E}[m]} - 1$  and  $\Lambda = -\frac{1}{\mathbb{E}[m]} \mathbf{b}' \Sigma_f$ .

Conversely, assume that (A.2) holds for some scalar  $\kappa$ , some vector  $\Lambda$  and for each asset  $i = 1, \dots, N$ . We search for  $a$  and  $\mathbf{b}$  such that (A.1) holds with  $m = a + \mathbf{b}'\mathbf{f}$ . Since

$$\mathbb{E}[r_i] = \kappa + \Lambda' \Sigma_f^{-1} \text{Cov}[\mathbf{f}, r_i],$$

it suffices to have:

$$\kappa = \frac{1}{\mathbb{E}[m]} - 1, \quad \mathbf{b} = -\mathbb{E}[m] \Sigma_f^{-1} \Lambda.$$

These equalities are satisfied by taking:

$$\begin{aligned} \mathbf{b} &= -\frac{1}{1 + \kappa} \Sigma_f^{-1} \Lambda, \\ a &= \frac{1}{1 + \kappa} (1 + \boldsymbol{\mu}'_f \Sigma_f^{-1} \Lambda). \end{aligned}$$

With this choice of  $a$  and  $\mathbf{b}$ , the random variable  $m = a + \mathbf{b}'\mathbf{f}$  is such that (A.2) holds for each asset  $i$ . Hence (A.1) holds too, and  $m$  is an SDF.

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### A.2 Proof of Proposition 2

The pay-offs  $c^*$  and  $f_k^*$  are defined as in the proposition. We let  $\bar{\mathbf{f}} = (1 \ f_1 \ \dots \ f_K)'$  and take a similar definition for  $\bar{\mathbf{f}}^*$ , so that  $\bar{\mathbf{f}}^* = \mathbb{E}[\bar{\mathbf{f}} \mathbf{X}'] \mathbb{E}[\mathbf{X} \mathbf{X}']^{-1} \mathbf{X}$ . We also let  $\bar{\mathbf{b}} = (c \ \mathbf{b})'$ .

For each pay-off  $x_i$  we have:

$$\begin{aligned} \mathbb{E}[m^* x_i] &= \mathbb{E}[(\bar{\mathbf{b}}' \bar{\mathbf{f}}^*) x_i] \\ &= \bar{\mathbf{b}}' \mathbb{E}[\bar{\mathbf{f}} \mathbf{X}'] \mathbb{E}[\mathbf{X} \mathbf{X}']^{-1} \mathbb{E}[x_i \mathbf{X}] \\ &= \bar{\mathbf{b}}' \mathbb{E}[\bar{\mathbf{f}} x_i] \\ &= \mathbb{E}[(\bar{\mathbf{b}}' \bar{\mathbf{f}}) x_i] \\ &= \mathbb{E}[m x_i], \end{aligned}$$

which shows that  $m^*$  is an SDF.

### A.3 Proof of Proposition 3

The second part of the proposition, namely that the return of the long-short MSR portfolio prices all assets, has already been shown.

We now focus on the first implication. Let  $r_0 = \mathbf{w}'_0 \mathbf{r}$  denote the return of the pricing portfolio. By definition of a pricing factor, constants  $a$  and  $b$  exist such that  $m = a + br_0$  is an SDF. Consider any portfolio with a return  $r_P$ . We use subscript 0 to refer to the pricing portfolio and  $P$  to the portfolio to price. By (2.2) and given that the risk-free rate satisfies  $r_d = \frac{1}{\mathbb{E}[m]} - 1$ , we have:

$$\begin{aligned} \mu_P &= r_d - \frac{1}{\mathbb{E}[m]} \text{Cov}[m, r_P], \\ \mu_0 &= r_d - \frac{1}{\mathbb{E}[m]} \text{Cov}[m, r_0]. \end{aligned}$$

From the second equation, we get:

$$\mu_0 - r_d = -\frac{b}{\mathbb{E}[m]} \mathbb{V}[r_0],$$

hence :

$$\frac{\mu_P - r_d}{\sigma_P} = \frac{\mu_0 - r_d}{\sigma_0} \rho_{0P}.$$

Since a correlation is less than 1, this implies that the portfolio  $P$  has lower Sharpe ratio than the portfolio 0.

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### A.4 Proof of Proposition 4

We first recall that the squared Sharpe ratios of the long-short MSR portfolios of individual constituents and indices are given by:

$$\lambda_{MSR}^2 = \tilde{\boldsymbol{\mu}}' \boldsymbol{\Sigma}^{-1} \tilde{\boldsymbol{\mu}}, \quad \lambda_{MSR,I}^2 = \tilde{\boldsymbol{\mu}}_I' \boldsymbol{\Sigma}_I^{-1} \tilde{\boldsymbol{\mu}}_I.$$

Since the indices excess returns satisfy  $\tilde{\boldsymbol{I}} = \boldsymbol{W}' \tilde{\boldsymbol{r}}_i$ , where  $\tilde{\boldsymbol{r}}$  is the vector of excess returns to the individual constituents, the moments of the indices are related to those of the constituents through:

$$\tilde{\boldsymbol{\mu}}_I = \boldsymbol{W}' \tilde{\boldsymbol{\mu}}, \quad \boldsymbol{\Sigma}_I = \boldsymbol{W}' \boldsymbol{\Sigma} \boldsymbol{W}.$$

To express  $\tilde{\boldsymbol{\mu}}$  and  $\boldsymbol{\Sigma}$  as functions of  $\tilde{\boldsymbol{\mu}}_I$  and  $\boldsymbol{\Sigma}_I$ , we introduce the  $K \times N$  matrix of betas from the regressions (3.6). It is given by  $\boldsymbol{\beta} = \boldsymbol{\Sigma}_I^{-1} \boldsymbol{W}' \boldsymbol{\Sigma}$ . Then, (3.6) implies that:

$$\tilde{\boldsymbol{\mu}} = \boldsymbol{\alpha} + \boldsymbol{\beta}' \tilde{\boldsymbol{\mu}}_I, \quad \boldsymbol{\Sigma} = \boldsymbol{\beta}' \boldsymbol{\Sigma}_I \boldsymbol{\beta} + \boldsymbol{\Sigma}_\eta,$$

where  $\boldsymbol{\Sigma}_\eta$  is the covariance matrix of the residuals. Hence:

$$\tilde{\boldsymbol{\mu}}' \boldsymbol{\Sigma}^{-1} \tilde{\boldsymbol{\mu}} = \boldsymbol{\alpha}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\alpha} + 2 \boldsymbol{\alpha}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\beta}' \tilde{\boldsymbol{\mu}}_I + \tilde{\boldsymbol{\mu}}_I' \boldsymbol{\beta} \boldsymbol{\Sigma}^{-1} \boldsymbol{\beta}' \tilde{\boldsymbol{\mu}}_I.$$

