

From Deterministic to Stochastic Life-Cycle Investing: Implications for the Design of Improved Forms of Target Date Funds

September 2010



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This research has benefited from the support of the UFG-LFP "Dynamic Allocation Models and New Forms of Target-Date Funds" research chair at EDHEC-Risk Institute. We would like to thank Noël Amenc, Xavier Lépine, Bacem Rezgui, as well as participants at the 2009 EDHEC Institutional Days seminar, for very useful comments. Any remaining errors are ours.

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The opinions expressed in this study are those of the authors and do not necessarily reflect those of EDHEC Business School. The authors can be contacted at research@edhec-risk.com.

Foreword

Target-date funds have experienced considerable success over the last ten years, notably through the impetus of retirement savings. However, this commercial success has been unable to hide the limitations of the approaches that are often chosen, typically involving an allocation that merely decreases the share of risky assets over time and disregards all the conceptual advances of the life-cycle approach, and as such fails to integrate the relative prices of the risky assets at different times.

The present research project is part of the UFG-LFP research chair at EDHEC-Risk Institute on "Dynamic Allocation Models and New Forms of Target-Date Funds." This research chair aims to combine state and time dependency and to show that, given a reasonable partition between the state of the world and the time horizon, it is possible, with a limited number of funds, to propose much more efficient target-date investment management.

We would particularly like to thank our partners at UFG-LFP for their support of this research.



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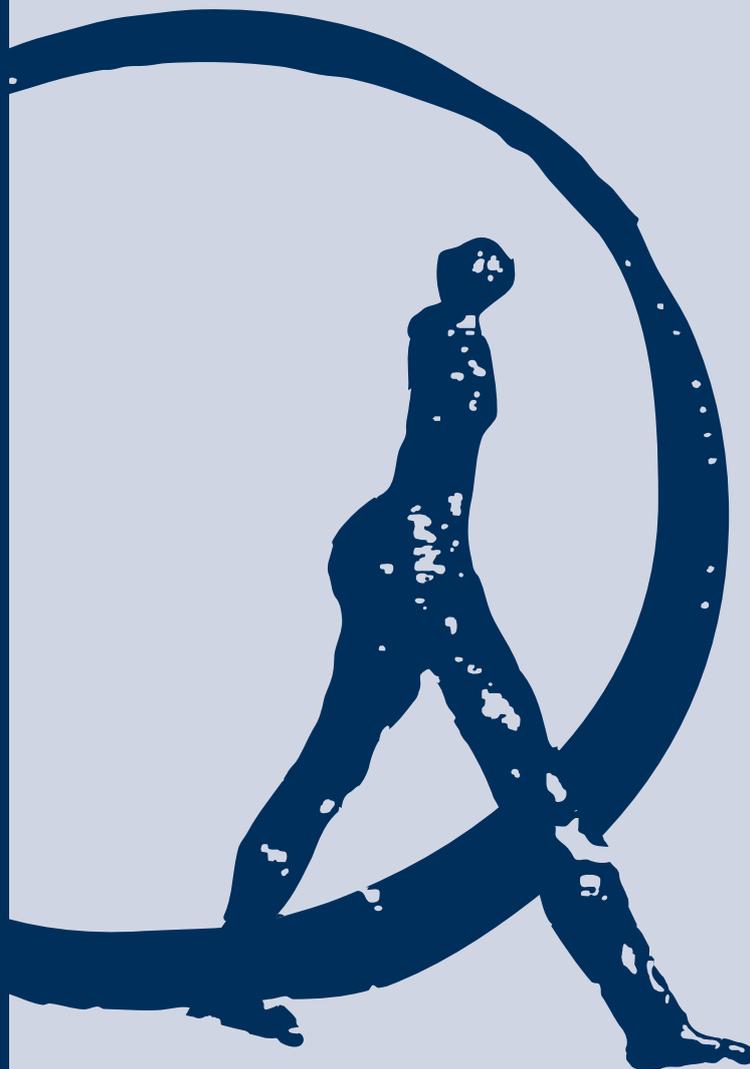
Abstract



Abstract

In an attempt to address the concern over financially illiterate individuals being increasingly responsible for investment decisions related to retirement risk, the financial industry has started to design dedicated mutual fund products known as *target date funds*. These funds, whose aim is to provide investors with one-stop solutions to their life-cycle investment needs, typically propose a deterministic decrease of equity allocation until a date called the *target date* of the fund. This approach, however, has been found inconsistent with the prescriptions of standard life-cycle investment models (Viceira and Field 2007). In this paper, we characterise in closed-form the optimal time- and state-dependent allocation strategy for a long-term investor preparing for retirement in the presence of interest-rate and inflation risks and a mean-reverting equity risk premium. We confirm that existing target date fund products are the wrong answer to the right question, and the opportunity cost involved in purely deterministic life-cycle strategies is found to be substantial for reasonable parameter values. Surprisingly, perhaps we also find that even reasonably fine partitions of the set of investors and market conditions, only marginally more complex than current partitions based solely on time horizon, allow substantial welfare gains compared to existing target date funds. Our results have important practical implications since they suggest that a parsimonious set of life-cycle investment benchmarks can be designed, which could serve as relatively accurate proxies for a whole range of retail investors' optimal long-term investment strategies.

1. Introduction



1. Introduction

A global trend towards the use of stricter, mark-to-market accounting rules, combined with an enhanced regulatory focus on risk management, has led corporations to transfer an increasing fraction of pension-related risks to individuals. As a consequence, the retirement system in most developed countries has undergone a substantial transformation in recent years, with a shift from defined-benefit (DB) plans to defined-contribution (DC) plans, and the development of Individual Retirement Accounts. In 2009, DC assets accounted for 42% of global pension assets, compared with 32% in 1999 (Towers Watson 2010), a share that is likely to keep increasing. As a result of this trend, employees must increasingly rely on their own saving and investment decisions to fund their retirement. This is a serious concern, not only because of the risk transfer but also because individual investors usually lack the expertise to make educated investment decisions and often show a great deal of inertia (Mitchell *et al.* 2006; Wise 2009). In response to this concern, the asset management industry has begun to design dedicated investment mutual fund products whose goal is to provide investors with one-stop solutions to their life-cycle investment needs. These funds, known as *target date funds* (TDFs) or *life-cycle funds*, typically rebalance the investments in the underlying funds so as to implement a deterministic decrease of equity allocations (also known as *glide path*) until a date called the *target date* or *target maturity* date of the fund. This prescription is somewhat reminiscent of the Shiller (2005) rule of thumb, which states that the allocation to equity should be given, roughly, by 100 minus the investor's age in years.

Embedding life-cycle investment decisions in a one-stop decision is certainly a valuable attempt to provide added value to unsophisticated investors who may otherwise make sub-optimal decisions and possibly stick to them. In fact, TDFs have experienced a great deal of success: from about \$1 billion in 1996, when Fidelity launched its own version of these funds, total TDF assets multiplied more than twelve-fold, to more than \$180 billion, between 2002 and March 2008. And growth is accelerating, as these funds attracted \$58 billion in 2007, compared with \$35 billion in 2006; in the first three months of 2008, equity funds experienced outflows of more than \$40 billion while TDFs gathered nearly \$15bn (Strategic Insight 2009). Life-cycle investing is unambiguously gaining traction in the industry, but the question of whether one can fully rationalise the deterministic glide paths implemented in most target date funds within the prescriptions of modern portfolio theory has not yet been resolved.

Long-term investment decisions in the presence of a stochastic opportunity set have been formally analysed by Merton (1971), who shows that the presence of risk factors that impact the productivity of wealth justifies the introduction of intertemporal hedging demands in an investor's optimal allocation. Subsequent papers have shown that, when maximising utility from her nominal wealth, an investor hedges only those state variables that impact the nominal short-term rate and the market prices of risk (Lioui and Poncet 2001; Detemple, Garcia, and Rindisbacher 2003). Other papers have solved explicitly the portfolio choice problem when only one state variable is stochastic (e.g., the

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nominal interest rate in Sørensen [1999], Lioui and Poncet [2001] and Munk and Sørensen [2004], or the stock index Sharpe ratio in Kim and Omberg [1996], Campbell and Viceira [1999] or Wachter [2002]). More realistic models have also been developed to account for the presence of both state variables. This challenge has been addressed in the context of both discrete-time VAR models (Campbell, Chan, and Viceira 2003), and continuous-time models, either by solving numerically the Hamilton-Jacobi-Bellman (HJB) equation obtained through dynamic programming (Brennan, Schwartz, and Lagnado 1997), or, more recently, by exploiting the affine structure of the model (in the sense of Dai and Singleton [2000] and Liu [2007]) and solving the HJB equation explicitly (Munk, Sørensen, and Vinther 2004) or quasi-explicitly up to the solution of a series of ordinary differential equations (Sangvinatsos and Wachter 2005). A related strand of the literature has focused on portfolio choice over the life-cycle when the investor earns a stochastic non-financial income (Viceira 2001; Cocco, Gomes, and Maenhout 2005 or Benzoni, Collin-Dufresne, and Goldstein 2007). In general, these papers focus on modelling labour income and thus assume constant investment opportunities. Munk and Sørensen (2010) relax this assumption by adding a mean-reverting short-term interest rate to a model with stochastic labour income.

In most of these papers, the fraction of wealth allocated to equities is usually shown to be a decreasing function of time-to-horizon, either because of the term structure of equity risk implied by the presence of mean reversion in equity returns or because of the need for young investors to compensate for the bond-like

nature of human capital.¹ Regardless of the explanation for why young investors should hold more stocks than older investors, these findings seem to provide at best partial justification for the glide path prescription of target date funds. In fact, recommending the same allocation for investors with a given time horizon regardless of market conditions cannot possibly be optimal in the presence of a stochastic opportunity set, and omitting such state-dependencies can lead to severe efficiency costs. As very clearly explained by Viceira and Field (2007), "long-term equity investors should invest more on average in equities than their short-horizon counterparts, but they should also consider periodic revisions of this allocation as market conditions change. It is logically inconsistent to count on reduced long-term risk while ignoring the variation in returns that produces it. This market-sensitive allocation policy is very different from the asset allocation policy of life-cycle funds, whose target mix moves mechanically away from stocks as an inverse function of investment horizon, regardless of market conditions. Thus mean-reversion arguments provide, if anything, only a partial justification for the roll down schedule characteristic of life-cycle funds." In addition to Viceira and Field (2007), a number of recent studies have documented the many shortcomings of standard forms of target date funds, including Basu and Brisbane (2009), Basu and Drew (2009), Booth and Yakoubov (2000), Cairns, Blake, and Dowd (2006), Lewis, Okunev, and White (2007), or Bodie, Detemple, and Rindisbacher (2009). For example, Cairns, Blake, and Dowd (2006) do a formal analysis of the welfare loss involved in following a deterministic strategy similar to actual target date fund strategies with respect to

1 - One notable exception is Benzoni, Collin-Dufresne, and Goldstein (2007), who question the bond-like nature of human capital, and show that the presence of co-integration of labour income and stock dividends results in a high long-term correlation between stocks and human capital. This substantially reduces the optimal demand for stocks by young investors, and even results in short positions in the equity market.

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the optimal stochastic life-cycle strategy, and they found the associated opportunity cost to be very substantial.

As a result, current forms of target date funds appear to be the wrong answer to the right question: namely, how to design long-term investment strategies for financially illiterate individuals who are increasingly responsible for investment decisions related to retirement risk. The limitations of current products extend far beyond the use of deterministic, as opposed to market-sensitive, asset allocation decisions. For example, life-cycle investment models suggest that the average decrease in equity allocation should be matched by an increased investment in the riskfree asset, as opposed to an increased investment in bonds, a feature not found in most TDFs.²

In addition, to focus solely on differences in time horizon, many TDFs do not account for differences in risk aversion, an obviously undesirable (and unneeded) simplification. Current forms of TDFs can be blamed for using not only the wrong allocation strategies but also the wrong building blocks. Instead of expressing allocation decisions in terms of stocks versus bonds, portfolio theory unambiguously suggests that proper allocation decisions should be expressed in terms of performance-seeking portfolio versus liability-hedging portfolio. On the one hand, constant maturity bond indices, commonly used by target date funds, are legitimate ingredients in the performance-seeking portfolio, where they will help diversify equity risk to generate the highest possible risk-reward ratio. On the other hand, liability-hedging portfolios should be composed of bond

portfolios with a duration matching the investor's residual time horizon. Bond indices have too long a duration for investors close to retirement and too short a duration for investors far from retirement; as a result they can hardly be regarded as safe assets for any possible investor. In addition to poor management of interest rate risk, current TDFs fail to manage inflation risk, management of which calls for the use of inflation-linked bonds, or some other assets with attractive inflation-hedging properties, in the liability-hedging portfolio.

The intended contribution of our paper with respect to the aforementioned literature is dual. First, we contribute to the life-cycle literature by proposing a comprehensive long-horizon dynamic allocation model with labour income in the presence of stochastic inflation and interest rates and a mean-reverting equity risk premium. Our model slightly extends Munk, Sørensen, and Vinther (2004) by relaxing the assumption of a perfect negative correlation between equity returns and risk premium uncertainty.³ We also consider real estate an important addition to the asset mix, in addition to stocks and bonds. Although our setting is rich enough to account for the aforementioned features, as well as the presence of an endowment stream (deterministic), we manage to obtain a quasi-explicit representation for the optimal portfolio strategy. Having quasi-analytical expressions for the optimal strategy turns out to be very useful in the analysis of the (sub)optimality of allocation strategies embedded in target date funds. We confirm that the opportunity cost involved in focusing on a deterministic allocation scheme is very substantial for reasonable parameter

2 - A partial explanation for this could be related to the fact that TDF providers do not have strong incentives to promote the use of low-fee money market funds.

3 - As such, our paper is closely related to Sangvinatsos and Wachter (2005).

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values. Implementing such extended forms of life-cycle investing strategies in a delegated money management context is a serious challenge, however, because it requires a narrower classification of plan participants based on factors other than the age of the participant. In retail money management, the challenge is to provide a parsimonious enough partition of the relevant subjective attributes (mainly age and risk aversion) and objective attributes (in particular, the current estimated risk premium provided by equities) that can exhaust the overall population of retail investors and market conditions. Against this backdrop, our second and main contribution is to propose a formal analysis of the speed at which realistic strategies that take into account the constraints related to limited customisation in retail money management converge to the true optimal fully customised strategy. Surprisingly, perhaps, we find that very reasonably fine partitions, perfectly consistent with implementation in retail money management, allow welfare gains substantially greater than those of deterministic life-cycle strategies. Our results are found to be robust with respect to the introduction of reasonable measurement errors in equity risk premium, the only unobservable quantity in the model. On the whole, our analysis has important potential implications for the target date fund industry since it suggests that a limited number of life-cycle investment benchmarks can be designed that will serve as relatively accurate proxies for truly optimal long-term retail investment vehicles in a retirement context.

The rest of the paper is organised as follows. In section 2, we introduce a formal optimal

allocation model for a long-term investor in the presence of labour income and preferences over real terminal wealth, as well as stochastic interest rates and an equity risk premium. Section 3 proposes a numerical analysis of the model, and provides measures of welfare costs associated with using sub-optimal strategies. Section 4 concludes and presents suggestions for further research. Technical details and proofs of the main results are relegated to dedicated appendices.

