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Life-Cycle Investing in Private Wealth Management

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Foreword

The present research paper is part of the La Française AM research chair at EDHEC-Risk Institute on "Dynamic Allocation Models and New Forms of Target-Date Funds." The goal of this chair is to study the implementation of asset management solutions that genuinely exploit the usefulness of dynamic allocation strategies within a life-cycle investing framework.

In the first-year research paper from the research chair, "From Deterministic to Stochastic Life-Cycle Investing: Implications for the Design of Improved Forms of Target Date Funds," we drew on the fact that target-date funds had been found inconsistent with the prescriptions of standard life-cycle investment model and characterised in closed-form the optimal time- and state-dependent allocation strategy for a long-term investor preparing for retirement in the presence of interest-rate and inflation risks and a mean-reverting equity risk premium. We confirmed that existing target date fund products are the wrong answer to the right question, and the opportunity cost involved in purely deterministic life-cycle strategies is found to be substantial for reasonable parameter values.

The present paper looks at the application of those findings in private wealth management and argues in favour of better target date funds based on stochastic life cycle investing, taking into account the presence of risk factors that impact not only asset returns, but also private investors' wealth levels.

We would like to extend warm thanks to our partners at La Française AM for their support of this research.



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Executive Summary



Executive Summary

Academic research has shown that asset-liability management (ALM) is the proper framework for analysing private clients' investment decisions because it allows for the integration of their specific time-horizon, constraints and objectives in the portfolio construction process (see Amenc *et al.* (2009) for a recent reference). While the ideal solution for ultra high-net worth clients and large family offices, such a highly customized approach cannot, however, be implemented for all private investors. In this context, it appears more than appropriate for the asset management industry to work towards the design of life-cycle funds that can allow for the incorporation of a class of private investors' horizon and objectives. Currently available target-date fund products, mostly oriented towards retail clients, are not a satisfactory answer to the problem because they are based on simplistic allocation schemes leading to a deterministic decrease in equity allocation regardless of market conditions (see Martellini and Milhau (2010)). We argue in this paper that financial innovation is needed to design better target date funds based on stochastic life cycle investing, taking into account the presence of risk factors that impact not only asset returns, but also private investors' wealth levels. One key element in private wealth management is the presence of income risk, which has a substantial impact on the optimal asset allocation strategy.

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1. Introduction

Asset-liability management (ALM) denotes the adaptation of the portfolio management process in order to handle the presence of various constraints relating to the commitments that represent the liabilities of an investor. Academic research has suggested that suitable extensions of portfolio optimisation techniques used by institutional investors, e.g., pension funds, would usefully be transposed to the context of private wealth management because they have been precisely engineered to allow for the incorporation of an investor's specific constraints, objectives and horizon in the portfolio construction process – all of which can be summarised in terms of a single state variable, the value of the "liability" portfolio (see Amenc *et al.* (2009) for a recent reference).

It should be noted at this stage that within the framework of private wealth management, we use a broad definition of "liabilities", which encompasses any commitment or spending objective, typically self-imposed (as opposed to exogenously imposed as in a pension fund context), that an investor is facing. Overall, it is not the performance of a particular fund nor that of a given asset class that will be the determinant factor in the ability to meet a private investor's expectations. The success or failure of the satisfaction of the investor's long term objectives is fundamentally dependent on an ALM exercise that aims at determining the proper strategic inter-classes allocation as a function of the investor's specific objectives and constraints, in addition to the investor's time-horizon. In other words, what will prove to be the decisive factor is the ability to design an

asset allocation solution that is a function of the particular kinds of risks to which the investor is exposed, as opposed to the market as a whole.

While the perfect solution for ultra high-net worth clients and large family offices, such a highly customized approach cannot, however, be implemented for all private investors. In this context, it appears more than appropriate for the asset management industry to work towards the design of life-cycle funds that can allow for the incorporation of a class of private investors' horizon and objectives. Currently available target-date fund products, mostly oriented towards retail clients, are not a satisfactory answer to the problem. One key limitation is that they are based on simplistic allocation schemes leading to a deterministic decrease in equity allocation regardless of market conditions, while academic research instead supports the emergence of extended forms of life-cycle strategies that adjust the allocation to equities, not only as a function of time-horizon but also as a function of the relative cheapness of equity markets (see for example Martellini and Milhau (2010) for a quantitative measure of the opportunity cost implied by the kinds of deterministic glide paths typically used by currently available forms of target date funds).