Substituting the expression for  $\boldsymbol{\beta}$  in the last two terms, we obtain:

$$\begin{aligned} \tilde{\boldsymbol{\mu}}' \boldsymbol{\Sigma}^{-1} \tilde{\boldsymbol{\mu}} &= \boldsymbol{\alpha}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\alpha} + 2 \boldsymbol{\alpha}' \boldsymbol{W} \boldsymbol{\Sigma}_I^{-1} \tilde{\boldsymbol{\mu}}_I + \tilde{\boldsymbol{\mu}}_I' \boldsymbol{\Sigma}_I^{-1} \boldsymbol{W}' \boldsymbol{\Sigma} \boldsymbol{\Sigma}^{-1} \boldsymbol{\Sigma} \boldsymbol{W} \boldsymbol{\Sigma}_I^{-1} \tilde{\boldsymbol{\mu}}_I \\ &= \boldsymbol{\alpha}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\alpha} + 2 \boldsymbol{\alpha}' \boldsymbol{W} \boldsymbol{\Sigma}_I^{-1} \tilde{\boldsymbol{\mu}}_I + \tilde{\boldsymbol{\mu}}_I' \boldsymbol{\Sigma}_I^{-1} \tilde{\boldsymbol{\mu}}_I, \end{aligned}$$

where the first equality uses the fact that  $\boldsymbol{\Sigma}_I = \boldsymbol{W}' \boldsymbol{\Sigma} \boldsymbol{W}$ . Hence:

$$\lambda_{MSR}^2 - \lambda_{MSR,I}^2 = 2 \boldsymbol{\alpha}' \boldsymbol{W} \boldsymbol{\Sigma}_I^{-1} \tilde{\boldsymbol{\mu}}_I + \tilde{\boldsymbol{\mu}}_I' \boldsymbol{\Sigma}_I^{-1} \tilde{\boldsymbol{\mu}}_I.$$

Moreover, we have  $\boldsymbol{\alpha} = \tilde{\boldsymbol{\mu}} - \boldsymbol{\beta}' \tilde{\boldsymbol{\mu}}_I$ , hence  $\boldsymbol{\alpha} = [\boldsymbol{I}_N - \boldsymbol{M}]' \tilde{\boldsymbol{\mu}}$  with  $\boldsymbol{M} = \boldsymbol{W} \boldsymbol{\beta} = \boldsymbol{W} \boldsymbol{\Sigma}_I^{-1} \boldsymbol{W}' \boldsymbol{\Sigma}$ .

We have:

$$\begin{aligned} \boldsymbol{\alpha}' \boldsymbol{W} \boldsymbol{\Sigma}_I^{-1} &= \tilde{\boldsymbol{\mu}}' [\boldsymbol{I}_N - \boldsymbol{M}] \boldsymbol{W} \boldsymbol{\Sigma}_I^{-1} \\ &= \tilde{\boldsymbol{\mu}}' [\boldsymbol{W} \boldsymbol{\Sigma}_I^{-1} - \boldsymbol{W} \boldsymbol{\Sigma}_I^{-1} \boldsymbol{W}' \boldsymbol{\Sigma} \boldsymbol{W} \boldsymbol{\Sigma}_I^{-1}] \\ &= \tilde{\boldsymbol{\mu}}' [\boldsymbol{W} \boldsymbol{\Sigma}_I^{-1} - \boldsymbol{W} \boldsymbol{\Sigma}_I^{-1}] \\ &= \mathbf{0}'_K, \end{aligned}$$

where the before-last equality uses the fact that  $\boldsymbol{\Sigma}_I = \boldsymbol{W}' \boldsymbol{\Sigma} \boldsymbol{W}$ . Hence we obtain  $\lambda_{MSR}^2 - \lambda_{MSR,I}^2 = \boldsymbol{\alpha}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\alpha}$ .

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### A.5 Proof of Proposition 5

As explained in Section 2.2.3, the realised excess returns on individual assets admit the following decomposition:

$$r_i - r_d = \beta'_i \mathbf{f} + \varepsilon_i, \quad i = 1, \dots, N.$$

Hence:

$$\tilde{\boldsymbol{\mu}} = \boldsymbol{\beta}' \mathbb{E}[\mathbf{f}], \quad \boldsymbol{\Sigma} = \boldsymbol{\beta}' \boldsymbol{\Sigma}_f \boldsymbol{\beta} + \boldsymbol{\Sigma}_\varepsilon. \quad (\text{A.3})$$

Moreover, it follows from Section 2.2.3 that the vector of factor risk premia satisfies  $\boldsymbol{\Lambda} = \mathbb{E}[\mathbf{f}] = \boldsymbol{\Lambda}$ . Because  $\mathbf{f} = \mathbf{W}' \tilde{\mathbf{R}}$ , we obtain  $\boldsymbol{\Lambda} = \mathbf{W}' \tilde{\boldsymbol{\mu}}$ .