1. Introduction

2. Life-Cycle Investment Decisions with Stochastic Interest and Inflation Rates and a Mean-Reverting Equity Risk Premium



2. Life-Cycle Investment Decisions with Stochastic Interest and Inflation Rates and a Mean-Reverting Equity Risk Premium

In this section, we solve the optimisation programme of an investor who faces stochastic investment opportunities, receives an income stream, and has preferences expressed over terminal real wealth.

2.1 State Variables and Asset Returns

We consider an investor with finite horizon date T . Uncertainty in the economy is represented through a standard probability space $(\Omega, \mathcal{A}, \mathbb{P})$ endowed with a filtration $\mathcal{F} \equiv (\mathcal{F}_t)_{t \in [0, T]}$ such that $\mathcal{F}_T = \mathcal{A}$. All processes relevant to decision making are assumed to be progressively measurable with respect to this filtration.

The sources of risk in our model are equity return uncertainty z^S , equity risk premium (Sharpe ratio) uncertainty z^λ , real estate return uncertainty z^Y , nominal interest rate uncertainty z^R , and consumer price index uncertainty z^Φ .⁴ The processes z^S, z^λ, z^Y, z^R , and z^Φ are standard Wiener processes such that the vector $(z^S, z^\lambda, z^Y, z^R, z^\Phi)$ is Gaussian. We let ρ_{ij} be the correlation between z^i and z^j , where i and j lie in the set of indices $\{R, \Phi, \lambda, S, Y\}$. Since no other source of uncertainty impacts investor's decisions, we can assume with no loss of generality that \mathcal{F} is the complete version of the filtration generated by $(z^S, z^\lambda, z^Y, z^R, z^\Phi)$.

We assume that the state variables evolve as:

$$dR_t = a(b - R_t) dt + \sigma_R dz_t^R \quad (2.1)$$

$$\frac{d\Phi_t}{\Phi_t} = \pi dt + \sigma_\Phi dz_t^\Phi \quad (2.2)$$

$$d\lambda_t^S = \kappa(\bar{\lambda} - \lambda_t^S) dt + \sigma_\lambda dz_t^\lambda \quad (2.3)$$

$$\frac{dS_t}{S_t} = (R_t + \sigma_S \lambda_t^S) dt + \sigma_S dz_t^S \quad (2.4)$$

$$\frac{dY_t}{Y_t} = (R_t + \sigma_Y \lambda_t^Y) dt + \sigma_Y dz_t^Y \quad (2.5)$$

or, in vector form:

$$dR_t = a(b - R_t) dt + \sigma'_R dz_t$$

$$\frac{d\Phi_t}{\Phi_t} = \pi dt + \sigma'_\Phi dz_t$$

$$d\lambda_t^S = \kappa(\bar{\lambda} - \lambda_t^S) dt + \sigma'_\lambda dz_t$$

$$\frac{dS_t}{S_t} = (R_t + \sigma_S \lambda_t^S) dt + \sigma'_S dz_t$$

$$\frac{dY_t}{Y_t} = (R_t + \sigma_Y \lambda_t^Y) dt + \sigma'_Y dz_t$$

where R denotes the nominal short-term interest rate, π is the assumed constant expected inflation rate, Φ is the price index, S is the stock index price, λ^S is its Sharpe ratio, and Y is a traded real estate index price. The asset mix includes at least the stock index S , the real estate index Y , and a nominal constant maturity zero-coupon bond B with a maturity denoted by τ .⁵ This ensures that the risks z^S, z^Y , and z^R are spanned. Hedging the risk z^λ is possible if changes in the stock value are perfectly anti-correlated to the changes in equity premium. This is equivalent to taking $\rho_{S\lambda} = -1$. Inflation risk z^Φ may be spanned by the introduction of an indexed constant maturity zero-coupon bond I . In the absence of such a bond, inflation risk is not entirely spanned, so the market is incomplete.

It is shown in appendix A that if τ denotes the constant maturity of the bonds, then the dynamics of B and I are given by:

4 - In appendix C.2, we also consider an extended version of the model that accounts for the presence of stochastic equity volatility, but we maintain the assumption of a constant volatility for simplicity of exposure throughout the paper.

5 - Real estate assets can prove useful ingredients in long-term investment strategies both for their inflation-hedging benefits and for their diversification benefits with respect to stocks and bonds. The main messages in the paper, however, would hold true in the absence of a real estate asset.

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$$\frac{dB_t}{B_t} = [R_t + \sigma_B(\tau)\lambda_R] dt + \sigma_B(\tau)' dz_t \quad (2.6)$$

$$\frac{dI_t}{I_t} = [R_t + \sigma_B(\tau)\lambda_R + \sigma_\phi\lambda_\phi] dt + \sigma_I(\tau)' dz_t \quad (2.7)$$

where

$$\sigma_B(\tau) = \sigma_B(\tau)\sigma_R, \quad \sigma_I(\tau) = \sigma_\phi + \sigma_B(\tau)\sigma_R$$

$$\text{and } \sigma_B(\tau) = b_a(\tau) = -\frac{1-e^{-\sigma\tau}}{\sigma}.$$

The volatility matrix of traded assets, σ_t , is obtained by concatenating the volatility vectors of these assets. If the inflation-indexed bond is not traded, this is a 5×3 matrix. If this bond is traded, σ_t will be of size 5×4 . We define λ_t as the unique market price of risk vector spanned by the assets. If the market is incomplete, i.e., if σ_t is not square and non-singular, there are infinitely many prices of risk. As shown by He and Pearson (1991), the market prices of risk are those vector processes of the form $\lambda + \nu$, where ν satisfies the condition $\sigma_t'\nu_t = 0$ for all t , as well as technical measurability and integrability requirements. Each ν defines a pricing kernel $M^{\lambda+\nu}$, through:

$$dM_t^{\lambda+\nu} = -M_t^{\lambda+\nu}[R_t dt + (\lambda_t + \nu_t)' dz_t]$$

As explained below, in this incomplete market setting, the optimisation programme is solved by computing the value of ν_t such that the optimal payoff can be replicated by a trading strategy involving available assets.

2.2 Human Capital

The investor is also assumed to receive an income flow $(e_t)_{t \in [0, T]}$, which, to avoid further increases in the complexity of the model, we take to be deterministic.⁶ Let θ_t be the vector of dollar amounts invested

in the traded assets, with the following convention: the first component is the wealth allocated to the nominal bond with constant maturity τ , the second component is the amount invested in stock, the third component is the investment in real estate, and θ_t has a fourth component only if the inflation-linked bond is traded, this fourth component being the dollar investment in this indexed bond. The remaining wealth is invested in the cash, also referred to as the "risk-free asset". Thus the financial wealth evolves as:

$$dA_t = (A_t R_t + \theta_t' \sigma_t' \lambda_t) dt + \theta_t' \sigma_t' dz_t + e_t dt$$

Since e is a deterministic income stream, it can be valued as a bond maturing at date T and paying a continuous coupon. We define the human capital of the investor as the current value of future income streams:

$$H_t = \int_t^T B(t, s) e_s ds$$

which is a function $\mathcal{H}(t, R_t)$. In particular, the volatility vector of H is proportional to σ_R :

$$\begin{aligned} \sigma_H &= \frac{\mathcal{H}_R(t, R_t)}{\mathcal{H}(t, R_t)} \sigma_R \\ &= \frac{1}{H_t} \left(\int_t^T B(t, s) b_a(s-t) e_s ds \right) \sigma_R \end{aligned}$$

and the portfolio strategy θ^H replicating H is given by:

$$\theta_t^H = \frac{1}{b_a(\tau)} \left(\int_t^T B(t, s) b_a(s-t) e_s ds \right) e_1$$

where $e_1 = (1, 0, 0, 0)'$.

The process $A+H$, which is the total wealth process, can be seen as the value of a

6 - Introducing income risk in this model is straightforward as long as this risk is spanned by traded assets. In the presence of unspanned income risk, however, the income stream cannot be valued as the dividend flow of a traded asset/portfolio, and one must introduce a utility-based investor-specific price (Henderson 2005; Munk 2000).

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self-financing strategy. Indeed, it evolves as:

$$d(A_t + H_t) = (A_t + H_t)R_t dt + [\theta_t + \theta_t^H]' \sigma_t' \lambda_t dt + [\theta_t + \theta_t^H]' \sigma_t' dz_t$$

In particular, the process $M^\lambda(A + H)$ follows a martingale, so for $t \leq T$ we have:

$$A_t + H_t = \mathbb{E}_t \left[\frac{M_T^\lambda}{M_t^\lambda} A_T \right] \quad (2.8)$$

2.3 Optimal Allocation in the General Incomplete Market Setting

Throughout the paper, we let $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ be the investor's constant relative risk aversion (CRRA) utility function. As in Brennan and Xia (2002), Munk, Sørensen, and Vinther (2004), and Sangvinatsos and Wachter (2005), we assume that she has preferences from real, as opposed to nominal, wealth. Hence the problem can be mathematically written as:

$$\max_{\theta} \mathbb{E} \left[U \left(\frac{A_T}{\Phi_T} \right) \right] \quad (2.9)$$

2.3.1 Derivation of the Optimal Strategy

We first solve the problem with four assets only: the nominal bond, the stock index, the real estate index, and the risk-free asset. Adding inflation-linked bonds to the asset mix would be strongly desirable since it would allow a perfect hedge against inflation risk. The capacity of the inflation-linked bond markets is limited, however, and we therefore focus in what follows on the general situation in which perfect inflation-hedging instruments are not available. For

completeness, we also report in appendix C.1 the optimal investment strategy when inflation-linked bonds are added to the menu of asset classes.

(2.9) is a dynamic problem since the control variable is the vector of dollar amounts θ . When we take the martingale approach developed by He and Pearson (1991) in incomplete market settings, this dynamic problem can be reformulated as a static problem for each pricing kernel $M^{\lambda+\nu}$:

$$\begin{aligned} \max_{A_T} \mathbb{E} \left[U \left(\frac{A_T}{\Phi_T} \right) \right], \\ \text{s.t. } \mathbb{E} \left[M_T^{\lambda+\nu} A_T \right] = A_0 + H_0 \quad (2.10) \end{aligned}$$

The form of the budget constraint follows from (2.8). The solution technique for the portfolio choice problem then proceeds as follows: first, a candidate optimal terminal payoff is computed from (2.10) for each value ν . Second, this payoff is priced using the pricing kernel $M^{\lambda+\nu}$. Applying Ito's lemma, one derives the diffusion term in this process and computes the value ν_t^* of ν_t such that this diffusion term is spanned by σ_S . In other words, one computes the value of ν_t such that the candidate optimal payoff can be replicated by a trading strategy involving only available assets. $M^{\lambda+\nu^*}$ is called the *minimax pricing kernel*. He and Pearson (1991) provide another useful characterisation of the minimax pricing kernel (which justifies the terminology): ν^* is the value of ν that minimises the value of the static problem (2.10) over all admissible values for ν .

Here the risky assets are the nominal bond, the stock index, and the real estate index. Hence the volatility matrix of the traded assets is (we can now drop the index t

2. Life-Cycle Investment Decisions with Stochastic Interest and Inflation Rates and a Mean-Reverting Equity Risk Premium

because the matrix is constant over time):⁷

$$\sigma = \begin{pmatrix} \sigma_B(\tau) & \sigma_S & \sigma_Y \end{pmatrix}$$

The spanned market price of risk vector is given by:

$$\lambda_t = \sigma(\sigma'\sigma)^{-1} \begin{pmatrix} b_a(\tau)\sigma_R\lambda_R \\ \sigma_S\lambda_t^S \\ \sigma_Y\lambda_Y \end{pmatrix}$$

Introducing the volatility matrix of traded risks:

$$\Sigma = \begin{pmatrix} \sigma_R & \sigma_S & \sigma_Y \end{pmatrix}$$

we can rewrite λ_t as:

$$\lambda_t = \Sigma(\Sigma'\Sigma)^{-1} \begin{pmatrix} \sigma_R\lambda_R \\ \sigma_S\lambda_t^S \\ \sigma_Y\lambda_Y \end{pmatrix}$$

To isolate the effect of the stochastic λ_t^S on λ_t , we will use the following decomposition:

$$\lambda_t = \Lambda_1 + \lambda_t^S \Lambda_2$$

where:

$$\Lambda_1 = \Sigma(\Sigma'\Sigma)^{-1} \begin{pmatrix} \sigma_R\lambda_R \\ 0 \\ \sigma_Y\lambda_Y \end{pmatrix},$$

$$\Lambda_2 = \Sigma(\Sigma'\Sigma)^{-1} \begin{pmatrix} 0 \\ \sigma_S \\ 0 \end{pmatrix}$$

7 - This matrix is constant because we have assumed a constant maturity bond. If the nominal bond had a fixed maturity T_0 , the volatility matrix would be time-dependent.

We also let

$$N = I_5 - \sigma(\sigma'\sigma)^{-1}\sigma' = I_5 - \Sigma(\Sigma'\Sigma)^{-1}\Sigma$$

be the matrix of the residual of the projection onto the columns of σ . Were the market dynamically complete, the matrix N would be zero. With these notations, we have the following result.

Proposition 1 Consider programme (2.9).

- The optimal payoff is given by:

$$A_T^* = \frac{A_0 + H_0}{\mathbb{E} \left[(M_T^{\lambda+\nu^*} \Phi_T)^{1-\frac{1}{\gamma}} \right]} \left(M_T^{\lambda+\nu^*} \right)^{-\frac{1}{\gamma}} \Phi_T^{1-\frac{1}{\gamma}} \quad (2.11)$$

where $M^{\lambda+\nu^*}$ is the minimax pricing kernel, and the optimal wealth process reads:

$$A_t^* = \frac{A_0 + H_0}{\mathbb{E} \left[(M_T^{\lambda+\nu^*} \Phi_T)^{1-\frac{1}{\gamma}} \right]} \Phi_t^{1-\frac{1}{\gamma}} \left(M_t^{\lambda+\nu^*} \right)^{-\frac{1}{\gamma}} g(t, R_t, \lambda_t^S) - H_t \quad (2.12)$$

with:

$$g(t, R_t, \lambda_t^S) = \exp \left[\frac{1-\gamma}{\gamma} \left[A_1(T-t) + A_2(T-t)R_t + A_3(T-t)\lambda_t^S + \frac{1}{2}A_4(T-t)(\lambda_t^S)^2 \right] \right] \quad (2.13)$$

- The functions A_1 , A_2 , A_3 , and A_4 are the solutions to a system of (coupled) ordinary differential equations (ODEs):

$$A_2(T-t) = \frac{1 - e^{-a(T-t)}}{a} \quad (2.14)$$

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$$A'_4(T-t) = \frac{\|\Lambda_2\|^2}{\gamma} + 2 \left[\frac{1-\gamma}{\gamma} \sigma'_\lambda \Lambda_2 - \kappa \right] A_4(T-t) + \frac{1-\gamma}{\gamma} \left[\sigma_\lambda^2 - (1-\gamma) \sigma'_\lambda N \sigma_\lambda \right] A_4(T-t)^2 \quad (2.15)$$

$$A'_3(T-t) = \Lambda'_2 \sigma_\phi + \frac{(\Lambda_1 - \sigma_\phi)' \Lambda_2}{\gamma} + \frac{1-\gamma}{\gamma} \sigma'_R \Lambda_2 A_2(T-t) + \left[\frac{1-\gamma}{\gamma} \sigma'_\lambda \Lambda_2 - \kappa \right] A_3(T-t) + \left[\kappa \bar{\lambda} + \frac{1-\gamma}{\gamma} \sigma'_\lambda (\Lambda_1 - \sigma_\phi) + \frac{(1-\gamma)^2}{\gamma} \sigma'_\phi N \sigma_\lambda \right] A_4(T-t) + \frac{1-\gamma}{\gamma} \left[\sigma_\lambda^2 - (1-\gamma) \sigma'_\lambda N \sigma_\lambda \right] A_3(T-t) A_4(T-t) + \frac{1-\gamma}{\gamma} \sigma_R \sigma_\lambda \rho_{R\lambda} A_2(T-t) A_4(T-t) \quad (2.16)$$

$$A'_1(T-t) = -\pi + \frac{\|\Lambda_1 - \sigma_\phi\|^2}{2\gamma} + \sigma'_\phi \Lambda_1 - \frac{(1-\gamma)^2}{2\gamma} \sigma'_\phi N \sigma_\phi + \left[ab + \frac{1-\gamma}{\gamma} \sigma'_R (\Lambda_1 - \sigma_\phi) \right] A_2(T-t) + \frac{1}{2} \sigma_\lambda^2 A_4(T-t) + \left[\kappa \bar{\lambda} + \frac{1-\gamma}{\gamma} \sigma'_\lambda (\Lambda_1 - \sigma_\phi) + \frac{(1-\gamma)^2}{\gamma} \sigma'_\phi N \sigma_\lambda \right] A_3(T-t) + \frac{1-\gamma}{2\gamma} \left[\sigma_\lambda^2 - (1-\gamma) \sigma'_\lambda N \sigma_\lambda \right] A_3(T-t)^2 + \frac{1-\gamma}{2\gamma} \sigma_R^2 A_2(T-t)^2 + \frac{1-\gamma}{\gamma} \sigma_R \sigma_\lambda \rho_{R\lambda} A_2(T-t) A_3(T-t) \quad (2.17)$$

with the initial conditions $A_1(0) = A_3(0) = A_4(0) = 0$.