Financial innovation is therefore needed to design better target date funds based on stochastic life cycle investing. Such funds could provide high net worth individuals with a much better answer to their long-term investment needs compared to existing balanced-fund approaches, because they can be designed to take

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into account the presence of risk factors that impact asset returns, as well as private investors' wealth levels. However, implementing optimal strategies in a delegated money management context is a serious challenge, since it requires a finer classification of private investors based on factors other than their age and risk-aversion. The challenge is in fact to design a parsimonious partition of the investors and states-of-nature that will allow for different allocation strategies. Broadly speaking, there are two sets of attributes that should be used to define the various categories of asset allocation decisions, namely the objective and the subjective attributes. The objective attributes apply to all investors and relate to market conditions, with a proposed asset allocation decision that will be a function of the following three state variables: the risk premium, the short-term interest rate and the volatility. Estimating the risk premium is notoriously an issue (Merton, 1980). But previous research (see e.g. Martellini and Milhau (2010)) has shown that a strategy that would only distinguish between a finite number of levels for this parameter (such as low, medium and high) would lead to much higher welfare than a deterministic target-date fund. The subjective attributes, on the other hand, are related to each particular investor, and include (in addition to age, which is currently the sole determinant in current TDF products) risk aversion as well as the income. Martellini and Milhau (2010) show that using an approximated horizon instead of the actual one only leads to small utility losses, unless the approximation is too rough. This property holds even if this approximation is

combined with the use of a proxy for the actual equity risk premium.

One key element that is missing in their analysis is the presence of non-financial income. Assuming away the presence of income risk might be a reasonable approximation for "old money" private clients, for whom the present value of future income is typically small compared to the current level of accumulated wealth. On the other hand it certainly is not appropriate for "new-money" affluent private clients, who are typically entrepreneurs who still enjoy a substantial stream of revenues. More generally, it is an extremely simplified assumption for most high net worth individuals, for whom income risk remains a substantial problem. For these clients, the present value of future income (human capital) is indeed typically large compared to financial wealth, and it is more than likely to have an effect on optimal portfolio decisions.

A first study of the effect of non-financial income on portfolio decisions is provided by Merton (1971), who shows that if income is deterministic, then investors would optimally choose a portfolio with the same relative weights allocated to the risky assets as investors of no income. But they would behave as if they had total wealth equal to the sum of financial wealth and human capital, which typically implies a more leveraged allocation than in the case of no income. Investors with stochastic non-financial income would optimally follow a similar strategy, but they also would seek to completely offset the implicit long position in the risk factors impacting their human capital by

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selling short hypothetical financial claims perfectly indexed on their future labour income. In practice, however, such strategies cannot be implemented, for at least two reasons. First, claims that perfectly replicate an investor's income stream are not traded. Even if some financial assets provide a partial hedge against labour income risk, some uninsurable risk still remains. Secondly, and more importantly perhaps, borrowing against future income raises moral hazard issues, since an investor cannot credibly pre-commit to receiving a given income pattern. This constraint may be violated even by optimal policies because these policies guarantee that the total wealth remains positive, but the financial wealth may go negative (see He and Pages (1993) for an example of such a situation). From the technical standpoint, however, solving a portfolio choice problem with unspanned income risk and/or liquidity constraint is challenging. The standard approach to such problems is the dynamic programming technique, based on the solution of the Hamilton-Jacobi-Bellman equation. This partial differential equation is non-linear in the value function and involves as many variables as there are stochastic variables in the model, which makes it typically difficult to solve, even numerically. One way of reducing the dimension of the problem is to replace the current pair of wealth and income by the wealth-to-income ratio in the arguments of the value function. This idea is exploited in Cairns *et al.* (2006), who provide numerical solutions in the presence of unspanned income risk and a stochastic risk-free rate, but they do not impose liquidity constraints. Munk and Sørensen (2010) solve the reduced

equation derived by Duffie *et al.* (1997), under the restriction that financial wealth must stay non negative, but assuming constant investment opportunities. In a discrete-time model, Viceira (2001) derives an approximate optimal policy based on the log-normal approximation proposed by Campbell (1993) (see also Koo (1998) for another discrete-time model). Some general insights into the properties of optimal policies can be obtained analytically, as in Duffie *et al.* (1997) and also in El Karoui and Jeanblanc-Picqué (1998), who show that in the presence of a liquidity constraint, the optimal strategy involves a long position in an American option – the purpose of which is to prevent wealth from going negative.¹ Analytical expressions for optimal portfolio rules are obtained by Henderson (2005) under the assumptions of CARA preferences, constant opportunity set and normally or log-normally distributed income process.

The assumption of a constant opportunity set is, however, difficult to justify in long-horizon contexts, where it is needed to recognise that risk and return parameters may evolve randomly over the investor's life-cycle. As shown by Merton (1973), a stochastic opportunity set gives birth to *hedging demands* for the risky assets. Stochastic investment opportunities also result in horizon-dependent, and often state-dependent, weights. A typical example is the mean-reverting equity risk premium, whose effects are studied in Kim and Omberg (1996) and Campbell and Viceira (1999) among many others.² From a general perspective, relaxing the assumption of a self-financed portfolio only reinforces the need to incorporate horizon and state dependencies. In this

1 - This result is similar to the one established by Cox and Huang (1989) for investors managing self-financed portfolios, who show that if the positivity constraint on terminal wealth is not binding, investors will have to invest in a European option in order to insure against shortfall risk.