The betas in (A.3) are given by  $\beta_i = \boldsymbol{\Sigma}_f^{-1} \boldsymbol{\nu}_i$ , where  $\boldsymbol{\nu}_i$  is the  $K \times 1$  vector of covariances between  $r_i - r_d$  and  $\mathbf{f}$ . Because  $\mathbf{f} = \mathbf{W}' \tilde{\mathbf{R}}$ , it follows that the  $K \times N$  matrix of betas is  $\boldsymbol{\beta} = \boldsymbol{\Sigma}_f^{-1} \mathbf{W}' \boldsymbol{\Sigma}$ . Hence, we have:

$$\begin{aligned} \boldsymbol{\beta} \boldsymbol{\Sigma}^{-1} \tilde{\boldsymbol{\mu}} &= \boldsymbol{\Sigma}_f^{-1} \mathbf{W}' \boldsymbol{\Sigma} \boldsymbol{\Sigma}^{-1} \tilde{\boldsymbol{\mu}} \\ &= \boldsymbol{\Sigma}_f^{-1} \mathbf{W}' \tilde{\boldsymbol{\mu}} = \boldsymbol{\Sigma}_f^{-1} \boldsymbol{\Lambda}. \end{aligned}$$

The weights of the long-short MSR portfolio are  $\mathbf{w}_{MSR} = \frac{\boldsymbol{\Sigma}^{-1} \tilde{\boldsymbol{\mu}}}{x}$ , where  $x = \mathbf{1}'_N \boldsymbol{\Sigma}^{-1} \tilde{\boldsymbol{\mu}}$ . Hence, the  $K \times 1$  vector of factor exposures of this portfolio is:

$$\boldsymbol{\beta}_{MSR} = \boldsymbol{\beta} \mathbf{w}_{MSR} = \frac{\boldsymbol{\beta} \boldsymbol{\Sigma}^{-1} \tilde{\boldsymbol{\mu}}}{x} = \frac{\boldsymbol{\Sigma}_f^{-1} \boldsymbol{\Lambda}}{x}$$

For the specific risk, we note that:

$$\begin{aligned} \boldsymbol{\Sigma}_\varepsilon \boldsymbol{\Sigma}^{-1} \tilde{\boldsymbol{\mu}} &= [\boldsymbol{\Sigma} - \boldsymbol{\beta}' \boldsymbol{\Sigma}_f \boldsymbol{\beta}] \boldsymbol{\Sigma}^{-1} \boldsymbol{\beta}' \boldsymbol{\Lambda} \\ &= \boldsymbol{\beta}' \boldsymbol{\Lambda} - [\boldsymbol{\Sigma} \mathbf{W} \boldsymbol{\Sigma}_f^{-1} \boldsymbol{\Sigma}_f \boldsymbol{\Sigma}_f^{-1} \mathbf{W}' \boldsymbol{\Sigma}] \times \boldsymbol{\Sigma}^{-1} \times [\boldsymbol{\Sigma} \mathbf{W} \boldsymbol{\Sigma}_f^{-1}] \boldsymbol{\Lambda} \\ &= \boldsymbol{\beta}' \boldsymbol{\Lambda} - \boldsymbol{\Sigma} \mathbf{W} \boldsymbol{\Sigma}_f^{-1} \mathbf{W}' \boldsymbol{\Sigma} \mathbf{W} \boldsymbol{\Sigma}_f^{-1} \boldsymbol{\Lambda}. \end{aligned}$$

But we have  $\boldsymbol{\Sigma}_f = \mathbf{W}' \boldsymbol{\Sigma} \mathbf{W}$ . Hence:

$$\boldsymbol{\Sigma}_\varepsilon \boldsymbol{\Sigma}^{-1} \tilde{\boldsymbol{\mu}} = \boldsymbol{\beta}' \boldsymbol{\Lambda} - \boldsymbol{\Sigma} \mathbf{W} \boldsymbol{\Sigma}_f^{-1} \boldsymbol{\Lambda} = \boldsymbol{\beta}' \boldsymbol{\Lambda} - \boldsymbol{\beta}' \boldsymbol{\Lambda} = 0.$$

As a result,  $\boldsymbol{\Sigma}_\varepsilon \mathbf{w}_{MSR} = \mathbf{0}$ , which shows that the MSR has zero specific risk.

By (A.3), the MSR return satisfies:

$$\begin{aligned} r_{MSR} &= \mathbf{w}'_{MSR} \mathbf{r} = \mathbf{w}'_{MSR} [r_d \mathbf{1}_N + \boldsymbol{\beta}' \mathbf{f} + \boldsymbol{\varepsilon}] \\ &= r_d + \boldsymbol{\beta}'_{MSR} \mathbf{f} + \mathbf{w}'_{MSR} \boldsymbol{\varepsilon}, \end{aligned}$$

and the last term is zero since it has zero expectation and variance.

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### A.6 Proof of Proposition 6

- 1 is equivalent to 2.

It is clear that 2 implies 1. Conversely, assume that the covariance matrix of the extended universe is singular. Then, a non-zero  $p \times 1$  vector  $\mathbf{x}$  and a constant  $c$  exist such that, almost surely:

$$\mathbf{x}' \begin{pmatrix} \mathbf{r} \\ \mathbf{R} \end{pmatrix} = c.$$

Partition  $\mathbf{x}$  as  $\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_a \end{pmatrix}$ , where  $\mathbf{x}_0$  is  $n \times 1$ . We obtain:

$$\mathbf{x}'_0 \mathbf{r} = \mathbf{x}'_a \mathbf{R} + c.$$

If  $\mathbf{x}_0$  was zero, then we would have  $\mathbf{x}'_a \mathbf{R} = -c$ , while  $\mathbf{x}_a$  would necessarily be non zero (because  $\mathbf{x}$  has to be non zero). This would contradict the fact that the new assets are not redundant with each other. Thus,  $\mathbf{x}_0$  is non zero. Similarly,  $\mathbf{x}_a$  is non zero.

- 2 is equivalent to 3.

This equivalence follows from the definition of the canonical correlation.