• The optimal strategy is given by:

$$\theta_t^* = (A_t^* + H_t) \left[\frac{1}{\gamma} (\sigma' \sigma)^{-1} \sigma' \lambda_t + \left(1 - \frac{1}{\gamma} \right) \frac{A_2(T-t)}{A_2(\tau)} e_1 - \left(1 - \frac{1}{\gamma} \right) \left[A_3(T-t) + A_4(T-t) \lambda_t^S \right] (\sigma' \sigma)^{-1} \sigma' \sigma_\lambda + \left(1 - \frac{1}{\gamma} \right) (\sigma' \sigma)^{-1} \sigma' \sigma_\phi \right] - \frac{1}{b_a(\tau)} \mathcal{H}_R(t, R_t) e_1 \quad (2.18)$$

Proof. See appendix B.1.

This result is completed by the following proposition, which provides an expression for the indirect utility function.

Proposition 2 The indirect utility function is:

$$J(t, A_t^*, R_t, \lambda_t^S, \Phi_t) = \frac{1}{1-\gamma} \left(\frac{A_t^* + H_t}{\Phi_t} \right)^{1-\gamma} g(t, R_t, \lambda_t^S)^y$$

Proof. See appendix B.1.

2.3.2 Analysis of the Optimal Portfolio

The optimal strategy (2.18) can also be written in terms of weights as:

$$\omega_t^* = \left(1 + \frac{H_t}{A_t^*} \right) \omega_t^{*,0} - \frac{1}{A_t^*} \theta_t^H \quad (2.19)$$

where $\omega_t^{*,0}$ is the optimal weight vector when there is no income and θ_t^H is the portfolio replicating the income stream (also known as income-hedging portfolio), as defined in subsection 2.1. The portfolio strategy $\omega_t^{*,0}$ is given by:

$$\omega_t^{*,0} = \frac{1}{\gamma} \omega_t^{PSP} - \left(1 - \frac{1}{\gamma} \right) \left[A_3(T-t) + A_4(T-t) \lambda_t^S \right] \omega^\lambda - \left(1 - \frac{1}{\gamma} \right) A_2(T-t) \omega^R + \left(1 - \frac{1}{\gamma} \right) \omega^\phi$$

where:

- $\omega_t^{PSP} = (\sigma' \sigma)^{-1} \sigma' \lambda_t$ is the performance-seeking portfolio (PSP);
- $\omega^R = (\sigma' \sigma)^{-1} \sigma' \sigma_R$ is the interest rate-hedging portfolio;
- $\omega^\lambda = (\sigma' \sigma)^{-1} \sigma' \sigma_\lambda$ is the equity premium-hedging portfolio;
- $\omega^\phi = (\sigma' \sigma)^{-1} \sigma' \sigma_\phi$ is the inflation hedging portfolio.

Another way of decomposing the optimal allocation for the individual investor would involve grouping the intertemporal hedging demand against interest-rate risk ω^R and

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the intertemporal hedging demand against inflation risk ω^Φ to identify a portfolio that would serve as a proxy for the safe asset for a long-term investor preparing for retirement, a portfolio that would typically be referred to as a *liability-hedging portfolio* in a pension fund context. In a complete market setting, this portfolio would be entirely invested in a pure discount inflation-linked bond with a maturity date matching the investor's horizon. In the absence of inflation-linked bonds, the liability-hedging portfolio can be written as $\omega^\Phi - A_2(T - t)\omega^R$. On the one hand, the term $-A_2(T - t)\omega^R$ is the (perfect) hedge against interest rate risk, which can be interpreted as the replicating portfolio for a pure discount bond with maturity T using the constant maturity bond index and cash. On the other hand, ω^Φ is the imperfect hedge against inflation risk, which can be interpreted as the portfolio of risky assets that exhibits the maximum correlation with respect to changes in inflation.

A number of additional comments are in order. First, $\omega_t^{*,0}$ is independent of the current wealth of the investor, a property typical of CRRA utility functions. We also confirm that the optimal strategy involves not only time dependencies but also state dependencies, in contrast to the heuristic deterministic glide paths of life cycle funds. In particular, the optimal allocation will change as a function of changes in the stock index Sharpe ratio. Given that the functions A_i are continuous at zero, the hedging demand ω^λ against the Sharpe ratio risk vanishes, as expected, when the time-to-maturity shrinks to zero. For a finite time-to-maturity, this demand is non-zero. In the specific case of a perfect anti-correlation between shocks to S

and shocks to λ^S , this hedging demand simplifies into a portfolio fully invested in the stock index. In the general case $\rho_{S\lambda} > -1$, some general properties can be analysed even in the absence of fully analytical expressions.⁸ In particular, we show in appendix B.2 that, for $\gamma > 1$, A_4 is a positive increasing function. This result implies that if the investor is more risk-averse than the logarithmic one (i.e., if $\gamma > 1$), then an increase in the expected stock return should lead to a decrease in the allocation to the portfolio hedging against Sharpe ratio risk. In particular, if $\rho_{S\lambda} = -1$, an increase in the Sharpe ratio should result in greater allocation to stocks. This effect is independent of the effect of a higher Sharpe ratio on the composition of the tangency portfolio.⁹

Second, for a vanishing time-to-maturity, the human capital and the hedging demand against income risk in (2.18) go to zero, so the investor behaves as if she were to receive no income:

$$\lim_{T-t \downarrow 0} [\omega_t^* - \omega_t^{*,0}] = 0$$

Moreover, the hedging demands against interest rate risk and equity premium risk cancel out as well. Eventually, we obtain that:

$$\lim_{T-t \downarrow 0} \omega_t^* = \frac{1}{\gamma} \omega_t^{PSP} + \left(1 - \frac{1}{\gamma}\right) \omega^\Phi$$

If there were no inflation risk, the investor would behave myopically, by investing only in the PSP and the cash. If inflation is stochastic, the investor keeps hedging this risk, even just before the horizon.

Third, the decomposition (2.19) sheds light on the impact of the non-financial income.

8 - In general, having a system of ordinary differential equations (ODEs) satisfied by the functions A_i greatly simplifies numerical simulations of the optimal strategy, which we shall use extensively in what follows. Closed-form expressions could possibly be obtained, but given their complexity they would arguably be of little help in understanding the impact of the various parameters (see, e.g., Munk, Sørensen, and Vinther [2004] for such expressions in a slightly different context, with two risky traded assets only).

9 - Since A_4 is an increasing function of time-to-maturity, the term $A_4(T - t)\lambda^S$ tends to make the allocation to stocks an increasing function of horizon for investors with $\gamma > 1$. However, the final effect of an increase in investment horizon on the weight also depends on $A_3(T - t)$.

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It shows that in the presence of deterministic income, the investor chooses the same portfolio as another investor with the same age and risk aversion characteristics who would receive no income but would have a wealth equal to $A + H$. This result holds more generally in the presence of stochastic income as long as income risk is spanned (Munk and Sørensen 2010).

We now turn to a numerical analysis of the model, in which we shall make the following assumptions. First, we take the income stream to be zero. Indeed, as explained above, an investor with non-zero income behaves as an investor without income and with greater wealth. Second, we take the stock index Sharpe ratio to be perfectly anti-correlated with stock index returns. This assumption is supported by empirical works that proxy the expected excess return as an affine function of the dividend yield (see, e.g., Barberis [2000] and Xia [2001], who find strongly negative correlations between excess returns and the dividend yield), and has been also made by Wachter (2002) and Munk, Sørensen, and Vinther (2004). It is not crucial but it clarifies the impact of the stochastic Sharpe ratio on the allocation: the hedging portfolio ω^λ is entirely invested in equities, so an increase in the Sharpe ratio leads to greater allocation to the stock index, both in the speculative portfolio and in the hedging demand. Although equity premium risk is spanned by the stock itself, the market is still incomplete because we do not necessarily assume that a perfect inflation-hedging instrument exists.

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The goal of this section is to provide a quantitative estimate for the utility loss incurred by using a heuristic, sub-optimal strategy such as that generally used by target date funds. To assess formally the utility loss incurred by using a sub-optimal strategy rather than the continuous-time optimal strategy, we compute the associated monetary utility loss (MUL), defined as the amount x , expressed as a percentage of initial wealth, such that the investor does not prefer one of the following two options to the other: (a) using the sub-optimal strategy starting with the initial capital A_0 ; (b) using the optimal (continuous-time) strategy starting with a lower initial capital $A_0 - x$. In the subsequent numerical exercise, all real-world strategies that we test are implemented using a quarterly time-step, and the expected utility is approximated using 50,000 simulated outcomes for the terminal payoff. This will provide a quantitative measure for the welfare loss for an investor relying on a sub-optimal strategy based on a deterministic glide path with respect to the ideal continuous-time and fully customised stochastic life-cycle strategy, as in Cairns, Blake, and Dowd (2006), for example. In addition to estimating monetary utility loss, we estimate for various optimal or heuristic strategies a *return-to-risk* (or *risk return*) ratio, which is the ratio of the mean to the standard deviation of the annualised real log return. Formally, this ratio is defined as:

$$\text{Risk return ratio} = \frac{\mathbb{E}[r_T]}{\sqrt{\mathbb{V}[r_T]}}$$

where $r_T = \frac{1}{T} \ln \left(\frac{A_T}{A_0} \frac{I(0,T)}{\Phi_T} \right)$ is the log return on real wealth. Obviously, outperformance as measured by this return-to-risk ratio is not strictly equivalent to

outperformance as measured in terms of expected utility of terminal wealth, which is the maximisation objective used to obtain the optimal strategy. From a mean-variance perspective, however, it is a reasonable indicator of quality for a given portfolio strategy.

Our base-case parameters for the stock price, the Sharpe ratio, and the price index processes are borrowed from Munk, Sørensen, and Vinther (2004). In particular, we set the correlation between the Brownian motions z^S and z^λ to -1 , which removes the incompleteness due to the Sharpe ratio. The maturity of the constant-maturity bond is taken equal to ten years.¹⁰ Parameters for the real estate process are taken from De Jong, Driessen, and Van Hemert (2007). All parameter values are displayed in table 1.¹¹

It must be acknowledged at this point that the continuous-time optimal strategy is a purely idealised strategy that could not be implemented in realistic situations, and as such proves to be an unfair benchmark for the heuristic deterministic glide path strategies. In fact, real-world implementations of stochastic life-cycle strategies in retail management of mutual funds will suffer from at least three limitations with respect to the idealised optimal strategy. First, the strategy will involve a discrete partition of the set of trading dates: the strategy will be implemented with discrete rather than continuous trading and with monthly or quarterly trading periods. The discrete implementation of the optimal strategy will involve a monetary utility loss, which must be measured before the inefficiency of heuristic target date fund strategies can be assessed. Second, the

10 - The choice of the maturity for the bond index has no impact, not at least in the context of a continuous-time implementation, since a bond index with a given constant maturity is a perfect substitute for a bond index with any other given constant maturity.

11 - The continuous-time parameters involved in our model could also be calibrated using a Kalman filter (Harvey 1991). This technique has been used, e.g., in Brennan and Xia (2002), Munk, Sørensen, and Vinther (2004), and Sangvinatsos and Wachter (2005).

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strategy will also involve a discrete partition of the set of investors. Indeed, in retail money management, the strategy will not be fully tailored to each retail investor's profile, and investors will instead have to be sorted in categories by age (or time horizon), in such a way that, in general, an investor with a given time horizon will be assigned to a strategy that will only be roughly optimal. For example, an investor with a time horizon equal to say nineteen or twenty-one years will be placed at the initial date in a category defined by a twenty-year time horizon, and the strategy that will be proposed to all investors in that particular category will be optimal only for investors with a time horizon of *exactly* twenty years.¹² Third, the strategy will involve a discrete partition of the set of market conditions: as opposed to using the exact risk premium estimate at each given point in time and for each state of the world, risk premia will usually be sorted into low, average, and high categories.

Our main focus is not to measure the welfare costs of strategies based on deterministic glide paths by taking the ideal continuous-time and continuous-state fully customised strategy as a benchmark. This would be an unfair competition because such an ideal strategy will never be implemented in practice. Instead, our contribution is to show that the welfare costs of strategies involving a discrete partition of the sets of trading dates, investor characteristics, and market conditions, are modest. After all, if it turns out that deterministic life-cycle strategies are substantially less inefficient than realistically implemented stochastic life-cycle strategies, then any hope for generating an improved version of target date funds will have to be re-assessed.

3.1 Introducing a Discrete Partition of the Set of Trading Dates

To provide a fairer benchmark to heuristic strategies, we first recognise that optimal stochastic life-cycle strategies would not be implemented in continuous-time. We therefore consider a discrete version of the optimal strategy, which we call "discretised optimal strategy", and in which the vector of weights at time t_i is taken to be $\omega_{t_i}^*$. In terms of the heuristic deterministic life-cycle strategy that gradually switches from stocks and real estate ("risky assets") to the bond index (the "safe" asset), we make the following assumption. Formally, the "risky" part of the allocation is taken to be a portfolio of 80% stocks and 20% real estate, and the "safe" part is fully invested in the bond index:

$$\omega^{\text{risky}} = (0, 0.8, 0.2)', \quad \omega^{\text{safe}} = (1, 0, 0)'$$

The weight vector of the deterministic life-cycle strategy is given by:

$$\omega_{t_i}^{\text{dlf}} = f(t_i)\omega^{\text{risky}} + [1 - f(t_i)]\omega^{\text{safe}}$$

where $f(t)$ is a deterministic function of time, decreasing from 80% at the initial date to 20% just before the horizon, by annual steps. Mathematically, if time is expressed in years, we have:

$$f(t_i) = 0.8 - \frac{0.6}{T-1} [t_i]$$

For example, in the case of a twenty-year horizon, the allocation to the risky part decreases by $0.6/19 \approx 316$ basis points per year. In unreported results, we have also tested other variants of deterministic schemes similar to those implemented in practice and have obtained very similar results in terms of the induced welfare loss.