2 - See also Brennan (1998), Barberis (2000) and Xia (2001) for incorporation of parameter uncertainty and learning effects in these studies.

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more general setting, the optimal asset allocation strategy involves a state-dependent allocation to three building-blocks: (i) a performance-seeking portfolio, heavily invested in equities, but also in bonds and alternative classes such as real estate, (ii) a liability-hedging portfolio, heavily invested in bonds for interest rate hedging motives, and also in real estate for inflation hedging motives, as well as (iii) an income-hedging portfolio, heavily invested in cash but also invested in equities, which exhibit appealing wage inflation hedging properties, particularly over long-horizons. In the early stages, the income-hedging fund is expected to be the dominant low-risk component of the investment strategy, but as the retirement date approaches, there is a gradual, albeit non-deterministic, switch from the income-hedging building block into the liability-hedging building block. Again, this switching only superficially resembles deterministic life cycle investing; instead of switching from high-risk assets to low-risk assets, as in the case of deterministic life cycle investing, the optimal stochastic lifestyle strategy involves a switch between different types of hedging demands; moreover this switch takes place in a stochastic state-dependent (as opposed to deterministic) manner, as a function of the current ratio of human capital to financial wealth.

Most of the models studied in the literature predict that the presence of uncertain income generally results in a higher demand for stocks by young investors, even in the presence of substantial transaction costs. Bodie *et al.* (1992) show that this effect is even larger when labour income is endogenous, since investors

have the option to increase their salary by working more in case they are faced with a fall in the stock market. Viceira (2001) shows that when labour income risk is orthogonal to equity risk, investors will optimally invest a larger fraction of their financial portfolio in stocks during their working life than during retirement. The intuition is that employed investors can rely on their labour earnings to finance their consumption needs, and are therefore ready to take more risk. It is only if the correlation between income and the stock market is very high that young agents will want to reduce their exposure to equity risk. Cocco *et al.* (2005) also report that the presence of a component orthogonal to equity risk in income risk makes human capital bond-like, which raises young investors' optimal demand for stocks. But these findings leave unanswered the questions of why holding profiles in equities exhibit a high degree of heterogeneity in practice, and why some young agents are in practice reluctant to invest in equities. The low demand for stocks can only be explained by extremely high correlation values between innovations to stock returns and labor income shocks or by the possibility of disastrous income shocks (see Cocco *et al.* (2005)). Taking into account liquidity constraints may help lower the large demands for risky assets that are generally implied by the models (see Koo (1998) and Munk (2000)). Benzoni *et al.* (2007) study a model that helps solve the puzzle: if labor income is co-integrated with dividends, then human capital is more stock-like than bond-like, so that the optimal allocation to equities is a hump-shaped function of the time-to-horizon. In particular, young investors

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may want to reduce their holdings in stocks.

Our paper complements these numerous studies by focusing on three questions that have important practical implications. First, we measure the welfare cost of ignoring non-financial income in the design of the long-term investment strategy, and find that this cost represents a significant fraction of an investor's total wealth. Taking into account sources of income other than financial gains is therefore important. Second, we focus on the practical implementation of strategies that meet this requirement. This question is by no means trivial, given the constraints faced by a fund provider. In general, it is impossible to offer fully customized solutions to clients, and only limited customization can be considered. But the optimal weights depend on the income profile of the investor for three reasons: (i) they depend on human capital, hence on the future income of the investor, (ii) they depend on financial wealth, hence on past income received by the agent, and (iii) they depend on the risk factors that affect the income process, and these risk factors are investor-specific. The question is therefore whether one can approximate the utility-maximising strategy by a policy that is compatible with limited customization, without incurring overly large utility losses. Third, we will take into account the implementation constraints that come from the estimation of the equity risk premium. This means that an additional approximation will be added to the strategies. We will show that partitioning the state-space of the equity risk premium into three values leads to a more reasonable estimation of the risk

premium, without significantly increasing the losses of expected utility with respect to the optimal strategy.

We first analyse a general life-cycle investing problem in the absence and in the presence of income risk, and we study our two research questions in the context of three specifications for the income process: first, a deterministic income stream, which implies bond-like human capital; second, the income stream related to the performance of a given stock (e.g., an entrepreneur or executive with revenues tied to the performance of a particular company), which implies stock-like human capital; third, an income stream that combines the bond-like and the stock-like features. The latter situation would, for example, be relevant for a top executive in an investment bank or asset management firm, with revenues strongly impacted by market performance. In each of these models, we measure the utility cost of ignoring non-financial income, and that of approximating the optimal strategy. Our results show that for a reasonably fine level of approximation, the welfare loss in the second case is lower than the welfare loss induced by simply ignoring any income that does not come from financial profits. This finding makes a case for life-cycle investment strategies that incorporate proper state and horizon dependencies. As such, our paper is related to papers that have assessed the sub-optimality of currently available deterministic target-date funds (see e.g. Cairns *et al.* (2006), Viceira and Field (2008), Martellini and Milhau (2010) and the references therein).