### A.7 Proof of Proposition 7

We take two non-zero vectors  $\mathbf{x}_0$  and  $\mathbf{x}_a$ , and we let  $\mathbf{u} = \Sigma_0^{-\frac{1}{2}} \mathbf{x}_0$ . By Cauchy-Schwartz inequality, we have:

$$\left| \left[ \Sigma_a^{-\frac{1}{2}} \Sigma_{a,0} \Sigma_0^{-\frac{1}{2}} \mathbf{u} \right]' \Sigma_a^{\frac{1}{2}} \mathbf{x}_a \right| \leq \sqrt{\mathbf{u}' \Sigma_0^{-\frac{1}{2}} \Sigma_{0,a} \Sigma_a^{-1} \Sigma_{a,0} \Sigma_0^{-\frac{1}{2}} \mathbf{u}} \times \sqrt{\mathbf{x}'_a \Sigma_a \mathbf{x}_a}, \quad (\text{A.4})$$

which can be rewritten as:

$$|\mathbf{x}'_0 \Sigma_{0,a} \mathbf{x}_a| \leq \sqrt{\mathbf{u}' \Sigma_0^{-\frac{1}{2}} \Sigma_{0,a} \Sigma_a^{-1} \Sigma_{a,0} \Sigma_0^{-\frac{1}{2}} \mathbf{u}} \times \sqrt{\mathbf{x}'_a \Sigma_a \mathbf{x}_a}.$$

Let  $\sqrt{\lambda_{\max}}$  be the maximum eigenvalue of the matrix  $\mathbf{M} = \Sigma_0^{-\frac{1}{2}} \Sigma_{0,a} \Sigma_a^{-1} \Sigma_{a,0} \Sigma_0^{-\frac{1}{2}}$ . Then, we have:

$$\begin{aligned} |\mathbf{x}'_0 \Sigma_{0,a} \mathbf{x}_a| &\leq \sqrt{\lambda_{\max}} \times \|\mathbf{u}\| \times \sqrt{\mathbf{x}'_a \Sigma_a \mathbf{x}_a} \\ &= \sqrt{\lambda_{\max}} \times \sqrt{\mathbf{x}'_0 \Sigma_0 \mathbf{x}_0} \times \sqrt{\mathbf{x}'_a \Sigma_a \mathbf{x}_a}. \end{aligned}$$

This shows that the canonical correlation is bounded from above by  $\sqrt{\lambda_{\max}}$ , and that the maximum is reached when  $\mathbf{u}$  is an associated eigenvector of  $\mathbf{M}$ .

Moreover, equality in (A.4) holds if  $\Sigma_a^{-\frac{1}{2}} \Sigma_{a,0} \Sigma_0^{-\frac{1}{2}} \mathbf{u}$  is parallel to  $\Sigma_a^{\frac{1}{2}} \mathbf{x}_a$ , i.e. if  $\mathbf{x}_a$  is parallel to  $\Sigma_a^{-1} \Sigma_{a,0} \Sigma_0^{-1} \mathbf{x}_0$ .

### A.8 Proof of Proposition 8

The computations are similar to those made in Appendix A.4 for the proof of Proposition 4. We use subscripts 0,  $a$  and 1 in the notations of moments (covariance matrix and vector

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of expected returns) to refer respectively to the universes  $\mathcal{I}_0$ ,  $\mathcal{I}_a$  and  $\mathcal{I}_1$ .

We first note that the covariance matrix in  $\mathcal{I}_1$  can be written as:

$$\Sigma_1 = \begin{pmatrix} \Sigma_0 & \Sigma_{0,a} \\ \Sigma_{a,0} & \Sigma_a \end{pmatrix}.$$

By definition of the regression parameters in (3.9), we have:

$$\Sigma_a = \beta' \Sigma_0 \beta + \Sigma_\varepsilon,$$

where  $\beta = \Sigma_0^{-1} \Sigma_{0,a}$ . Hence, the blockwise inversion formula implies that:

$$\Sigma_1^{-1} = \begin{pmatrix} \Sigma_0^{-1} + \Sigma_0^{-1} \Sigma_{0,a} \Sigma_\varepsilon^{-1} \Sigma_{a,0} \Sigma_0^{-1} & -\Sigma_0^{-1} \Sigma_{0,a} \Sigma_\varepsilon^{-1} \\ -\Sigma_\varepsilon^{-1} \Sigma_{a,0} \Sigma_0^{-1} & \Sigma_\varepsilon^{-1} \end{pmatrix}. \quad (\text{A.5})$$

Moreover, the GMV, MRR and MSR portfolios are given by:

$$\mathbf{w}_{GMV} = \frac{\Sigma^{-1} \mathbf{1}}{\mathbf{1}' \Sigma^{-1} \mathbf{1}}, \quad \mathbf{w}_{MRR} = \frac{\Sigma^{-1} \boldsymbol{\mu}}{\mathbf{1}' \Sigma^{-1} \boldsymbol{\mu}}, \quad \mathbf{w}_{MSR} = \frac{\Sigma^{-1} \tilde{\boldsymbol{\mu}}}{\mathbf{1}' \Sigma^{-1} \tilde{\boldsymbol{\mu}}}.$$

Hence, the coefficients that fully characterise the efficient frontier can be written as:

$$\begin{aligned} \sigma_{GMV}^2 &= \frac{1}{\mathbf{1}' \Sigma^{-1} \mathbf{1}}, & \zeta_{MRR} &= \sqrt{\boldsymbol{\mu}' \Sigma^{-1} \boldsymbol{\mu}}, \\ \zeta_{MSR} &= \sqrt{\tilde{\boldsymbol{\mu}}' \Sigma^{-1} \tilde{\boldsymbol{\mu}}}, & \mu_{GMV} &= \frac{\mathbf{1}' \Sigma^{-1} \boldsymbol{\mu}}{\mathbf{1}' \Sigma^{-1} \mathbf{1}}. \end{aligned}$$

Hence:

$$\begin{aligned} \frac{1}{\sigma_{GMV,1}^2} - \frac{1}{\sigma_{GMV,0}^2} &= \mathbf{1}'_N \Sigma_1^{-1} \mathbf{1}_N - \mathbf{1}'_n \Sigma_0^{-1} \mathbf{1}_n, \\ \zeta_{MRR,1}^2 - \zeta_{MRR,0}^2 &= \boldsymbol{\mu}'_1 \Sigma_1^{-1} \boldsymbol{\mu}_1 - \boldsymbol{\mu}'_0 \Sigma_0^{-1} \boldsymbol{\mu}_0, \\ \frac{\mu_{GMV,1}}{\sigma_{GMV,1}^2} - \frac{\mu_{GMV,0}}{\sigma_{GMV,0}^2} &= \mathbf{1}'_N \Sigma_1^{-1} \boldsymbol{\mu}_1 - \mathbf{1}'_n \Sigma_0^{-1} \boldsymbol{\mu}_0. \end{aligned}$$