12 - This imperfect partitioning of the set of investors should, in principle, also be done by risk aversion in addition to by time horizon. On the other hand, because risk aversion is not an observable parameter, measuring a welfare loss implied by a discrete partition is arguably not as straightforward as it is for an observable parameter such as time horizon. In fact, investors will usually have to position themselves in various categories (high, moderate, and low risk-aversion, say), which then will be assumed to perfectly represent the investors' unobservable risk aversion, without necessarily incurring an explicit welfare cost related to imperfect tailoring of the strategy to this dimension. In the numerical analysis, we focus on the values $\gamma = 10, 15, \text{ and } 20$. In practice, the choice of the risk aversion parameter levels is typically calibrated to avoid undesirable extreme or short positions in the asset classes.

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Figure 1 provides a measure of welfare loss for this deterministic life-cycle strategy with respect to the optimal idealised continuous-time strategy for three values of the risk aversion parameter and an initial time horizon of twenty years (throughout the paper, we maintain the assumption of an initial time horizon of twenty years; in unreported results, we have tested time horizons of ten and thirty years, and have obtained qualitatively similar results). The welfare loss appears great, some 80% for all tested risk aversions. By contrast, quarterly rebalancing constraints imply modest losses of 6.0% for $\gamma = 10$, 4.2% for $\gamma = 15$, and 2.9% for $\gamma = 20$ on the utility that would result from a continuous-time implementation of the optimal strategy. On the whole, these results suggest that there is ample room to improve current heuristic target date funds. Before turning to an assessment of the impact of the introduction of a realistic discrete partition of the set of investors and market conditions, we also consider a variant of the optimal strategy implemented in discrete time, in which the real estate index is excluded from the menu of traded assets. Obviously, ignoring the real estate index implies a substantial welfare loss, reaching 43.3% for $\gamma = 10$, 31.9% for $\gamma = 15$, and 25.5% for $\gamma = 20$. These results confirm that an investable real estate index can be a useful addition to the menu of asset classes in the context of long-term investment strategies. In fact, real estate can, in general, be a useful addition to the performance-seeking portfolio because it provides diversification benefits, as well as to the inflation-hedging portfolio.¹³

13 - Anari and Kolari (2002) provide empirical evidence that house prices are a stable inflation hedge in the long run.

3.2 Introducing a Discrete Partition of the Set of Investors and Market Conditions

We now explain how the real-world implementation constraints can be integrated in the simulation of the various life-cycle investing strategies. In particular, we describe how the partition is done in terms of objective investment attributes, *i.e.*, market conditions as summarised in terms of values for the Sharpe ratio, and subjective investment attributes, in this case time horizon.

We first describe the partition of the set of market conditions as a function of the stock index Sharpe ratio, which is assumed to follow the mean-reverting process (2.3). As is well known from the properties of Ornstein-Uhlenbeck processes, the long-term mean and variance are $\bar{\lambda}$ and $\sigma_{\lambda}^2/(2\kappa)$ respectively. The crudest possible partition of market conditions would involve replacing all realisations of the process λ^S by the constant long-term mean $\bar{\lambda}$. A somewhat finer partition of the set of market conditions would involve distinguishing between high, moderate, and low risk premia. In what follows, we use the following threshold values:

$$\begin{aligned} \lambda_{\text{inf}} &= \bar{\lambda} - 2 \times 1.96 \sqrt{\frac{\sigma_{\lambda}^2}{2\kappa}}, \quad \bar{\lambda}, \\ \lambda_{\text{sup}} &= \bar{\lambda} + 2 \times 1.96 \sqrt{\frac{\sigma_{\lambda}^2}{2\kappa}} \end{aligned} \quad (3.1)$$

The classification rule is that the realisation of λ^S at a given date and for a given state of the world will be replaced by the closest of these threshold three values (if λ^S is equally distant from two of these values, it is replaced with the lower one). Therefore, the second level of partition

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contains three classes, which are:

$$\left] -\infty, \frac{\lambda_{\text{inf}} + \bar{\lambda}}{2} \right],$$

$$\left] \frac{\lambda_{\text{inf}} + \bar{\lambda}}{2}, \frac{\bar{\lambda} + \lambda_{\text{sup}}}{2} \right],$$

$$\left] \frac{\bar{\lambda} + \lambda_{\text{sup}}}{2}, \infty \right[$$

and three standard values: λ_{inf} , $\bar{\lambda}$, and λ_{sup} . The coefficient 2×1.96 in (3.1) has been chosen to ensure that the central class will contain 95% of the long-term observations of the Sharpe ratio. Values outside this interval are considered outliers, so refining the partition outside the central range would be irrelevant. Therefore, one subsequently refines the partition by dividing the central interval into two equal sub-intervals, a process that will eventually converge to a continuous set of partitions. For example, the third level of partition involves five threshold values for the Sharpe ratio:

$$\lambda_{\text{inf}}, \quad \frac{\lambda_{\text{inf}} + \bar{\lambda}}{2}, \quad \bar{\lambda},$$

$$\frac{\bar{\lambda} + \lambda_{\text{sup}}}{2}, \quad \lambda_{\text{sup}}$$

and five related classes. More generally, it can be verified that for $k \geq 2$ the partition of index k has $2^{k-1} + 1$ standard values. Let $\lambda_j^{(k)}$, for $1 \leq j \leq 2^{k-1} + 1$ denote these values. In particular, the values $\lambda_j^{(2)}$ for $1 \leq j \leq 3$ are given by (3.1). Mathematically, the values for this partition can be computed as:

$$\lambda_j^{(k)} = \frac{1}{2^{k-2}} \left[\left(2^{k-2} - r \right) \lambda_{i+1}^{(2)} + r \lambda_{i+2}^{(2)} \right],$$

$$1 \leq j \leq 2^{k-1} \tag{3.2}$$

with the unique decomposition $j - 1 = 2^{k-2}i + r$ and $0 \leq r \leq 2^{k-2} - 1$. The last value is:

$$\lambda_{2^{k-1}+1}^{(k)} = \lambda_3^{(2)}$$

In terms of partitioning the set of investors by time horizon $T - t$, the imperfect customisation inherent to retail money management implies replacing the actual time-to-horizon with some approximate value. In what follows, we will consider investors with time horizons ranging from zero to thirty years. As a result, the crudest possible partition would involve using the same average value $\bar{T} = 15$ years for all investors, regardless of their actual time horizon. Here, all investors, including the one who happens to have a time horizon originally equal to fifteen years, will incur a welfare loss as a result of this imperfect partition. After all, for a given investor, say the one with an initial time horizon of fifteen years, residual time-to-horizon will decrease as time goes by, while the strategy will continue to have a constant fifteen-year horizon. A slightly finer partition would involve defining three possible time horizons:

$$T_{\text{inf}} = 0, \quad \bar{T}, \quad T_{\text{sup}} = 30$$

More generally, the partition of index k would involve $2^{k-1} + 1$ threshold values, very similarly to what has been explained above for the partition of Sharpe ratio levels. In the limit of k going to infinity, a set of different portfolio strategies corresponding to a continuum of horizons will be offered to investors so that at all points in time the investment strategy proposed to them will be perfectly consistent with their respective residual time horizons. For finite k values, on the other hand, some utility loss will be incurred, and our motivation is to analyse the speed of

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convergence to zero of the welfare loss as a function of the fineness/coarseness of the partition measured by k .

Because risk aversion is also an important parameter that should affect the optimal allocation strategy, we present three different life-cycle strategy benchmarks for three different degrees of risk aversion. As explained above, we do not analyse the impact of a discrete partition of the set of investors by risk aversion; because this parameter is not observable, investors are assumed to be perfectly represented by a self-selected risk aversion profile.

3.2.1 Impact of a Discrete Partition of the Set of Investors

Figure 2 analyses the impact of partitioning the set of times-to-horizon. It displays the MUL for the strategy based on the partition as a function of the index k . As could be expected, this MUL converges to the MUL of the discretised optimal strategy as k grows to infinity. Surprisingly, perhaps, we find that the speed of convergence is very high. In fact, for $k = 4$, which corresponds to nine classes, the MUL for the partitioned strategy already almost coincides with the MUL for the fully customised strategy implemented in discrete time. Even assuming a constant horizon of fifteen years for all investors at all dates leads to a welfare loss that is still lower than the welfare loss incurred with heuristic deterministic life-cycle investing strategies. In figure 3, we confirm that return-to-risk ratios are approximately twice as good for the optimal discretised strategy as for the heuristic deterministic strategies, and the speed of convergence is again very high. For $k = 3$, which corresponds to five classes, the return-to-risk ratio obtained with the optimal strategy

implemented with a discrete partition of the set of time horizon is already very close to what would be achieved by the fully customised version. On the whole, these results seem to confirm that current forms of TDFs are highly inefficient investment strategies, which could be improved in realistic environments integrating retail money management constraints.

3.2.2 Impact of a Discrete Partition of the Set of Market Conditions

We now turn to an analysis of the impact of partitioning the set of Sharpe ratios. This is the purpose of figure 4, which shows the MUL for strategies using an approximate value for the Sharpe ratio. Again, the utility loss decreases with the index of the partition, and it rapidly converges to the MUL for the optimal discretised strategy. The main difference from figure 2 is that the MUL for the initial partition, which corresponds to a deterministic strategy based on a constant $\lambda^S = \bar{\lambda}$ value, is already low, since it is 7.6% for $\gamma = 20$ and 13.8% for $\gamma = 10$. This result has important practical implications, since it suggests that, if for some reason the mutual fund industry were to restrict itself to purely deterministic strategies, the kind of deterministic strategies currently used could be substantially improved. On the other hand, restricting the set of strategies to purely deterministic strategies is not desirable, as the marginal gain from introducing state dependencies in the allocation strategy can become substantial for some reasonable parameter values. To illustrate this effect, figure 6 shows the MUL for strategies based on partitions when the parameter λ^S exhibits greater variability. Such greater variability can be obtained by multiplying the standard

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deviation σ_λ by two, which results in multiplying the long-term mean volatility, $\sigma_\lambda/\sqrt{2\kappa}$, also by two. In fact, the base-case value of the long-term mean volatility $\sigma_\lambda/\sqrt{2\kappa} = 13.5\%$ was relatively low, and increasing it could very well provide a more accurate description of the actual underlying process. In this case, we then obtain that the MUL for the initial partition (corresponding to $k = 1$, which implies a deterministic strategy) ranges from 24.4% (for $\gamma = 20$) to 36.3% (for $\gamma = 10$), which indicates a substantial welfare loss from ignoring time-variation in the equity premium. The corresponding values in terms of return-to-risk ratio (see figure 7) also show substantial value to be added by switching from a deterministic strategy to a stochastic one that fully incorporates the impact of a time-varying opportunity set. When $k = 4$ (which corresponds to nine classes), on the other hand, the MUL obtained becomes less than 10%. From figure 6, it can also be noted that the convergence of the MUL towards the discretised optimal strategy is slightly slower than in figure 4, since the quasi-equality of the two utility losses holds only as of $k = 5$. The same can be said of the return-to-risk ratio, as appears from figures 5 and 7. If one introduces even more variability in the Sharpe ratio process, the results become accordingly more spectacular. In figures 8 and 9, we have set the volatility to twice its base-case value and we have divided the base-case value of the speed of mean reversion by two.¹⁴ In this case, the MULs for the deterministic strategies (*i.e.*, the strategies obtained for $k = 1$) exceed 40% of the initial capital for all tested degrees of risk aversion. In line with these results, the improvement in the Sharpe ratio (see figure 9) is even greater than in figure 7. These findings confirm that the assumption

of a constant Sharpe ratio involves a strong opportunity cost when the actual process exhibits significant variability.

3.2.3 Combined Impact of a Discrete Partition of the Set of Investors and Market Conditions

Having considered the separate effects of partitioning the set of investors and market conditions, we turn in figure 10 to the analysis of the joint impact of discretising both dimensions. Roughly speaking, the final effect is the result of the individual effects of the two partitions. In the base case, the utility loss is a decreasing function of the index k and is indistinguishable from the MUL of the discretised optimal strategy for $k \geq 4$. On the whole, our results suggest that, even with a reasonably low total number of partitions, perfectly consistent with implementation constraints in retail money management, real-world implementations of truly optimal strategies would represent a very significant improvement over the heuristic strategies currently used by the TDF industry.

3.3 Impact of an Imperfect Observation of the Equity Risk Premium

The previous strategies, including those based on partitions of the set of market conditions, rely on perfect knowledge of the equity Sharpe ratio λ^S , which is a key input in the expression of the optimal strategy. In practice, however, the instantaneous expected return on equity indices is not directly observable. The literature on stock return predictability has proposed several proxies for the expected return, including the dividend yield (Campbell and Viceira 1999, Barberis 2000, Xia 2001; Menzly, Santos, and Veronesi 2004).

14 - As a result, the long-term standard deviation of the process λ^S has been multiplied by 2.83.

3. Numerical Analysis of the Model

However, the predictive power of the dividend-price ratio is rather weak, which leaves substantial uncertainty over the actual value of λ^S . In this context, one naturally wonders whether or not observing λ^S imperfectly has a great impact on the superiority of optimal life-cycle investing strategies to heuristic allocation schemes. As an initial step towards addressing this problem, we run the following experiment: we assume that the agent observes a noisy version $\hat{\lambda}_t^S$, given $\hat{\lambda}_t^S = \lambda_t^S + \varepsilon_t$, of the equity index Sharpe ratio. The error term ε_t is taken to be white noise with mean zero (which means that the observation is unbiased) and variance denoted by σ_ε^2 . Greater variance means a noisier observation of λ^S . Then, life-cycle strategies are implemented with $\hat{\lambda}_t^S$ used as a (noisy) proxy for the true value of the Sharpe ratio.¹⁵ Figures 12 and 13 show the MULs induced by strategies based on partitions of both the set of times-to-horizon and the set of Sharpe ratios, assuming that a noisy version of the Sharpe ratio is observed. Two values of the standard deviation of the white noise ε are considered: a value equal to a respective two and five times the long-term volatility of the Sharpe ratio, which corresponds to $\sigma_\varepsilon = 30\%$ and $\sigma_\varepsilon = 67.4\%$ respectively. As expected, we obtain that for any number of classes in the partition but $k = 1$ the MULs are higher than in the corresponding figure 10, in which the expected excess return was perfectly observed.¹⁶ We also find that even after accounting for the presence of relatively high measurement error in the equity risk premium ($\sigma_\varepsilon = 67.4\%$) the MULs are still substantially lower than those obtained for heuristic glide paths used in TDFs.