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The remainder of the paper is organised as follows. In section 2, we present a model for life-cycle investing that takes into account mean-reversion in the interest rates and in equity risk premium. Non-financial income is introduced in section 3, and different specifications for the income payments are studied in section 4. In section 5, we measure the opportunity cost of ignoring non-financial income and we describe approximations of the optimal strategy that help reduce this cost. Section 6 concludes.

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2. Optimal Allocation Strategies Over the Life-Cycle Without Non-Financial Income



2. Optimal Allocation Strategies Over the Life-Cycle Without Non-Financial Income

In this section, we present the solution to a portfolio choice problem without non-financial income. This setting can be appropriate for "old-money" super affluent private clients, for which the present value of future revenues is very small compared to the current wealth level. The model that we consider is similar to the one studied in Martellini and Milhau (2010).

2.1 The Economy

Uncertainty is modeled through a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We also fix a finite investment horizon T , which can be interpreted as the retirement date.

The nominal short-term interest rate r is assumed to follow the Vasicek model (Vasicek, 1977):

$$dr_t = a(b - r_t) dt + \sigma^r dz_t^r.$$

As shown by Vasicek (1977), if a constant price of interest rate risk λ^r exists, then the price at time t of a zero-coupon maturing at date T can be computed as:

$$B(t, T) = \exp[-D(T - t)r_t + h_1(T - t)],$$

where:

$$D(T - t) = \frac{1 - e^{-a(T-t)}}{a}$$

is the duration, and:

$$h_1(T - t) = \frac{(\sigma^r)^2}{2a^2} \left[T - t - 2D(T - t) + \frac{1 - e^{-2a(T-t)}}{2a} \right] + \left(b - \frac{\sigma^r \lambda^r}{a} \right) [D(T - t) - (T - t)].$$

An application of Ito's lemma shows that the dynamic evolution of the bond price is:

$$\frac{dB(t, T)}{B(t, T)} = [r_t - D(T - t)\sigma^r \lambda^r] dt - D(T - t)\sigma^r dz_t^r.$$

A constant-maturity bond is a roll-over of zero-coupon bonds that all have the same time-to-maturity τ . This bond has the same dynamics as a zero-coupon with fixed maturity, but the decreasing time-to-maturity $T - t$ is replaced by a constant τ :

$$\frac{dB_t}{B_t} = [r_t - D(\tau)\sigma^r \lambda^r] dt - D(\tau)\sigma^r dz_t^r.$$

In what follows, we will assume that the investor can trade in the constant-maturity bond (simply referred to as the bond). Since the yield curve is driven by one factor only, she is therefore able to replicate all zero-coupon bonds with fixed maturity by mixing the bond and the cash, whose value is the continuously compounded nominal short-term rate.³

Also traded is a stock index S (with dividends re-invested), which evolves as:

$$\frac{dS_t}{S_t} = (r_t + \sigma^S \lambda_t^S) dt + \sigma^S dz_t^S.$$

The quantity λ^S is the conditional Sharpe ratio. Following the abundant literature that has documented time-variation in expected excess returns, we assume this ratio to be stochastic:

$$d\lambda_t^S = \kappa(\bar{\lambda} - \lambda_t^S) dt + \sigma^\lambda dz_t^\lambda. \quad (2.1)$$

This mean-reverting model is the same as in Kim and Omberg (1996), and it implies a mean-reverting expected excess return as in Campbell and Viceira (1999). Sharpe ratio risk is not necessarily spanned by the stock and the bond, which may induce market incompleteness. One way to remove this source of incompleteness is to assume a perfectly negative correlation between unexpected stock returns and the innovations to the Sharpe ratio

3 - Of course, this property holds because we have introduced no holding constraints on the cash and the constant-maturity bond. In particular, short positions of arbitrary size are allowed. In practice, such constraints exist, but the universe of fixed-income securities is wider than what is assumed in our stylized model. As a consequence, it is reasonable to assume that the investor can delegate the replication of zero-coupon bonds to fixed-income managers. Introducing explicitly these assets in our model would be feasible, but would complicate notations without providing more insights.