To obtain the expressions of Proposition 8, we expand the scalar products by using the expression (A.5) for  $\Sigma_1^{-1}$ . For instance, for the change in inverse variance, we have:

$$\begin{aligned} \mathbf{1}'_N \Sigma_1^{-1} \mathbf{1}_N &= \begin{pmatrix} \mathbf{1}_n \\ \mathbf{1}_K \end{pmatrix}' \Sigma_1^{-1} \begin{pmatrix} \mathbf{1}_n \\ \mathbf{1}_K \end{pmatrix} \\ &= \mathbf{1}'_n \Sigma_0^{-1} \mathbf{1}_n + \mathbf{1}'_K \Sigma_\varepsilon^{-1} \mathbf{1}_K - 2 \mathbf{1}'_n \Sigma_0^{-1} \Sigma_{0,a} \Sigma_\varepsilon^{-1} \mathbf{1}_K \\ &\quad + \mathbf{1}'_n \Sigma_0^{-1} \Sigma_{0,a} \Sigma_\varepsilon^{-1} \Sigma_{a,0} \Sigma_0^{-1} \mathbf{1}_n. \end{aligned}$$

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Given that  $\beta = \Sigma_0^{-1} \Sigma_{0,a}$ , we obtain:

$$\begin{aligned} \frac{1}{\sigma_{GMV,1}^2} &= \frac{1}{\sigma_{GMV,0}^2} + \mathbf{1}'_K \Sigma_\varepsilon^{-1} \mathbf{1}_K \\ &\quad - 2\mathbf{1}'_n \beta \Sigma_\varepsilon^{-1} \mathbf{1}_K + \mathbf{1}'_n \beta \Sigma_\varepsilon^{-1} \beta' \mathbf{1}_n \\ &= \frac{1}{\sigma_{GMV,0}^2} + [\beta' \mathbf{1}_n - \mathbf{1}_K]' \Sigma_\varepsilon^{-1} [\beta' \mathbf{1}_n - \mathbf{1}_K]. \end{aligned}$$

To compute  $[\zeta_{MRR,1}^2 - \zeta_{MRR,0}^2]$  and  $\left[ \frac{\mu_{GMV,1}}{\sigma_{GMV,1}^2} - \frac{\mu_{GMV,0}}{\sigma_{GMV,0}^2} \right]$ , we follow a similar procedure, decomposing the vector of expected returns of the extended universe as:

$$\mu_1 = \begin{pmatrix} \mathbf{0}_n \\ \alpha \end{pmatrix} + \begin{pmatrix} \mathbf{I}_n \\ \beta' \end{pmatrix} \mu_0.$$

The same decomposition can be applied to expected excess returns, in order to compute the change in the squared Sharpe ratio.

### B. Minimum Linear Torsion Matrix

The solution to the optimisation problem (3.5) is derived by Meucci et al. (2013) and Carli et al. (2014). In this appendix, we simply recall their result without proof.

Carli et al. (2014) show that the optimal matrix  $A$  can be written as:

$$A = PL^{-1}UV'D, \tag{B.1}$$

where notations are defined as follows:

- $D$  is the diagonal matrix of standard deviations, i.e. the square root of the diagonal matrix of variances;
- $L$  is the diagonal matrix of the square roots of the eigenvalues of  $\Sigma$ , ranked by decreasing order, and  $P$  is an orthogonal matrix such that  $\Sigma = PL^2P'$ ;
- $U$  and  $V$  are two orthogonal matrices coming from the singular value decomposition of the auxiliary matrix  $M = LP'D$ . This means that we have  $M = USV'$ , where  $S$  is the diagonal matrix of singular values ranked by decreasing order.

Taking inverses of both sides of the equality  $LP'D = USV'$ , we obtain:

$$L^{-1} = P'DVS^{-1}U'$$

Substituting this equality in (B.1), we get:

$$A = DVS^{-1}V'D.$$

This expression shows in particular that  $A$  is symmetric.

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### C. Test Statistics

In this appendix we write the detailed expressions for the test statistics that we use in our in-sample and out-of-sample tests.

#### C.1 Mean-Variance Spanning Tests

The regression (3.9) is run in the time series for each new asset, with index  $k \in \mathcal{I}_\alpha$ :

$$R_{kt} = \alpha_k + \beta'_k \mathbf{r}_t + \varepsilon_{kt}, \quad t = 1, \dots, T$$

and the intercepts and betas are collected in a  $K \times 1$  vector  $\alpha$  and a  $n \times K$  matrix  $\beta$ . If  $\mathcal{L}(\alpha, \beta)$  denotes the likelihood of the new returns  $(R_{11}, \dots, R_{1T}, \dots, R_{K1}, \dots, R_{KT})$  conditional on the original returns  $(R_{11}, \dots, R_{1T}, \dots, R_{n1}, \dots, R_{nT})$ , the likelihood ratio statistic for the joint test of the restrictions (5.1) is defined as (see Lütkepohl (1993) Chap. C.5):

$$LR = -2 \ln \frac{\max_{H_0} \mathcal{L}(\alpha, \beta)}{\max \mathcal{L}(\alpha, \beta)}.$$

The numerator is the maximum likelihood subject to the constraints (5.1) and the denominator is the unconstrained maximum. Kan and Zhou (2012) (KZ12) show that this statistic can be written as a function of the sample estimates for the risk and return parameter of the GMV and the MRR (see their Eq. (33)):

$$LR = -T \ln U,$$

where  $T$  is the sample size and  $U$  is given by:

$$\begin{aligned} U &= \frac{\hat{c}_0 + \hat{d}_0}{\hat{c}_1 + \hat{d}_1}, \\ \hat{d}_i &= \hat{a}_i \hat{b}_i - \hat{c}_i^2, \quad i = 0, 1, \\ \hat{a}_i &= \hat{\zeta}_{MRR, i}^2, \quad \hat{b}_i = \frac{\hat{\zeta}_{GMV, i}}{\hat{\sigma}_{GMV, i}^2}, \quad \hat{c}_i = \frac{1}{\hat{\sigma}_{GMV, i}^2}, \quad i = 0, 1. \end{aligned}$$

We recall that subscripts 0 and 1 refer respectively to the original and the extended universes, that  $\zeta$  denotes the risk-return ratio  $\frac{\mu}{\sigma}$  and that MRR and GMV stand for the "maximum risk-return ratio" and "global minimum variance" portfolios.