A comparison of figures 12 and 13 shows, as expected, that the MUL is an increasing

function of the variance of the noise. More interestingly, the increase in the MUL that follows from introducing measurement error is found to be larger for high values of the index k . This finding can be explained by the fact that, for low k values, the relative coarseness of the partition implies greater robustness since misclassification risk in this case is relatively limited. For $k = 2$, for example, there are only three admissible values, corresponding to low, medium, and high expected returns, and in most cases the imperfect observation of the true value for the equity risk premium will not have any impact on which expected return (low, medium, or high) should be used. For higher values of the index k , misclassification risk is a greater concern. In other words, although obtaining an accurate estimate of the equity risk premium is difficult, assessing whether the current value corresponds to a high, intermediate, or low risk premium is an easier task, and achieving such a more modest objective is sufficient to generate reasonably good proxies for optimal strategies. On the whole, the superiority of optimal life-cycle investing strategies implemented with reasonably fine partitions to the heuristic glide paths currently used in the industry is found to be robust with respect to the introduction of reasonable measurement errors in equity risk premium estimates.

15 - Another problem is the presence of estimation risk on the parameters of the equity risk-premium process, as opposed to the realised value for the process.

16 - For $k = 1$, the actual value of the Sharpe ratio is irrelevant since this stochastic quantity is treated as a constant equal to $\bar{\lambda}$.

4. Conclusion



4. Conclusion

In this paper, we characterise in closed-form the optimal time- and state-dependent allocation strategy for a long-term investor preparing for retirement in the presence of stochastic interest and inflation rates and a mean-reverting equity risk premium. We find that the opportunity cost involved in purely deterministic life-cycle strategies such as those implemented by available target date funds is, for reasonable parameter values, substantial. Our results also suggest that a parsimonious partition of the set of investors and market conditions can be done, a partition that allows a relatively accurate approximation of the true optimal strategy and is compatible with real-world implementation constraints in retail money management. Our results have important potential implications for the design of improved forms of *target date funds*. More financial innovation is critically needed at this stage to design better target date funds that could help investors cope with their long-term financial planning problems (see Bodie, Detemple, and Rindisbacher [2009] for a related discussion). On the whole, our results suggest that there is ample room for added value between one-(allocation)-size-fits-all (investors of the same age) solutions and do-it-yourself approaches to life-cycle investment decisions.

Our work can be extended in a number of directions. On the one hand, we have made the simplifying assumption of constant equity volatility. Incorporating time variation in equity volatility, in addition to time variation in equity returns, can be done at the cost of a relatively modest increase in the complexity of the model

(see appendix C.2). Another important ingredient missing from our model, as well as from most of the literature on life-cycle investing, is how the presence of short-term performance constraints faced by investors would affect optimal long-term allocation decisions. Incorporating explicit mechanisms for ensuring some protection against short-term downside equity risk would be a desirable ingredient for most long-term investors. Recent research provides useful insight in that direction, since it suggests that a general analytical representation of the relationship between optimal long-term strategies in the presence and in the absence of short-term constraints can be identified, which allows one to disentangle the impact of short-term constraints from the impact of return predictability on the optimal allocation decision (Martellini and Milhau 2010).

Appendices



Appendices

A. Dynamics of Constant-Maturity Bonds

We first need the expressions for the prices of a nominal zero-coupon bond and an inflation-indexed zero-coupon bond with finite maturity T_0 . The price for the nominal zero-coupon bond is standard given that the nominal term structure is driven by a single factor, the nominal short-term rate R , which follows an Ornstein-Uhlenbeck process (Vasicek 1977):

$$B(t, T_0) = \exp[b_a(T_0 - t)R_t + c(T_0 - t)] \quad (\text{A.1})$$

where:

$$b_a(s) = -\frac{1 - e^{-as}}{a}, \quad c(s) = \left(b - \frac{\sigma_R \lambda_R}{a}\right) \left[\frac{1 - e^{-as}}{a} - s\right] + \frac{\sigma_R^2}{2a^2} \left[s - 2\frac{1 - e^{-as}}{a} + \frac{1 - e^{-2as}}{2a}\right]$$

The price of the inflation-indexed zero-coupon bond follows from similar computations:

$$I(t, T_0) = \Phi_t \exp \left[b_a(T_0 - t)R_t - \left(b - \frac{\sigma_R \lambda_R}{a} + \frac{\sigma_R \sigma_\Phi \rho_{R\Phi}}{a} \right) [T_0 - t + b_a(T_0 - t)] + (\pi - \sigma_\Phi \lambda_\Phi)(T_0 - t) + \frac{\sigma_R^2}{2a^2} [T_0 - t + 2b_a(T_0 - t) - b_{2a}(T_0 - t)] \right]$$

The value of the nominal bond with constant maturity τ is the value of the self-financing strategy which, at each date t , is fully invested in zero-coupon bonds maturing at date $t + \tau$. The log-return on this strategy is given by:

$$d \log B_t = \log \frac{B(t + dt, t + \tau)}{B(t, \tau)}$$

Using (A.1) and ignoring terms of order > 1 , we obtain:

$$d \log B_t = -b'_a(\tau)R_t dt + b_a(\tau) dR_t - c'(\tau) dt$$

By absence of arbitrage opportunities, the drift term must be equal to $R_t + b_a(\tau)\sigma_R\lambda_R$, whence (2.6). The dynamics (2.7) is obtained in a similar way.

B Proofs of the Main Propositions

B.1 Proof of Propositions 1 and 2

For notational brevity, we define the state variable vector \mathbf{X}_t as $\mathbf{X}_t = (R_t, \lambda_t^S)'$ and the parameters μ_X and σ_X as:

$$\mu_X = \begin{pmatrix} ab \\ \kappa \bar{\lambda} \end{pmatrix} + \begin{pmatrix} -a & 0 \\ 0 & -\kappa \end{pmatrix} \mathbf{X}_t, \quad \sigma_X = \begin{pmatrix} \sigma_R & \sigma_\lambda \end{pmatrix}$$

so that \mathbf{X} evolves as

$$d\mathbf{X}_t = \mu_X dt + \sigma_X' dz_t.$$

We now turn to the solution of the static program. The optimal terminal payoff is $A_T^* = (\eta M_T^{\lambda+\nu^*})^{-\frac{1}{\nu}} \Phi_T^{1-\frac{1}{\nu}}$ where η is a Lagrange multiplier. The budget constraint yields

$$\eta^{-\frac{1}{\nu}} = \frac{A_0 + H_0}{\mathbb{E} \left[(M_T^{\lambda+\nu^*} \Phi_T)^{1-\frac{1}{\nu}} \right]}$$

Following He and Pearson (1991), we assume that conditions for $(\Phi, M^{\lambda+\nu^*}, \mathbf{X})$ to be a Markov process are satisfied. Hence, the optimal wealth process can be written as:

$$A_t^* = \eta^{-\frac{1}{\nu}} \mathbb{E}_t \left[\frac{(M_T^{\lambda+\nu^*} \Phi_T)^{1-\frac{1}{\nu}}}{M_t^{\lambda+\nu^*}} \right] - H_t = F(t, M_t^{\lambda+\nu^*}, \Phi_t, \mathbf{X}_t) - H_t$$

We assume that F is separable in the following sense:

$$F(t, M_t^{\lambda+\nu^*}, \Phi_t, \mathbf{X}_t) = \eta^{-\frac{1}{\nu}} \Phi_t^{1-\frac{1}{\nu}} (M_t^{\lambda+\nu^*})^{-\frac{1}{\nu}} g(t, \mathbf{X}_t) \quad (\text{B.1})$$

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where $g(t, \mathbf{X}_t)$ is thus equal to

$$\mathbb{E}_t \left[\left(\frac{M_T^{\lambda+\nu^*} \Phi_T}{M_t^{\lambda+\nu^*} \Phi_t} \right)^{1-\frac{1}{\gamma}} \right]$$

Matching the diffusion term of A^* and the diffusion term of $F(t, M_t^{\lambda+\nu^*}, \Phi_t, \mathbf{X}_t) - H_t$ yields the volatility vector of A^* :

$$\begin{aligned} \sigma_{A^*} = & \left(1 + \frac{H_t}{A_t^*} \right) \left[\frac{1}{\gamma} (\lambda_t + \nu_t^*) + \frac{\sigma_X g_x}{g} + \left(1 - \frac{1}{\gamma} \right) \sigma_\Phi \right] \\ & - \frac{H_t}{A_t^*} \sigma_H \end{aligned} \quad (\text{B.2})$$

Since A^* is the value process of some trading strategy, its volatility vector must be spanned by the volatility matrix of traded securities, σ . This can be written as $[\mathbf{I}_5 - \sigma(\sigma'\sigma)^{-1}\sigma']\sigma_{A^*} = \mathbf{0}$. From this constraint, and given that σ_H is spanned by σ , we get the vector ν_t^* given in the proposition:

$$\nu_t^* = \mathbf{N} \left[(1 - \gamma)\sigma_\Phi - \gamma \frac{\sigma_X g_x}{g} \right] \quad (\text{B.3})$$

where $\mathbf{N} = \mathbf{I}_5 - \sigma(\sigma'\sigma)^{-1}\sigma'$.

Then we match the drift terms of $A^* + H$ and $F(t, M_t^{\lambda+\nu^*}, \Phi_t, \mathbf{X}_t)$. The drift term of F follows from application of Ito's lemma to the r.h.s. of (B.1). In order to preclude arbitrage opportunities, it must be the case that:

$$\mu_{A^*} = R_t + \frac{e_t}{A_t^*} + \sigma_{A^*}'(\lambda_t + \nu_t^*)$$

This leads to:

$$\begin{aligned} R_t + & \left[\frac{1}{\gamma} (\lambda_t + \nu_t^*) + \frac{\sigma_X g_x}{g} + \left(1 - \frac{1}{\gamma} \right) \sigma_\Phi \right]' (\lambda_t + \nu_t^*) \\ = & \frac{g_t}{g} + \left(1 - \frac{1}{\gamma} \right) \pi + \frac{1}{\gamma} \left[R_t + \frac{\|\lambda_t + \nu_t^*\|^2}{2} \right] \\ + & \frac{g'_x}{g} \mu_X + \frac{1}{2g} \text{tr}[g_{xx} \sigma_X' \sigma_X] + \frac{1}{\gamma} \left(1 - \frac{1}{\gamma} \right) (\lambda_t + \nu_t^*)' \sigma_\Phi \\ + & \left(1 - \frac{1}{\gamma} \right) \sigma_\Phi' \frac{\sigma_X g_x}{g} + \frac{1}{\gamma} \frac{g'_x \sigma_X' (\lambda_t + \nu_t^*)}{g} \end{aligned}$$

Rearranging terms and substituting (B.3) into this equation, we arrive at:

$$\begin{aligned} 0 = & \frac{g_t}{g} + \frac{1-\gamma}{\gamma} (R_t - \pi + \sigma_\Phi' \lambda_t) + \frac{1-\gamma}{2\gamma^2} \|\lambda_t - \sigma_\Phi\|^2 \\ & + \frac{g'_x}{g} \left[\mu_X + \frac{1-\gamma}{\gamma} \sigma_X' (\lambda_t - \sigma_\Phi) \right] \\ & + \frac{1}{2g} \text{tr}[g_{xx} \sigma_X' \sigma_X] - \frac{(1-\gamma)^3}{2\gamma^2} \sigma_\Phi' \mathbf{N} \sigma_\Phi \\ & - \frac{1-\gamma}{2} \frac{g'_x \sigma_X' \mathbf{N} \sigma_X g_x}{g^2} \\ & + \frac{(1-\gamma)^2}{\gamma} \frac{\sigma_\Phi' \mathbf{N} \sigma_X g_x}{g} \end{aligned} \quad (\text{B.4})$$

This PDE and the associated terminal condition $g(T, x) = 1$ involve only t and x , which, *ex post*, justifies our assumption (B.1) on the separability of F . Equation (B.4) involves the real short-term interest rate, $r_t = R_t - \pi + \sigma_\Phi' \lambda_t$ and the real market price of risk vector, $\lambda_t - \sigma_\Phi$. The optimal portfolio strategy is then obtained as:

$$\theta_t^* = A_t^* (\sigma'\sigma)^{-1} \sigma' \sigma_{A^*}$$

where the volatility vector of optimal wealth, σ_{A^*} , is given in (B.2).

The indirect utility function is defined by:

$$J(t, A_t^*, \mathbf{X}_t, \Phi_t) = \frac{1}{1-\gamma} \mathbb{E}_t \left[\left(\frac{A_T^*}{\Phi_T} \right)^{1-\gamma} \right]$$

Given the optimal payoff, we get that:

$$\begin{aligned} J(t, A_t^*, \mathbf{X}_t, \Phi_t) &= \frac{\eta^{1-\frac{1}{\gamma}}}{1-\gamma} \mathbb{E}_t \left[\left(M_T^{\lambda+\nu^*} \Phi_T \right)^{1-\frac{1}{\gamma}} \right] \\ &= \frac{1}{1-\gamma} \left(\frac{A_t^* + H_t}{\Phi_t} \right)^{1-\gamma} g(t, \mathbf{X}_t)^\gamma \end{aligned}$$

The form for the function g must still be found. The model written in equations (2.1) through (2.5) is affine in the sense that the drifts of the state variables R and λ^S are affine functions of these variables, and their volatilities are constant. Following the bond pricing literature in the context

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of affine term-structure models (Dai and Singleton 2000), it is reasonable to assume that g is an exponential affine function of state variables. This solution technique is also used in Liu (2007). We thus consider a candidate g of the form (2.13) and we impose the initial constraints $A_i(0) = 0$ for $i = 1, \dots, 4$. We then take the relevant partial derivatives of g and substitute them into (B.4). The r.h.s. of (B.4) can then be seen as an affine function of $(R_t, \lambda_t^S, (\lambda_t^S)^2)$ with time-dependent coefficients. Writing that each coefficient must be zero, we obtain a series of ODEs involving the A_i functions. Cancelling the R_t term yields:

$$A_2'(T - t) = 1 - \alpha A_2(T - t)$$

This ODE can be solved in closed form with the initial condition $A_2(0) = 0$ which leads to (2.14).

The $(\lambda_t^S)^2$ term must also be zero, whence:

$$A_4'(T - t) = \frac{\|\Lambda_2\|^2}{\gamma} + 2 \left[\frac{1 - \gamma}{\gamma} (\sigma'_X \Lambda_2)_2 - \kappa \right] A_4(T - t) + \frac{1 - \gamma}{\gamma} \left[\sigma_\lambda^2 - (1 - \gamma) \sigma'_\lambda N \sigma_\lambda \right] A_4(T - t)^2$$

where $(\sigma'_X \Lambda_2)_2 = \sigma'_\lambda \Lambda_2$.

Similarly, cancelling the λ_t^S term yields the ODE satisfied by A_3 , and cancelling the constant term leads to the ODE (2.17).