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($\rho^{S\lambda} = -1$). This assumption is made in Wachter (2002) and it can be justified on the grounds of the strongly negative values reported in empirical work (e.g. Campbell and Viceira (1999), who report a correlation of -96% between realized returns and expected excess returns, and Xia (2001), who reports a value of -93%). We will make this assumption in what follows, so the market is complete. For notational convenience, it is useful to introduce a 3-dimensional Brownian motion \mathbf{z} and the volatility vectors. The innovation to each variable is therefore rewritten as:

$$\sigma^j dz_t^j = (\sigma^j)' dz_t, \quad j = r, S, \lambda.$$

With these notations, the dynamic evolution equations become:

$$\begin{aligned} dr_t &= a(b - r_t) dt + (\sigma^r)' dz_t, \\ d\lambda_t^S &= \kappa(\bar{\lambda} - \lambda_t^S) dt + (\sigma^\lambda)' dz_t, \\ \frac{dS_t}{S_t} &= (r_t + \sigma^S \lambda_t^S) dt + (\sigma^S)' dz_t, \\ \frac{dB_t}{B_t} &= [r_t - D(\tau)\sigma^r \lambda^r] dt + [\sigma^B(\tau)]' dz_t, \end{aligned}$$

where the volatility vector of the bond is:

$$\sigma^B(\tau) = -D(\tau)\sigma^r.$$

The volatility matrix of traded assets is obtained by writing the volatility vectors of the stock and the bond side by side:

$$\sigma = \begin{pmatrix} \sigma^B(\tau) & \sigma^S \end{pmatrix}$$

(note that this matrix is constant). The market price of risk vector λ_t is given by:

$$\lambda_t = \sigma (\sigma' \sigma)^{-1} \begin{pmatrix} -D(\tau)\sigma^r \lambda^r \\ \sigma^S \lambda_t^S \end{pmatrix} = \Lambda_1 + \lambda_t^S \Lambda_2.$$

Hence the vector of expected excess returns on the risky assets is equal to $\sigma' \lambda_t$.

The decomposition of λ_t as $\Lambda_1 + \lambda_t^S \Lambda_2$ aims at isolating the effect of the stochastic Sharpe ratio λ_t^S on the market price of risk vector. Since the market is complete, the market price of risk vector is unique. Another consequence of market completeness (Harrison and Kreps, 1979) is the existence of a unique pricing kernel in the economy, which is given by:⁴

$$M_t^0 = \exp \left[- \int_0^t \left(r_s + \frac{\|\lambda_s\|^2}{2} \right) ds - \int_0^t \lambda_s' dz_s \right] \quad \text{for } t \leq T \quad (2.2)$$

We assume that the investor has access to full information about the various risks in the economy. Technically, this hypothesis means that at any date t all decisions made are conditional on the sigma-algebra generated by the Brownian motion \mathbf{z} up to time t . Let \mathcal{F}_t denote this sigma-algebra, and $\mathbb{E}_t[\cdot]$ the corresponding conditional expectation operator, for t between 0 and T . Over the period $[0, T]$ the investor trades dynamically in the available assets, allocating the weights \mathbf{w}_t to the locally risky assets and $1 - \mathbf{w}_t' \mathbf{1}$ to the cash (we remind that $\mathbf{1}$ is the vector of size n full of ones). For the time being, we consider only self-financing strategies, where no cash is infused or withdrawn—an assumption that will be relaxed in section 3. As a consequence, the gain or loss of the portfolio is due only to the change in the values of financial assets, so the budget constraint can be written as:

$$\begin{aligned} \frac{dA_t}{A_t} &= \mathbf{w}_t' [(r_t \mathbf{1} + \sigma' \lambda_t) dt + \sigma' dz_t] \\ &\quad + (1 - \mathbf{w}_t' \mathbf{1}) r_t dt. \end{aligned} \quad (2.3)$$

This process can be interpreted as the value of the financial assets that the investor holds in preparation for retirement.

4 - The pricing kernel is also coined "stochastic discount factor" or "state-price deflator" in the literature.

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2.2 Optimal Portfolio Choice with Mean-Reverting Interest Rates and Sharpe Ratio

We now present the optimal strategy for an investor who is facing the budget constraint (2.3), and concerns about terminal wealth A_T . The optimisation program can be mathematically written as:⁵

$$\max_{A_T} \mathbb{E} [U (A_T)]. \quad (2.4)$$

for some utility function U . In this paper, we will maintain the assumption of a Constant Relative Risk Aversion (CRRA, or power) utility function:

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}.$$

Problem (2.4) can be solved by dynamic programming techniques, as explained in the seminal paper of Merton (1969), or via the convex duality technique of Cox and Huang (1989). We follow the latter approach. It consists of mapping the dynamic portfolio choice problem (2.4) into a "static" problem, where the control variable is the terminal value of the portfolio, rather than the whole process of portfolio weights:

$$\max_{\mathbf{w}} \mathbb{E} [U (A_T)]$$

The budget constraint is also made static, with (2.3) being replaced by a single equality:

$$\mathbb{E} [M_T A_T] = A_0.$$

The optimal portfolio rule is then obtained as the strategy that replicates the optimal terminal wealth. The following proposition gives the solution to (2.4).