The exact small-sample distribution of the statistics can be derived under the following assumptions on the regression model (3.9):

- if  $\varepsilon_t = (\varepsilon_{1t} \dots \varepsilon_{Kt})'$  is the vector of residuals at date  $t$  for  $t = 1, \dots, T$ , then the random variables  $\varepsilon_1, \dots, \varepsilon_T$  are independent and identically distributed with multivariate normal distribution, zero mean and covariance matrix  $\Sigma_\varepsilon$ ;
- we have  $T \geq p = n + K$ ;
- the  $T \times n$  matrix of regressors  $\mathbf{Z} = \begin{pmatrix} \mathbf{r}_1 & \dots & \mathbf{r}_T \end{pmatrix}'$  is such that  $\mathbf{Z}'\mathbf{Z}$  is non-singular.

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Then, Jobson and Korkie (1989) show that:

$$\begin{cases} \text{if } K \geq 2, & \frac{T-p}{K} \left( \frac{1}{\sqrt{U}} - 1 \right) \sim F_{2K, 2(T-p)}, \\ \text{if } K = 1, & \frac{T-p}{2} \left( \frac{1}{U} - 1 \right) \sim F_{2, T-p}, \end{cases}$$

where  $F_{d_1, d_2}$  denotes the Fisher-Snedecor distribution (also known as F-distribution) with  $d_1$  and  $d_2$  degrees of freedom.

KZ12 also introduce a two-step spanning test, in which the first stage is a test of  $\alpha = \mathbf{0}_K$  and the second is a test of  $\beta' \mathbf{1}_n = \mathbf{1}_K$  conditional on  $\alpha = \mathbf{0}_K$ . The test statistics are inspired by the F-test of Gibbons et al. (1989), and their null small-sample distributions are derived by KZ12:

$$\begin{cases} F_1 = \frac{T-p}{K} \times \frac{\hat{a}_1 - \hat{a}_0}{1 + \hat{a}_0} \sim F_{K, T-p}, \\ F_2 = \frac{T-p+1}{K} \times \left[ \frac{\hat{c}_1 + \hat{d}_1}{\hat{c}_0 + \hat{d}_0} \times \frac{1 + \hat{a}_0}{1 + \hat{a}_1} - 1 \right] \sim F_{K, T-p+1}. \end{cases}$$

### C.2 Equality of Analytics

We briefly present the test procedures that we use to compare the expected returns, volatilities and Sharpe ratios of two portfolios. All these tests obey the same principle. The test statistic is defined as:

$$\theta = \sqrt{T} \frac{\hat{\Delta}_{st}}{\hat{\sigma}_{\Delta, st}},$$

where  $\hat{\Delta}_{st}$  is a function of the ex-post analytics of the two portfolios and  $\hat{\sigma}_{\Delta, st}$  is a consistent estimator of the asymptotic variance of  $\hat{\Delta}_{st}$ . The distribution of  $\theta$  is then approximated by its asymptotic distribution, which is  $\mathcal{N}(0, 1)$ .

For each test,  $\hat{\Delta}_{st}$  has the form  $f(\hat{\mathbf{v}})$ , where  $\hat{\mathbf{v}} = (\hat{m}_1 \quad \hat{m}_2 \quad \hat{g}_1 \quad \hat{g}_2)'$ ,  $\hat{m}_i$  is the mean return of portfolio  $i$  and  $\hat{g}_i$  is the mean squared return, for  $i = 1, 2$  (for the Sharpe ratio test, returns are replaced by returns in excess of the risk-free rate). The function  $f$  takes a different form in each test:

$$\begin{aligned} f(\mathbf{v}) &= v_1 - v_2 && \text{for the mean test;} \\ f(\mathbf{v}) &= \log(v_3 - v_1^2) - \log(v_4 - v_2^2) && \text{for the volatility test;} \\ f(\mathbf{v}) &= \frac{m_1}{\sqrt{v_3 - v_1^2}} - \frac{m_2}{\sqrt{v_4 - v_2^2}} && \text{for the Sharpe ratio test;} \end{aligned}$$

If  $\mathbf{v}$  denotes the vector of true moments (expected returns and expected squared returns), we assume that:

$$\sqrt{T}(\hat{\mathbf{v}} - \mathbf{v}) \xrightarrow{d} \mathcal{N}(0, \Psi),$$

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where  $\Psi$  is the asymptotic covariance matrix of  $\sqrt{T}\hat{\mathbf{v}}$ . By the delta-method, we have the following convergence:

$$\sqrt{T}[f(\hat{\mathbf{v}}) - f(\mathbf{v})] \xrightarrow{d} \mathcal{N}(0, \nabla f(\mathbf{v})' \Psi \nabla f(\mathbf{v})),$$

where  $\nabla f(\mathbf{v})$  denotes the gradient of  $f$  evaluated at the vector  $\mathbf{v}$ . Hence, if we have a consistent estimator  $\hat{\Psi}$ , the asymptotic variance of  $\hat{\Delta}_{st}$  can be consistently estimated as:

$$\hat{\sigma}_{\Delta, st} = \sqrt{\nabla f(\hat{\mathbf{v}})' \hat{\Psi} \nabla f(\hat{\mathbf{v}})}.$$

The gradient can be computed analytically for each test. Among the two methods of estimation of  $\Psi$  proposed by Ledoit and Wolf (2011), we use a heteroscedasticity and autocorrelation robust kernel estimator.