B.2 A_4 Is Positive and Increasing

We rewrite the ODE (2.15) as:

$$A_4'(T - t) = \phi_1 + \phi_2 A_4(T - t) + \phi_3 A_4(T - t)^2$$

where:

$$\phi_1 = \frac{\|\Lambda_2\|^2}{\gamma},$$

$$\phi_3 = \frac{1 - \gamma}{\gamma} \left[\sigma_\lambda^2 - (1 - \gamma) \sigma'_\lambda N \sigma_\lambda \right]$$

For $\gamma > 1$, ϕ_3 is negative. ϕ_1 is non-negative, but is non-zero, because Λ_2 itself is non-zero by definition (indeed, we have $\Lambda_2 = \Sigma(\Sigma'\Sigma)^{-1} \begin{pmatrix} 0 & \sigma_S & 0 \end{pmatrix}'$ and $\sigma_S \neq 0$). Hence, the equation $\phi_1 + \phi_2 x + \phi_3 x^2 = 0$ has two roots x_1 and x_2 , which are negative and positive respectively. We have $A_4(0) = 0$, hence $A_4(0)$ lies between x_1 and x_2 , so $A_4'(0)$ is positive. Assume that A_4 takes negative values after time 0, and introduce the first time when A_4 becomes negative, namely:

$$u = \inf \{ t \in [0, T] ; A_4(t) < 0 \}$$

By assumption, u is finite. Moreover, because $A_4'(0)$ is positive and $A_4(0) = 0$, there exists a small interval $]0, \varepsilon_1]$ over which A_4 is positive. Hence u is positive. As a consequence, we have $A_4(u) = 0$, so that $A_4'(u)$ is positive, and there exists a small ε_2 such that A_4 is negative on the interval $[u - \varepsilon_2, u[$. This is inconsistent with the definition of u . Hence we have $A_4(t) \geq 0$ over $]0, T]$. In particular, $A_4(t)$ lies strictly between x_1 and x_2 ; hence $A_4'(t)$ is positive. Given that $A_4(0) = 0$, this implies that $A_4(t)$ is in fact positive over $]0, T]$.

C. Extensions

C.1 Introducing Inflation-Indexed Bonds

This subsection briefly describes the results when an inflation-indexed bond of constant maturity τ is used for the purpose of hedging against inflation risk. The volatility matrix of traded assets becomes:

$$\sigma = \begin{pmatrix} \sigma_B(\tau) & \sigma_S & \sigma_Y & \sigma_I(\tau) \end{pmatrix}$$

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and the spanned market price of risk vector is:

$$\lambda_t = \sigma(\sigma'\sigma)^{-1} \begin{pmatrix} b_a(\tau)\sigma_R\lambda_R \\ \sigma_S\lambda_t^S \\ \sigma_Y\lambda_Y \\ b_a(\tau)\sigma_R\lambda_R + \sigma_\Phi\lambda_\Phi \end{pmatrix}$$

Then the inflation-hedging portfolio becomes $\omega^\Phi = (0, 0, 0, 1)' - (1, 0, 0, 0)'$, and the optimal strategy in the absence of labour income is now:

$$\omega_t^{*,0} = \frac{1}{\gamma}\omega_t^{PSP} - \left(1 - \frac{1}{\gamma}\right) [A_3(T-t) + A_4(T-t)\lambda_t^S] \omega^\lambda + \left(1 - \frac{1}{\gamma}\right) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{C.1}$$

In the presence of labour income, it is still given by (2.19). If the stock index is perfectly anti-correlated with its Sharpe ratio, the market is complete, and portfolio (C.1) simplifies again since $\omega^\lambda = (0, 1, 0, 0)'$.

C.2 Introducing Volatility Risk

The implications of stochastic volatility in option pricing have been studied in many papers. Heston (1993) develops a method based on the inversion of Fourier transform to price European calls or puts written on a stock whose instantaneous variance follows a mean-reverting square-root process. However, fewer papers have documented the impact of stochastic volatility in asset allocation. The main reason is in fact that agents who maximise utility from nominal wealth do not hedge

volatility risk for itself. They are concerned only with those state variables that drive the nominal short-term rate and the market price of risk vector (see, e.g., Detemple and Rindisbacher [2010] for a recent reference). Hence, they hedge against volatility risk only if one of these two variables is explicitly linked to the volatility. Liu (2007) computes the optimal allocation to the stock in a special case of the Heston model where the Sharpe ratio is proportional to the volatility. Kraft (2005) considers a more general version of this model, where the Sharpe ratio is proportional to the volatility raised to some positive power. Chacko and Viceira (2005) solve for the optimal portfolio when the Sharpe ratio is proportional to the precision (*i.e.*, the inverse of the volatility). Introducing stochastic volatility in our model is in fact straightforward as long as we assume autonomous dynamics for the Sharpe ratio of the stock. Let us consider a modified version of the economy described by equations (2.1) to (2.5), where the instantaneous volatility of the stock, which we now denote $\sqrt{V_t}$, is a stochastic process evolving as in the Heston model:

$$dV_t = \alpha(\beta - V_t) dt + \sigma_V\sqrt{V_t} dz_t^V$$

The new dynamics of the stock is thus:

$$dS_t = S_t \left[\left(R_t + \sqrt{V_t}\lambda_t^S \right) dt + \sqrt{V_t} dz_t^S \right]$$

and the other dynamics are left unchanged. In vector notations, one can write the diffusion term of S as $\sqrt{V_t}\sigma'_S dz_t$, where σ_S is the unit volatility vector of S and z is a six-dimensional Brownian motion.

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One can then show that the new optimal portfolio is given by:

$$\begin{aligned} \omega_t^* &= \frac{1}{\gamma} (\sigma_t' \sigma_t)^{-1} \sigma_t' \lambda_t + \left(1 - \frac{1}{\gamma}\right) \frac{A_2(T-t)}{A_2(\tau)} e_1 \\ &\quad - \left(1 - \frac{1}{\gamma}\right) \left[A_3(T-t) + A_4(T-t) \lambda_t^S \right] (\sigma_t' \sigma_t)^{-1} \sigma_t' \sigma_\lambda \\ &\quad + \left(1 - \frac{1}{\gamma}\right) (\sigma_t' \sigma_t)^{-1} \sigma_t' \sigma_\phi \end{aligned}$$

where σ_t is the stochastic volatility matrix of the traded assets, namely

$$\sigma_t = \begin{pmatrix} \sigma_B(\tau) & \sqrt{V_t} \sigma_S & \sigma_Y \end{pmatrix}$$

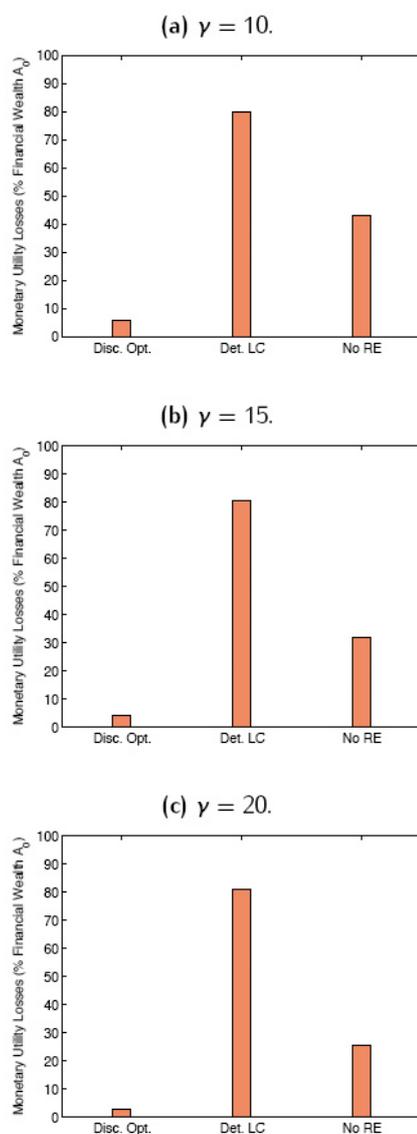
if the indexed bond is not traded, and

$$\sigma_t = \begin{pmatrix} \sigma_B(\tau) & \sqrt{V_t} \sigma_S & \sigma_Y & \sigma_I(\tau) \end{pmatrix}$$

otherwise.

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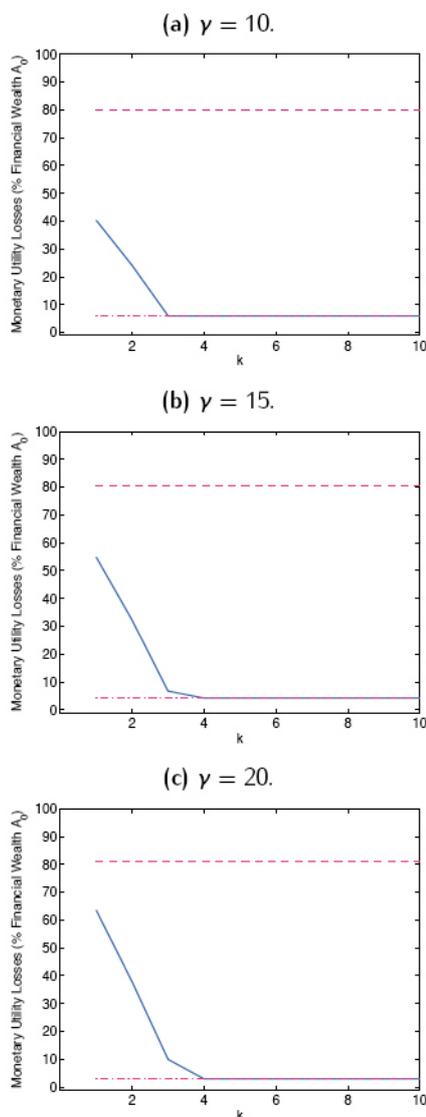
Figure 1: Monetary utility losses from following heuristic strategies



These figures represent monetary utility losses from following various heuristic strategies. All utility losses are computed with respect to the indirect utility from implementing the optimal strategy in continuous time. "Disc. Opt." refers to the optimal strategy implemented on a discretised basis; "Det. LC" corresponds to a deterministic life-cycle strategy based on a "rule of thumb" close to current practice; "No RE" refers to a variant of the optimal strategy implemented in discrete time, where the real estate index is excluded from the menu of traded assets. Unless otherwise indicated, parameters are fixed at their base-case values (see table 1).

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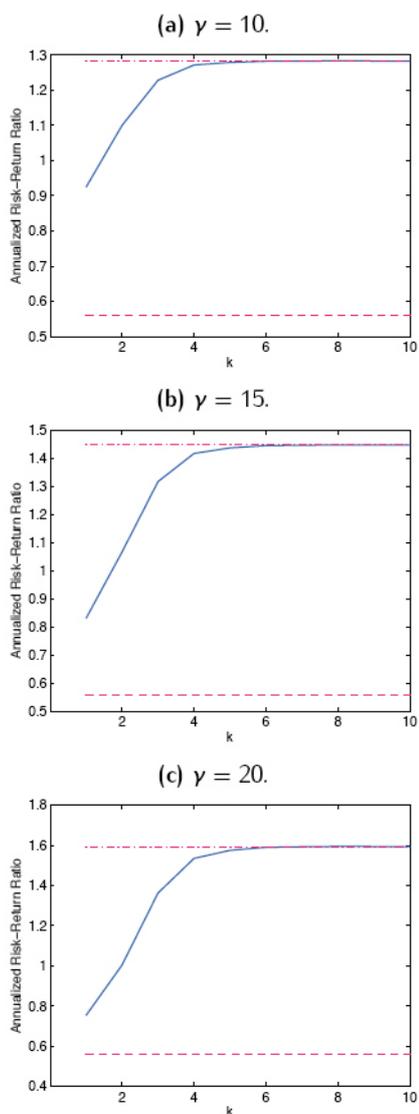
Figure 2: Monetary utility losses from partitioning the set of times-to-horizon



The solid line represents monetary utility losses (MUL) from following strategies based on partitions of the set of times-to-horizon. The dash-dot line represents the MUL from implementing the optimal strategy in discrete time, and the dashed line is the MUL from implementing a deterministic life-cycle strategy. Unless otherwise indicated, parameters are set at their base-case values (see table 1). k denotes the index of the partition, so that for $k \geq 2$, the number of classes in the partition is $2^{k-1} + 1$. γ is the relative risk aversion coefficient of the investor.

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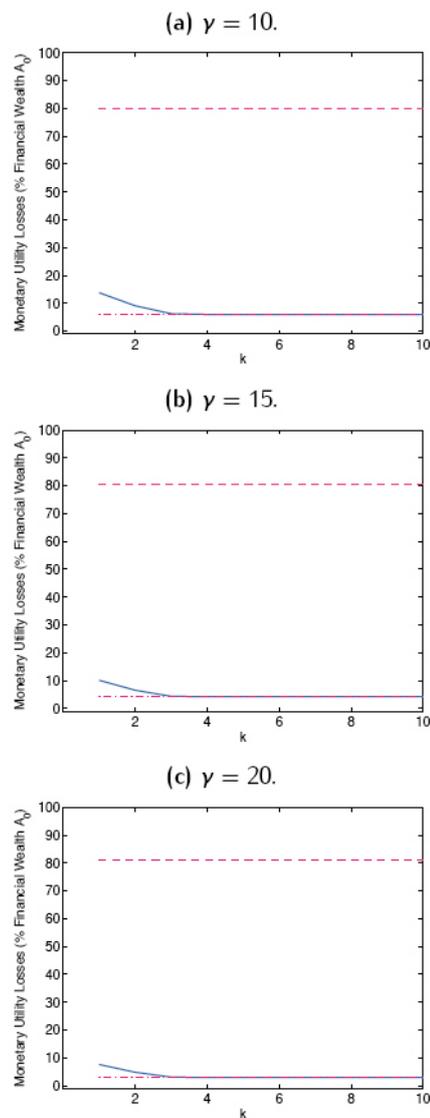
Figure 3: Risk-return ratios for strategies based on a partition of the set of times-to-horizon



The solid lines represent strategies based on partitions of the set of times-to-horizon. The dash-dot lines represent the optimal strategy implemented in discrete time, and the dashed lines represent a deterministic life-cycle strategy. The risk/return ratio is annualised. Unless otherwise indicated, parameters are set at their base-case values (see table 1). k denotes the index of the partition, so that for $k \geq 2$, the number of classes in the partition is $2^{k-1} + 1$. γ is the relative risk aversion coefficient of the investor.

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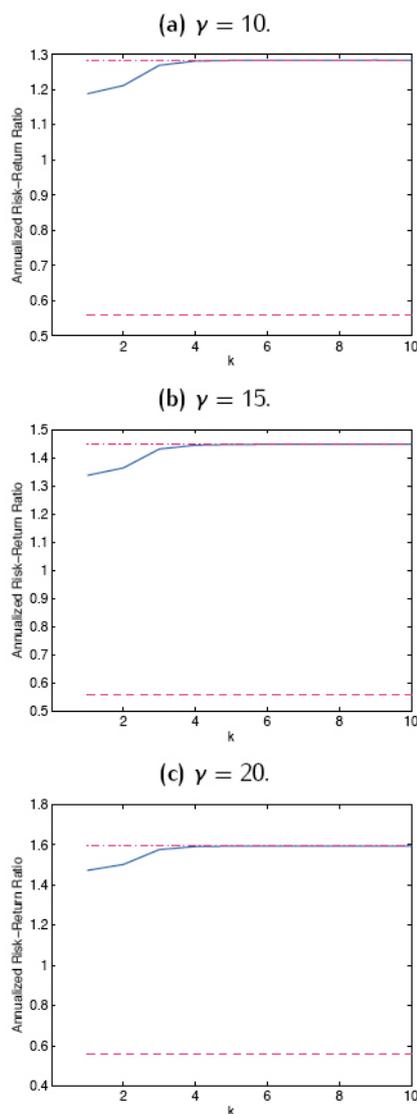
Figure 4: Monetary utility losses from partitioning the set of Sharpe ratios – low variability



The solid line represents monetary utility losses (MUL) from following strategies based on partitions of the set of Sharpe ratios. The dash-dot line represents the MUL from implementing the optimal strategy in discrete time, and the dashed line is the MUL from implementing a deterministic life-cycle strategy. Unless otherwise indicated, parameters are set at their base-case values (see table 1). k denotes the index of the partition, so that for $k \geq 2$, the number of classes in the partition is $2^{k-1} + 1$. γ is the relative risk aversion coefficient of the investor.