Proposition 1 *The solution to (2.4) is described by:*

• *The optimal wealth process:*

$$A_t^0 = \frac{A_0}{g(0, \lambda_0^S) B(0, T)^{1-\frac{1}{\gamma}}} (M_t^0)^{-\frac{1}{\gamma}} B(t, T)^{1-\frac{1}{\gamma}} g(t, \lambda_t^S);$$

• *The optimal portfolio weights:*

$$\begin{aligned} \mathbf{w}_t^0 &= \frac{1}{\gamma} \frac{\lambda_t^{PSP}}{\sigma_t^{PSP}} \mathbf{w}_t^{PSP} + \left(1 - \frac{1}{\gamma}\right) \mathbf{w}_t^B \\ &\quad + \left(1 - \frac{1}{\gamma}\right) [C_2(T-t) \\ &\quad + C_3(T-t)\lambda_t^S] \frac{\sigma^\lambda}{\sigma^S} \mathbf{w}_t^{HD}, \end{aligned} \quad (2.5)$$

where:

$$\mathbf{w}_t^{PSP} = \frac{(\sigma' \sigma)^{-1} \sigma' \lambda_t}{\mathbf{1}' (\sigma' \sigma)^{-1} \sigma' \lambda_t},$$

$$\mathbf{w}_t^B = (\sigma' \sigma)^{-1} \sigma' \sigma^B (T-t),$$

$$\mathbf{w}_t^{HD} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\frac{\lambda_t^{PSP}}{\sigma_t^{PSP}} = \mathbf{1}' (\sigma' \sigma)^{-1} \sigma' \lambda_t.$$

The function g is given by:

$$g(t, \lambda) = \exp \left[\frac{1-\gamma}{\gamma} \left[C_1(T-t) + C_2(T-t)\lambda + \frac{1}{2} C_3(T-t)\lambda^2 \right] \right],$$

where C_1 , C_2 and C_3 are solutions to the following system of ordinary differential equations:

$$\begin{aligned} C_3'(s) &= \frac{\|\mathbf{\Lambda}_2\|^2}{\gamma} - 2 \left[\frac{1-\gamma}{\gamma} \sigma^\lambda + \kappa \right] C_3(s) \\ &\quad + \frac{1-\gamma}{\gamma} (\sigma^\lambda)^2 C_3(s)^2, \end{aligned}$$

5 - It is standard in related literature to assume that the investor is concerned with terminal real wealth, rather than nominal wealth. The focus on real wealth is motivated by the fact that inflation erodes purchasing power over the long run. Investors are therefore willing to hold inflation-hedging securities. We do not include inflation in our model, because the focus of our study is on the impact of income risk on portfolio choice, not on the impact of stochastic market parameters. The portfolio choice problem with inflation risk has been studied in several recent papers (see e.g. Campbell and Viceira (2001) and Brennan and Xia (2002); see also Martellini and Milhau (2010) for a version of the model including inflation risk).

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$$C'_2(s) = \frac{[\mathbf{\Lambda}_1 - \sigma^B(T-t)]' \mathbf{\Lambda}_2}{\gamma} - \left[\frac{1-\gamma}{\gamma} \sigma^\lambda + \kappa \right] C_2(s) \\ + \left[\kappa \bar{\lambda} + \frac{1-\gamma}{\gamma} (\sigma^\lambda)' [\mathbf{\Lambda}_1 - \sigma^B(T-t)] \right] C_3(s) \\ + \frac{1-\gamma}{\gamma} (\sigma^\lambda)^2 C_2(s) C_3(s),$$

$$C'_1(s) = \frac{\|\mathbf{\Lambda}_1 - \sigma^B(T-t)\|^2}{2\gamma} + \frac{1}{2} (\sigma^\lambda)^2 C_3(s) \\ + \left[\kappa \bar{\lambda} + \frac{1-\gamma}{\gamma} (\sigma^\lambda)' [\mathbf{\Lambda}_1 - \sigma^B(T-t)] \right] C_2(s) \\ + \frac{1-\gamma}{2\gamma} (\sigma^\lambda)^2 C_2(s)^2,$$

with the terminal conditions $C_1(0) = C_2(0) = C_3(0) = 0$.

Proof. See appendix A.1.

The formula for the optimal portfolio weights involves a standard three-fund separation result. The three funds are the performance-seeking portfolio (PSP) $\mathbf{w}_t^{\text{PSP}}$, the portfolio replicating a zero-coupon bond maturing at date T denoted by \mathbf{w}_t^{B} , and a hedging portfolio against Sharpe ratio risk, \mathbf{w}_t^{HD} . For the logarithmic investor ($\gamma = 1$), only the PSP is present. The other two funds arise in fact because investment opportunities are stochastic in this model: they are intertemporal hedging demands in the sense of Merton (1973). The first hedging demand is a demand for the zero-coupon that matches the investor's horizon, which is the risk-free asset over the entire investment period. As explained above, the zero-coupon maturing at date T , if it is not readily available, can be replicated by investing in constant-maturity bonds and cash:

$$\mathbf{w}_t^{\text{B}} = \frac{D(T-t)}{D(\tau)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

As explained by Detemple and Rindisbacher (2010), the second hedging demand aims at hedging the fluctuations in the density of a T -forward probability measure.⁶ Since we have assumed a perfect negative correlation between the innovations to S and the innovations to λ^S , the latter hedging demand is entirely invested in stocks. As shown by Wachter (2003), an investor with infinite risk aversion would invest only in that zero-coupon (the weights allocated to the PSP and the portfolio hedging λ^S shrink to zero). For finite levels of risk aversion, the optimal portfolio rule involves horizon effects, through the adjustment of duration of the constant-maturity bond, and the functions C_2 and C_3 . It also involves state-dependencies, since the weight allocated to stocks is an increasing function of the current Sharpe ratio λ_t^S .