# Appendices

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# About Lyxor Asset Management



# About Lyxor Asset Management

## LYXOR ASSET MANAGEMENT

### The power to perform in any market

Lyxor Asset Management, the fully-owned Asset Manager of Société Générale Group, was founded in 1998 and counts 600 professionals worldwide managing and advising \$127.9bn<sup>1</sup> of assets. Lyxor offers customised investment management services in ETFs & Indexing, Alternatives & Multi-Management and Absolute Return & Solutions. Driven by acknowledged research, advanced risk-management and a passion for client satisfaction, Lyxor's investment specialists strive to deliver sustainable performance across all asset classes.

## CUSTOMISE YOUR INVESTMENT SOLUTIONS, MANAGE YOUR RISKS, REACH OPTIMAL PERFORMANCE

### A client driven organisation

From the outset, three key values have continuously been at the core of every solution at Lyxor: innovation, transparency and flexibility. The professionalism of Lyxor's experts and our flexible and receptive organisational structure guarantee an institutional quality of service and long-standing relationships with its clients. Lyxor brings together the spirit of a start-up, the responsiveness of an entrepreneur and the reliability of a rapidly expanding global player. An entrepreneurial approach and a spirit of innovation - a tireless quest for tailor-made, flexible solutions

### Experience driven by acknowledged research

Recognised in the industry and among academics alike for its research in macroeconomics, quantitative and alternative investments, Lyxor is also known for the publication of proprietary portfolio

management models and white papers.

- 20 researchers respected in the world of academic and professional research.
- Extensive experience in managing risk and volatility, options and hedge funds.

### Advanced risk management culture

ETFs & Indexing, Absolute Return, Alternative investments, Lyxor's investment solutions all have to address risk issues. Lyxor's ability to offer transparent and sustainable sources of performance results from its established experience in risk management.

- A 30-strong independent Risk Department.
- Constant monitoring of pricing, valuations, investment limits and market risks.

## ETFs & INDEXING

Standing among the most experienced ETF providers, Lyxor ETF ranks 3<sup>rd</sup> in Europe with more than \$53.7bn of ETF assets under management and 2<sup>nd</sup> in terms of liquidity. With 220 ETFs listed on 13 regulated exchanges across the world, Lyxor provides investors with a highly flexible opportunity to diversify their allocation across all asset classes (equities, bonds, money markets, commodities). With the Lyxor ETF quality charter adopted in 2011, Lyxor ETF also commits to deliver the highest standards in terms of index tracking quality, risk control, liquidity and transparency.

AUM: \$57.6bn
45 professionals
220 ETFs listed on 13 regulated exchanges
n°3 player in Europe, n°6 worldwide
7 of the most efficient European ETFs in the market

## ABSOLUTE RETURN & SOLUTIONS

Lyxor's Absolute Return & Solutions teams deliver investment solutions for investors looking for regular positive returns, well

<sup>1</sup> - Equivalent to €116.3bn - Assets under management and advisory as of 31 May 2015

# About Lyxor Asset Management

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identified and controlled risks and limited correlation to markets.

AUM: \$31.8bn
50 professionals
"Best Performing Diversified Global Fund" by UCITS Hedge Awards 2015
"Best Managed Futures (CTA) Fund" by Hedgeweek Global Awards 2015
Pioneer in Smart Beta Strategies

## ALTERNATIVES & MULTI-MANAGEMENT

With more than 17 years of experience in managing fund portfolios spanning the full spectrum of liquidity, risk/return and strategy options for institutional clients, Lyxor is a global leader in Multi-Management.

AUM: \$29.6bn
80 professionals
Among the world's leading Managed Account Platforms
17-year track record in manager selection
250 mutual funds and 100 hedge funds on its platforms

# About Lyxor Asset Management

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# About EDHEC-Risk Institute



# About EDHEC-Risk Institute

Founded in 1906, EDHEC is one of the foremost international business schools. Accredited by the three main international academic organisations, EQUIS, AACSB, and Association of MBAs, EDHEC has for a number of years been pursuing a strategy of international excellence that led it to set up EDHEC-Risk Institute in 2001.

This institute now boasts a team of over 95 permanent professors, engineers and support staff, as well as 48 research associates from the financial industry and affiliate professors.

## The Choice of Asset Allocation and Risk Management

EDHEC-Risk structures all of its research work around asset allocation and risk management. This strategic choice is applied to all of the Institute's research programmes, whether they involve proposing new methods of strategic allocation, which integrate the alternative class; taking extreme risks into account in portfolio construction; studying the usefulness of derivatives in implementing asset-liability management approaches; or orienting the concept of dynamic "core-satellite" investment management in the framework of absolute return or target-date funds.

## Academic Excellence and Industry Relevance

In an attempt to ensure that the research it carries out is truly applicable, EDHEC has implemented a dual validation system for the work of EDHEC-Risk. All research work must be part of a research programme, the relevance and goals of which have been validated from both an academic and a business viewpoint by the Institute's advisory board. This board is made up of internationally recognised researchers, the Institute's business partners, and representatives of major international institutional investors. Management of the research programmes respects a rigorous validation process, which guarantees the scientific quality and the operational usefulness of the programmes.

Six research programmes have been conducted by the centre to date:

- Asset allocation and alternative diversification
- Style and performance analysis
- Indices and benchmarking
- Operational risks and performance
- Asset allocation and derivative instruments
- ALM and asset management

These programmes receive the support of a large number of financial companies. The results of the research programmes are disseminated through the EDHEC-Risk locations in Singapore, which was established at the invitation of the Monetary Authority of Singapore (MAS); the City of London in the United Kingdom; Nice and Paris in France; and New York in the United States.