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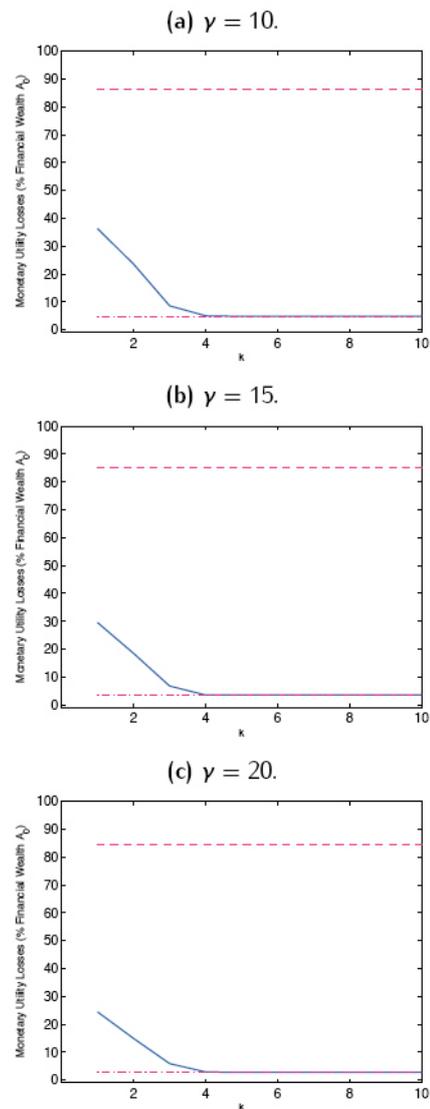
Figure 5: Risk/return ratios for strategies based on a partition of the set of Sharpe ratios – low variability



The solid lines represent strategies based on partitions of the set of Sharpe ratios. The dash-dot lines represent the optimal strategy implemented in discrete time, and the dashed lines represent a deterministic life-cycle strategy. The risk-return ratio is annualised. Unless otherwise indicated, parameters are set at their base-case values (see table 1). k denotes the index of the partition, so that for $k \geq 2$, the number of classes in the partition is $2^{k-1} + 1$. γ is the relative risk aversion coefficient of the investor.

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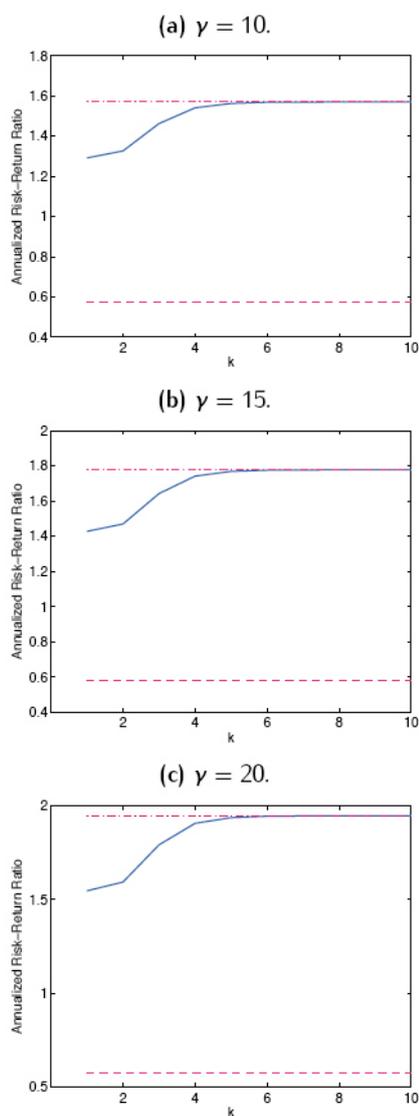
Figure 6: Monetary utility losses from partitioning the set of Sharpe ratios – medium variability



The solid line represents monetary utility losses (MUL) from following strategies based on partitions of the set of Sharpe ratios. The dash-dot line represents the MUL from implementing the optimal strategy in discrete time, and the dashed line is the MUL from implementing a deterministic life-cycle strategy. Unless otherwise indicated, parameters are set at their base-case values (see table 1), except for the standard deviation σ_λ , which was taken to be 9.4%. k denotes the index of the partition, so that for $k \geq 2$, the number of classes in the partition is $2^{k-1} + 1$. γ is the relative risk aversion coefficient of the investor.

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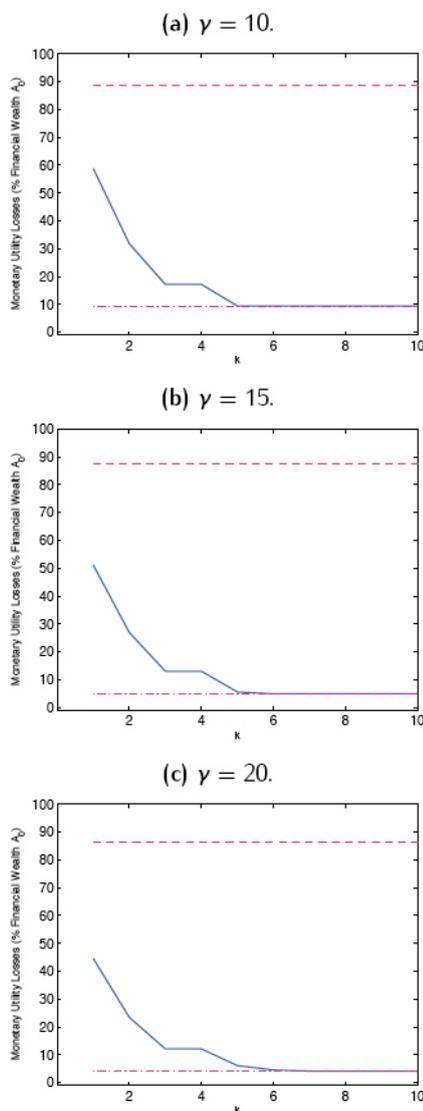
Figure 7: Risk/return ratios for strategies based on a partition of the set of Sharpe ratios - medium variability



The solid line represents the annualised risk/return ratio over T years for strategies based on partitions of the set of Sharpe ratios. The dash-dot line represents the optimal strategy implemented in discrete time, and the dashed line represents a deterministic life-cycle strategy. Unless otherwise indicated, parameters are set at their base-case values (see table 1), except for the standard deviation σ_{λ} , which was taken to be 9.4%. k denotes the index of the partition, so that for $k \geq 2$, the number of classes in the partition is $2^{k-1} + 1$. γ is the relative risk aversion coefficient of the investor.

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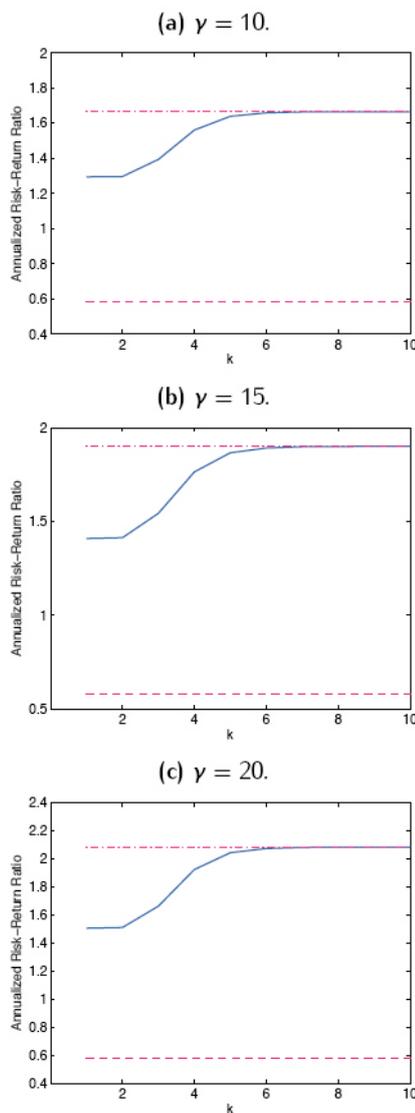
Figure 8: Monetary utility losses from partitioning the set of Sharpe ratios – high variability



The solid line represents monetary utility losses (MUL) from following strategies based on partitions of the set of Sharpe ratios. The dash-dot line represents the MUL from implementing the optimal strategy in discrete time, and the dashed line is the MUL from implementing a deterministic life-cycle strategy. Unless otherwise indicated, parameters are set at their base-case values (see table 1), except for the standard deviation σ_λ , which was taken to be 9.4%, and the long-term mean κ , which was set to 3.04%. k denotes the index of the partition, so that for $k \geq 2$, the number of classes in the partition is $2^{k-1} + 1$. γ is the relative risk aversion coefficient of the investor.

Appendices

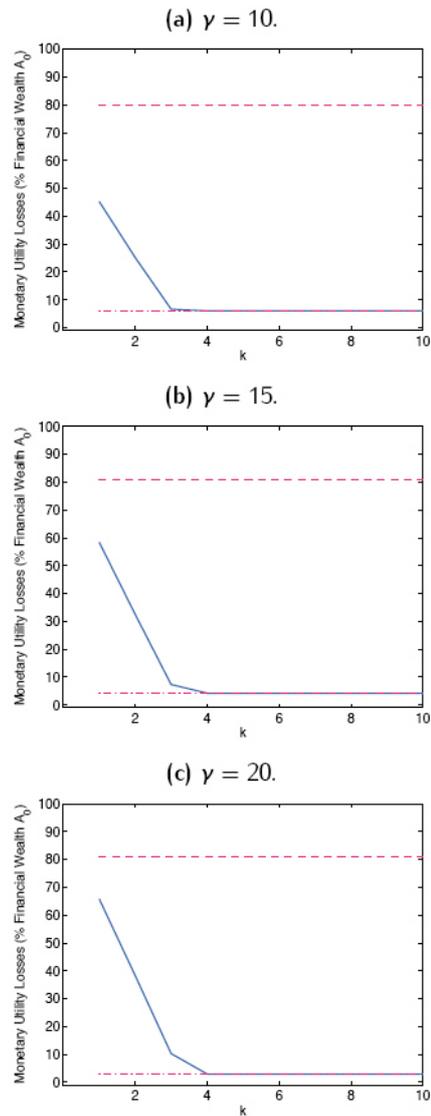
Figure 9: Risk/return ratios for strategies based on a partition of the set of Sharpe ratios – high variability



The solid line represents the annualised risk/return ratio over T years for strategies based on partitions of the set of Sharpe ratios. The dash-dot line represents the optimal strategy implemented in discrete time, and the dashed line represents a deterministic life-cycle strategy. Unless otherwise indicated, parameters are set at their base-case values (see table 1), except for the standard deviation σ_λ , which was taken to be 9.4%, and the long-term mean κ , which was set to 3.04%. k denotes the index of the partition, so that for $k \geq 2$, the number of classes in the partition is $2^{k-1} + 1$. γ is the relative risk aversion coefficient of the investor.

Appendices

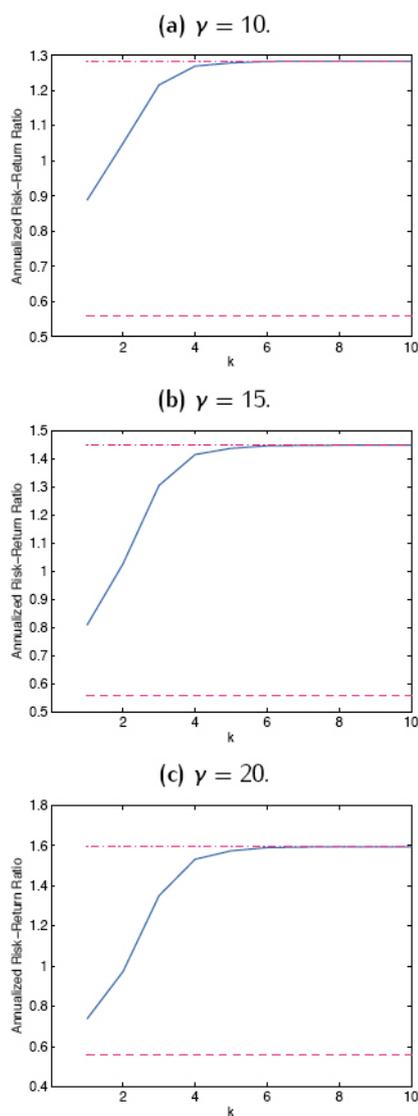
Figure 10: Monetary utility losses from partitioning both the set of times-to-horizon and the set of Sharpe ratios



The solid line represents monetary utility losses (MUL) from following strategies based on partitions of the set of Sharpe ratios and the set of horizons. The solid line represents strategies based on such partitions. The dash-dot line represents the MUL from implementing the optimal strategy in discrete time, and the dashed line is the MUL from implementing a deterministic life-cycle strategy. Unless otherwise indicated, parameters are set at their base-case values (see table 1). k denotes the index of the partition of the set of Sharpe ratios, so that the number of classes in the partition is $2^{k-1} + 1$. γ is the relative risk aversion coefficient of the investor.

Appendices

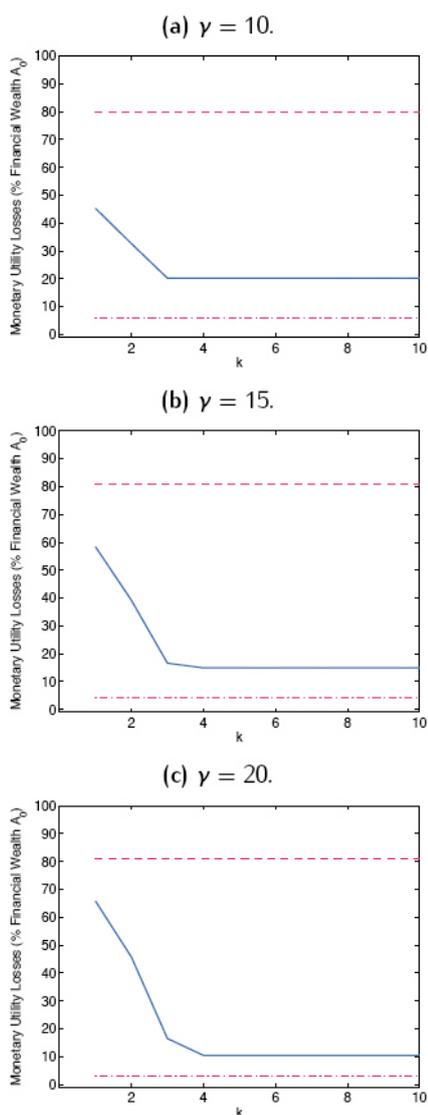
Figure 11: Risk/return ratios for strategies based on partitions of the set of times-to-horizon and of the set of Sharpe ratios



The solid lines represent strategies based on partitions of the set of times-to-horizon and of the set of Sharpe ratios. The dash-dot line represents the optimal strategy implemented in discrete time, and the dashed line represents a deterministic life-cycle strategy. The risk/return ratio is annualised. Unless otherwise indicated, parameters are set at their base-case values (see table 1). k denotes the index of the partition, so that for $k \geq 2$, the number of classes in the partition is $2^{k-1} + 1$. γ is the relative risk aversion coefficient of the investor.