6 - The density of the T -forward neutral measure with respect to the physical probability measure \mathbb{P} is given by: $\frac{d\mathbb{Q}^T}{d\mathbb{P}} = \frac{M_t^T}{B(0,T)}$.

2. Optimal Allocation Strategies Over the Life-Cycle Without Non-Financial Income

3. Introducing Non-Financial Income



3. Introducing Non-Financial Income

As mentioned in the introduction, income risk is an important factor in asset allocation decisions for "new-money" private clients and high net worth individuals. In this section, we introduce non-financial income in the analysis and derive optimal portfolio rules that take this feature into account.

3.1 Human Capital

We consider an investor who receives a non negative income and sets aside a fraction of this income in their financial portfolio. For simplicity, we assume that the income is received at deterministic dates, denoted t_1, \dots, t_n with $t_1 < \dots < t_n < T$. Between dates t_i and t_{i+1} , the portfolio is self-financing, and thus evolves as in (2.3). Then, at date t_i , the investor makes a contribution e_{t_i} to the financial portfolio. Modelling the saving decision is not the subject of this paper, so we simply assume that the contribution is equal to a constant and exogenous fraction of the income, as in Cairns *et al.* (2006). We take A to be right-continuous, so that A_{t_i} denotes the financial wealth just after the contribution has been made. Denoting with A_{t_i-} the wealth just before, we get that:

$$A_{t_i} = A_{t_i-} + e_{t_i}.$$

Wealth dynamics can be summarised in a single equation as:⁷

$$dA_t = A_t [(r_t + \mathbf{w}'_t \boldsymbol{\sigma}' \boldsymbol{\lambda}_t) dt + \mathbf{w}'_t \boldsymbol{\sigma}' dz_t] + \sum_{i=1}^n e_{t_i} \mathbb{1}_{\{t=t_i\}}. \tag{3.1}$$

The random contributions e_{t_i} may be affected by other sources of risk than those spanned by the traded assets. If this is the case, the market is incomplete. Finding an optimal strategy in this case is a difficult problem,

and it seems that only approximated solutions can be expected, except for very specific models.⁸ The case where the income payments are replicable is by far the most tractable. We emphasise that since we model contributions as constant fractions of labour income, it is equivalent to assume that income payments are replicable or that contributions are replicable. In this situation, there exist zero-coupon bonds maturing at dates t_1, \dots, t_n and paying exactly e_{t_1}, \dots, e_{t_n} . Then, the price of a bond paying the coupons e_{t_1}, \dots, e_{t_n} is uniquely defined, and is given by:

$$H_t = \sum_{t_i > t} B_e(t, t_i),$$

where $B_e(t, t_i) = \mathbb{E}_t \left[\frac{M_t^0}{M_{t_i}^0} e_{t_i} \right]$ is the price of the i^{th} zero-coupon bond. The present value of future contributions, H_t is called the human capital. It is equal to zero after date t_n , and its value drops by e_{t_i} after each date t_i . Although H needs not be pathwise decreasing – due to the effect of stochastic variables on the prices of zero-coupon bonds – it will generally be larger for long time-to-horizons and is zero by definition after t_n , which is the last payment date. As will be shown below, a key ingredient in the computation of the optimal allocation is the ratio of human capital to financial wealth. This ratio measures how much an individual's wealth is represented by future contributions, relative to financial wealth. The sum of financial wealth and human capital, $A_t + H_t$, is called the total wealth. The following proposition shows that the total wealth process can be viewed as the value of a self-financing strategy.

Proposition 2 Assume that for each $i = 1, \dots, n$, there exists a self-financing portfolio strategy $(\mathbf{w}_t^i)_{t \leq t_i}$ that replicates

7 - Technically, the indicator function $\mathbb{1}_{\{t=t_i\}}$ in (3.1) can also be written as $dD_t^{t_i}$, where D^{t_i} denotes the Dirac measure at date t_i , defined by $D^{t_i}([0, t]) = \mathbb{1}_{\{0 < t_i \leq t\}}$. D^{t_i} is also the distributional derivative of the Heaviside function $t \mapsto \mathbb{1}_{\{t_i \leq t\}}$
 8 - Henderson (2005) derives optimal portfolio rules for an investor with partially unspanned income risk, but who assumes constant investment opportunities.