EDHEC-Risk has developed a close partnership with a small number of sponsors within the framework of research chairs or major research projects:

- *Core-Satellite and ETF Investment, in partnership with Amundi ETF*
- *Regulation and Institutional Investment, in partnership with AXA Investment Managers*
- *Asset-Liability Management and Institutional Investment Management, in partnership with BNP Paribas Investment Partners*
- *Risk and Regulation in the European Fund Management Industry, in partnership with CACEIS*
- *Exploring the Commodity Futures Risk Premium: Implications for Asset Allocation and Regulation, in partnership with CME Group*

## About EDHEC-Risk Institute

- Asset-Liability Management in Private Wealth Management, *in partnership with Coutts & Co.*
- Asset-Liability Management Techniques for Sovereign Wealth Fund Management, *in partnership with Deutsche Bank*
- The Benefits of Volatility Derivatives in Equity Portfolio Management, *in partnership with Eurex*
- Structured Products and Derivative Instruments, *sponsored by the French Banking Federation (FBF)*
- Optimising Bond Portfolios, in partnership with the French Central Bank (*BDF Gestion*)
- Asset Allocation Solutions, *in partnership with Lyxor Asset Management*
- Infrastructure Equity Investment Management and Benchmarking, *in partnership with Meridiam and Campbell Lutyens*
- Investment and Governance Characteristics of Infrastructure Debt Investments, *in partnership with Natixis*
- Advanced Modelling for Alternative Investments, *in partnership with Newedge Prime Brokerage*
- Advanced Investment Solutions for Liability Hedging for Inflation Risk, *in partnership with Ontario Teachers' Pension Plan*
- The Case for Inflation-Linked Corporate Bonds: Issuers' and Investors' Perspectives, *in partnership with Rothschild & Cie*
- Solvency II, *in partnership with Russell Investments*
- Structured Equity Investment Strategies for Long-Term Asian Investors, *in partnership with Société Générale Corporate & Investment Banking*

The philosophy of the Institute is to validate its work by publication in international academic journals, as well as to make it available to the sector through its position papers, published studies, and conferences.

Each year, EDHEC-Risk organises three conferences for professionals in order to present the results of its research, one in London (EDHEC-Risk Days Europe), one in Singapore (EDHEC-Risk Days Asia), and one in New York (EDHEC-Risk Days North America) attracting more than 2,500 professional delegates.

EDHEC also provides professionals with access to its website, [www.edhec-risk.com](http://www.edhec-risk.com), which is entirely devoted to international asset management research. The website, which has more than 65,000 regular visitors, is aimed at professionals who wish to benefit from EDHEC's analysis and expertise in the area of applied portfolio management research. Its monthly newsletter is distributed to more than 1.5 million readers.

### EDHEC-Risk Institute: Key Figures, 2013–2014

Nbr of permanent staff	97
Nbr of research associates	26
Nbr of affiliate professors	28
Overall budget	€13,500,000
External financing	€10,100,000
Nbr of conference delegates	1,782
Nbr of participants at EDHEC-Risk Institute Executive Education seminars	1,576

## About EDHEC-Risk Institute

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### Research for Business

The Institute's activities have also given rise to executive education and research service offshoots. EDHEC-Risk's executive education programmes help investment professionals to upgrade their skills with advanced risk and asset management training across traditional and alternative classes. In partnership with CFA Institute, it has developed advanced seminars based on its research which are available to CFA charterholders and have been taking place since 2008 in New York, Singapore and London.

In 2012, EDHEC-Risk Institute signed two strategic partnership agreements with the Operations Research and Financial Engineering department of Princeton University to set up a joint research programme in the area of risk and investment management, and with Yale School of Management to set up joint certified executive training courses in North America and Europe in the area of investment management.

As part of its policy of transferring know-how to the industry, EDHEC-Risk Institute has also set up ERI Scientific Beta. ERI Scientific Beta is an original initiative which aims to favour the adoption of the latest advances in smart beta design and implementation by the whole investment industry. Its academic origin provides the foundation for its strategy: offer, in the best economic conditions possible, the smart beta solutions that are most proven scientifically with full transparency in both the methods and the associated risks.

# EDHEC-Risk Institute Publications and Position Papers (2012-2015)



# EDHEC-Risk Institute Publications (2012–2015)

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## 2015

- Goltz, F., and V. Le Sourd. Investor Interest in and Requirements for Smart Beta ETFs (April).
- Amenc, N., S. Badaoui, F. Goltz, V. Le Sourd and A. Lodh. Alternative Equity Beta Investing: A Survey (March).
- Amenc, N., K. Gautam, F. Goltz, N. Gonzalez, and J.P Schade. Accounting for Geographic Exposure in Performance and Risk Reporting for Equity Portfolios (March).
- Amenc, N., F. Ducoulombier, F. Goltz, V. Le Sourd, A. Lodh and E. Shirbini. The EDHEC European Survey 2014 (March).
- Deguest, R., L. Martellini, V. Milhau, A. Suri and H. Wang. Introducing a Comprehensive Risk Allocation Framework for Goals-Based Wealth Management (March).
- Blanc-Brude, F., and M. Hasan. The Valuation of Privately-Held Infrastructure Equity Investments (January).

## 2014

- Coqueret, G., R. Deguest, L. Martellini, and V. Milhau. Equity Portfolios with Improved Liability-Hedging Benefits (December).
- Blanc-Brude, F., and D. Makovsek. How Much Construction Risk do Sponsors take in Project Finance. (August).
- Loh, L., and S. Stoyanov. The Impact of Risk Controls and Strategy-Specific Risk Diversification on Extreme Risk (August).
- Blanc-Brude, F., and F. Ducoulombier. Superannuation v2.0 (July).
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For more information, please contact:  
Carolyn Essid on +33 493 187 824  
or by e-mail to: [carolyn.essid@edhec-risk.com](mailto:carolyn.essid@edhec-risk.com)

**EDHEC-Risk Institute**

393 promenade des Anglais  
BP 3116 - 06202 Nice Cedex 3  
France  
Tel: +33 (0)4 93 18 78 24

**EDHEC Risk Institute—Europe**

10 Fleet Place, Ludgate  
London EC4M 7RB  
United Kingdom  
Tel: +44 207 871 6740

**EDHEC Risk Institute—Asia**

1 George Street  
#07-02  
Singapore 049145  
Tel: +65 6438 0030

**ERI Scientific Beta—Japan**

East Tower 4th Floor, Otemachi First Square  
1-5-1 Otemachi, Chiyoda-ku  
Tokyo 100-0004 – Japan  
Tel: +81 352 191 418

**ERI Scientific Beta—North America**

One Boston Place, 201 Washington Street  
Suite 2608/2640 – Boston, MA 02108  
United States of America  
Tel: +1 857 239 8891

**EDHEC Risk Institute—France**

16-18 rue du 4 septembre  
75002 Paris  
France  
Tel: +33 (0)1 53 32 76 30