Appendices

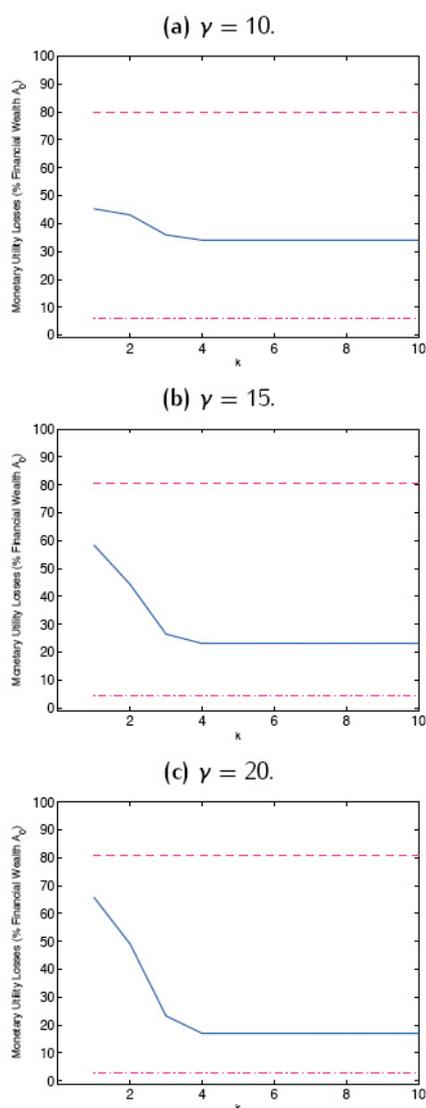
Figure 12: Monetary utility losses for strategies based on partitions of the set of times-to-horizon and of the set of Sharpe ratios when the Sharpe ratio is imperfectly observed – low volatility of the noise process



The solid lines represent strategies based on partitions of the set of times-to-horizon and of the set of Sharpe ratios. The dash-dot line represents the optimal strategy implemented in discrete time, and the dashed line represents a deterministic life-cycle strategy. Unless otherwise indicated, parameters are set at their base-case values (see table 1), and the Sharpe ratio is observed with a noise whose standard deviation is twice the long-term standard deviation of the stochastic λ^S . k denotes the index of the partition, so that for $k \geq 2$, the number of classes in the partition is $2^{k-1} + 1$. γ is the relative risk aversion coefficient of the investor.

Appendices

Figure 13: Monetary utility losses for strategies based on partitions of the set of times-to-horizon and of the set of Sharpe ratios when the Sharpe ratio is imperfectly observed – high volatility of the noise process



The solid lines represent strategies based on partitions of the set of times-to-horizon and of the set of Sharpe ratios. The dash-dot line represents the optimal strategy implemented in discrete time, and the dashed line represents a deterministic life-cycle strategy. Unless otherwise indicated, parameters are set at their base-case values (see table 1), and the Sharpe ratio is observed with a noise whose standard deviation is five times the long-term standard deviation of the stochastic λ^S . k denotes the index of the partition, so that for $k \geq 2$, the number of classes in the partition is $2^{k-1} + 1$. γ is the relative risk aversion coefficient of the investor.

Appendices

Table 1: Base-case parameters

Stock index	
σ_S	0.1468
S_0	100
Sharpe ratio	
σ_λ	0.047
κ	0.0608
$\bar{\lambda}$	0.4414
λ_0	0.4414
Nominal rate	
σ_R	0.0195
λ_R	-0.2747
a	0.0395
b	0.0369
R_0	0.0369
Price index	
σ_Φ	0.0081
π	0.0357
Φ_0	1
Real estate	
σ_Y	0.596
λ_Y	0.734
Y_0	100
Correlations	
ρ_{RS}	0.0845
$\rho_{S\Phi}$	-0.0678
$\rho_{R\Phi}$	0.0032
$\rho_{R\lambda}$	-0.0845
$\rho_{\Phi\lambda}$	0.0678
$\rho_{S\lambda}$	-1
ρ_{SY}	-0.0134
ρ_{RY}	-0.0036
$\rho_{\Phi Y}$	0.088
$\rho_{Y\lambda}$	0.0134

This table displays the base-case parameter values. The dynamics of the state variables are given in equations (2.1), (2.2), (2.3), (2.4), and (2.5).

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About EDHEC-Risk Institute

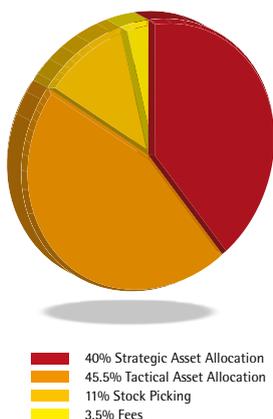


About EDHEC-Risk Institute

Founded in 1906, EDHEC is one of the foremost French business schools. Accredited by the three main international academic organisations, EQUIS, AACSB, and Association of MBAs, EDHEC has for a number of years been pursuing a strategy for international excellence that led it to set up EDHEC-Risk in 2001. With 47 professors, research engineers and research associates, EDHEC-Risk has the largest asset management research team in Europe.

The Choice of Asset Allocation and Risk Management

EDHEC-Risk structures all of its research work around asset allocation and risk management. This issue corresponds to a genuine expectation from the market. On the one hand, the prevailing stock market situation in recent years has shown the limitations of diversification alone as a risk management technique and the usefulness of approaches based on dynamic portfolio allocation. On the other, the appearance of new asset classes (hedge funds, private equity, real assets), with risk profiles that are very different from those of the traditional investment universe, constitutes a new opportunity and challenge for the implementation of allocation in an asset management or asset/liability management context. This strategic choice is applied to all of the centre's research programmes, whether they involve proposing new methods of strategic allocation, which integrate the alternative class; taking extreme risks into account in portfolio construction; studying the usefulness of derivatives in implementing asset/liability management approaches; or orienting the concept of dynamic "core-satellite" investment management in the framework of absolute return or target-date funds.



Source EDHEC (2002) and Ibbotson, Kaplan (2000)

An Applied Research Approach

In an attempt to ensure that the research it carries out is truly applicable, EDHEC has implemented a dual validation system for the work of EDHEC-Risk. All research work must be part of a research programme, the relevance and goals of which have been validated from both an academic and a business viewpoint by EDHEC-Risk's advisory board. This board is made up of internationally recognised researchers, the centre's business partners and representatives of major international institutional investors. The management of the research programmes respects a rigorous validation process, which guarantees the scientific quality and the operational usefulness of the programmes.

Six research programmes have been undertaken :

- Asset allocation and alternative diversification
- Style and performance analysis
- Indices and benchmarking
- Operational risks and performance
- Asset allocation and derivative instruments
- ALM and asset management

These programmes receive the support of a large number of financial companies. The results of the research programmes are disseminated through the three EDHEC-Risk locations in London, Nice, and Singapore.

In addition, EDHEC-Risk has developed close partnerships with a small number of sponsors within the framework of research chairs. These research chairs involve a three-year commitment by EDHEC-Risk and the sponsor to research themes on which the parties to the chair have agreed.

About EDHEC-Risk Institute

The following research chairs have been endowed :

- Regulation and Institutional Investment, *in partnership with AXA Investment Managers (AXA IM)*
- Asset/Liability Management and Institutional Investment Management, *in partnership with BNP Paribas Investment Partners*
- Risk and Regulation in the European Fund Management Industry, *in partnership with CACEIS*
- Structured Products and Derivative Instruments, *sponsored by the French Banking Federation (FBF)*
- Private Asset/Liability Management, *in partnership with ORTEC Finance*
- Dynamic Allocation Models and New Forms of Target-Date Funds, *in partnership with UFG*
- Advanced Modelling for Alternative Investments, *in partnership with Newedge Prime Brokerage*
- Asset/Liability Management Techniques for Sovereign Wealth Fund Management, *in partnership with Deutsche Bank*
- Core-Satellite and ETF Investment, *in partnership with Amundi ETF*
- The Case for Inflation-Linked Bonds: Issuers' and Investors' Perspectives, *in partnership with Rothschild & Cie*

The philosophy of the institute is to validate its work by publication in international journals, but also to make it available to the sector through its position papers, published studies and conferences.

Each year, EDHEC-Risk organises a major international conference for institutional investors and investment management professionals with a view to presenting the results of its research: EDHEC-Risk Institutional Days.

EDHEC also provides professionals with access to its website, www.edhec-risk.com, which is entirely devoted to international asset management research. The website, which has more than 35,000 regular visitors, is aimed at professionals who wish to benefit from EDHEC's analysis and expertise in the area of applied portfolio management research. Its monthly newsletter is distributed to more than 400,000 readers.

EDHEC-Risk Institute: Key Figures, 2008-2009

Number of permanent staff	47
Number of research associates	17
Number of affiliate professors	5
Overall budget	€8,700,000
External financing	€5,900,000
Number of conference delegates	1,950
Number of participants at EDHEC-Risk Executive Education seminars	371

Research for Business

EDHEC-Risk's activities have also given rise to executive education and research service offshoots.

EDHEC-Risk's executive education programmes help investment professionals to upgrade their skills with advanced risk and asset management training across traditional and alternative classes.

About EDHEC-Risk Institute

The EDHEC-Risk Institute PhD in Finance

The EDHEC-Risk Institute PhD in Finance at EDHEC Business School is designed for professionals who aspire to higher intellectual levels and aim to redefine the investment banking and asset management industries. It is offered in two tracks: a residential track for high-potential graduate students, who hold part-time positions at EDHEC Business School, and an executive track for practitioners who keep their full-time jobs. Drawing its faculty from the world's best universities and enjoying the support of the research centre with the greatest impact on the European financial industry, the EDHEC-Risk Institute PhD in Finance creates an extraordinary platform for professional development and industry innovation.

The EDHEC-Risk Institute MSc in Risk and Investment Management

The EDHEC-Risk Institute Executive MSc in Risk and Investment Management is designed for professionals in the investment management industry who wish to progress, or maintain leadership in their field, and for other finance practitioners who are contemplating lateral moves. It appeals to senior executives, investment and risk managers or advisors, and analysts. This postgraduate programme is designed to be completed in seventeen months of part-time study and is formatted to be compatible with professional schedules.

The programme has two tracks: an executive track for practitioners with significant investment management experience and an apprenticeship track for selected high-potential graduate students who have recently joined the industry. The programme is offered in Asia—from Singapore—and in Europe—from London and Nice.

FTSE EDHEC-Risk Efficient Indices

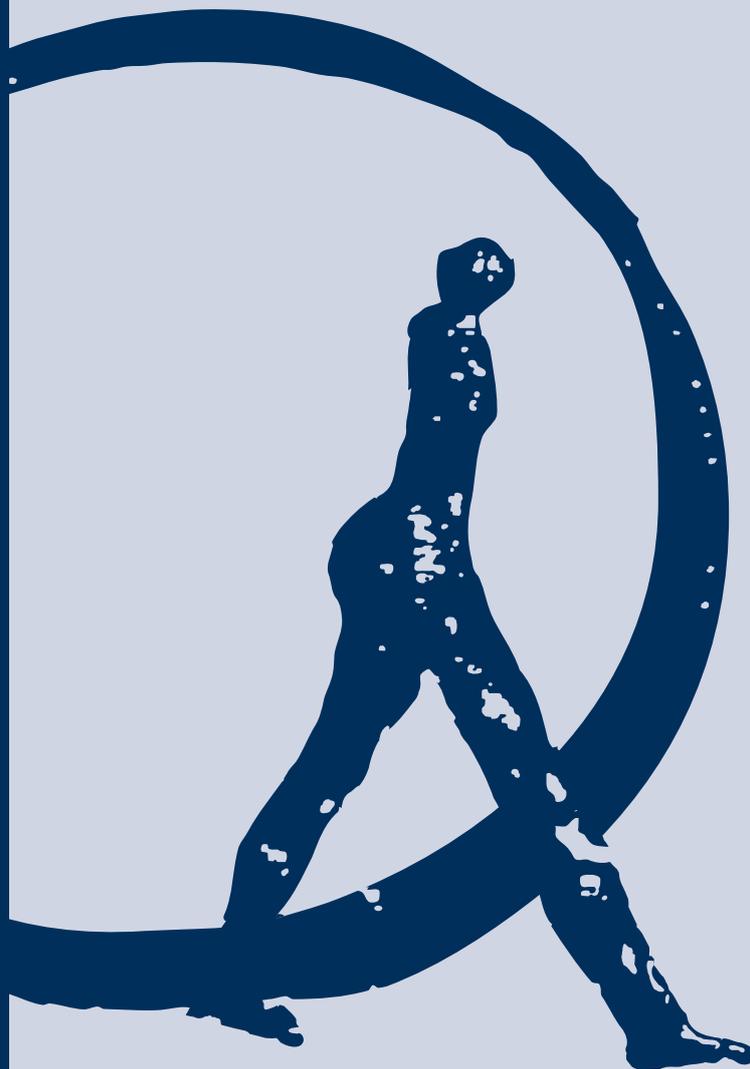
FTSE Group, the award winning global index provider, and EDHEC-Risk Institute launched the first set of FTSE EDHEC Risk Efficient Indices at the beginning of 2010. Initially offered for the UK, the Eurobloc, the USA, Developed Asia-Pacific, ex. Japan, and Japan, the index series aims to capture equity market returns with an improved risk/reward efficiency compared to cap-weighted indices. The weighting of the portfolio of constituents achieves the highest possible return-to-risk efficiency by maximising the Sharpe ratio (the reward of an investment per unit of risk).

EDHEC-Risk Alternative Indexes

The different hedge fund indexes available on the market are computed from different data, according to diverse fund selection criteria and index construction methods; they unsurprisingly tell very different stories. Challenged by this heterogeneity, investors cannot rely on competing hedge fund indexes to obtain a "true and fair" view of performance and are at a loss when selecting benchmarks. To address this issue, EDHEC-Risk was the first to launch composite hedge fund strategy indexes as early as 2003.

The thirteen EDHEC-Risk Alternative Indexes are published monthly on www.edhec-risk.com and are freely available to managers and investors.

About UFG-LFP



About UFG-LFP

UFG-LFP: Together, giving new meaning to finance

UFG-LFP is an asset management firm that offers both real estate and securities expertise. Its primary shareholder is CMNE (80.45%), followed by both company directors and employees (14.53%) and various institutional investors (MACSF and Groupe Monceau : 5.02%).

The new group occupies a pole position with regards to mutual fund assets, thematic assets, sustainable investment and alternative funds of hedge funds. The group is additionally a key player in the overall real estate market, be it asset or property management. UFG-LFP also has a private equity activity. UFG-LFP answers to a wide customer base including institutional investors, banks, distribution platforms and financial advisors. It seeks to expand its offer internationally and to meet private customer requirements.

UFG-LFP currently manages over 32 billion euros in assets. It aspires to "give new meaning to finance" through its commitment to understanding overall trends and their impact on both the economy and financial markets and through its capacity to create solutions, truly adapted to consumer requirements, as illustrated notably by its socially responsible, multiple class asset, management offer.



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2007

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