3. Introducing Non-Financial Income

the contribution e_{t_i} . Then the total wealth $A + H$ is the value of a self-financing strategy for which the weight vector is:

$$\frac{A_t \mathbf{w}_t + H_t \beta_t^H \mathbf{w}_t^H}{A_t + H_t},$$

where:

$$\begin{aligned} H_t &= \sum_{t_i > t} B_e(t, t_i) = \mathbb{E}_t \left[\sum_{t_i > t} \frac{M_{t_i}^0}{M_t^0} e_{t_i} \right], \\ \beta_t^H &= \frac{1}{H_t} \sum_{t_i > t} B_e(t, t_i) \mathbf{1}' \mathbf{w}_{t_i}^i, \\ \mathbf{w}_t^H &= \frac{1}{H_t \beta_t^H} (\boldsymbol{\sigma}' \boldsymbol{\sigma})^{-1} \boldsymbol{\sigma}' \boldsymbol{\sigma}_t^H = \frac{1}{\sum_{t_i > t} B_e(t, t_i) \mathbf{1}' \mathbf{w}_{t_i}^i} \sum_{t_i > t} B_e(t, t_i) \mathbf{w}_{t_i}^i. \end{aligned}$$

Proof. See appendix A.2.

One consequence of this proposition is that the total wealth deflated by the pricing kernel, follows a martingale. In particular, we have, for $t \leq T$:⁹

$$A_t = \mathbb{E}_t \left[\frac{M_T^0}{M_t^0} A_T \right] - H_t. \quad (3.2)$$

When the income payments are not replicable, the market is incomplete and there exist infinitely many pricing kernels. As shown by He and Pearson (1991), these pricing kernels are those processes of the form:

$$\begin{aligned} M_t &= \exp \left[- \int_0^t \left(r_s + \frac{\|\boldsymbol{\lambda}_s + \boldsymbol{\nu}_s\|^2}{2} \right) \right. \\ &\quad \left. ds - \int_0^t (\boldsymbol{\lambda}_s + \boldsymbol{\nu}_s)' dz_s \right], \quad t \leq T, \end{aligned}$$

where the process $\boldsymbol{\nu}$ is such that $\boldsymbol{\sigma}' \boldsymbol{\nu}_t = 0$ almost surely for all t . In a slight abuse of notation, we will denote any of these pricing kernels with M_t . Each of these gives rise to a value for the stream of contributions, through:

$$H_t = \mathbb{E}_t \left[\sum_{t_i > t} \frac{M_{t_i}}{M_t} e_{t_i} \right].$$

In particular, the human capital is no longer uniquely defined.

3.2 Optimal Portfolio Strategies with Non-Financial Income

We now solve for optimal portfolio strategies in the presence of non-financial income. For the sake of generality, in this section we do not assume that the income payments are replicable, so the market may be incomplete. The optimization program reads:

$$\max_{\mathbf{w}} \mathbb{E} [U(A_T)], \text{ subject to (3.1)}$$

As explained in He and Pearson (1991), such a dynamic program can be reformulated as a static program where the control variable is the terminal wealth, and the budget constraint is expressed in terms of the present value of the terminal wealth. There is one static program for each pricing kernel M : $\max_{A_T} \mathbb{E} [U(A_T)]$,

$$\text{such that } \mathbb{E}[M_T A_T] = A_0 + H_0, \quad (3.3)$$

where H_0 is the human capital at date 0 computed with the pricing kernel M . The equivalence between the static program and the original dynamic program is obtained for the so-called *minimax* pricing kernel, which we denote with M^* in the following proposition.

Proposition 3 Let M^* denote the *minimax pricing kernel* in (3.3). Then the optimal wealth process is given by:

$$A_t^* = \frac{A_0 + H_0^*}{G_0^* B(0, T)^{1-\frac{1}{\gamma}}} (M_t^*)^{-\frac{1}{\gamma}} B(t, T)^{1-\frac{1}{\gamma}} G_t^* - H_t^*,$$

9 - Conversely, it can be shown that if A is the value of some portfolio strategy such that $M(A + H)$ is a martingale, then the dynamics of A are necessarily of the form (3.1). This technical result is used to substitute the static budget constraint (3.2) for the dynamic one (3.1) in the utility maximization program.

3. Introducing Non-Financial Income

where:

$$G_t^* = \mathbb{E}_t \left[\left(\frac{M_T^*}{M_t^* B(t, T)} \right)^{1-\frac{1}{\gamma}} \right],$$

$$H_t^* = \mathbb{E}_t \left[\sum_{t_i > t} \frac{M_{t_i}^*}{M_t^*} e_{t_i} \right]. \quad (3.4)$$

The optimal portfolio strategy reads:

$$\mathbf{w}_t^* = \left(1 + \frac{H_t^*}{A_t^*} \right) \left[\frac{\lambda_t^{PSP}}{\gamma \sigma_t^{PSP}} \mathbf{w}_t^{PSP} + \left(1 - \frac{1}{\gamma} \right) \mathbf{w}_t^B + \mathbf{w}_t^{G^*} - \frac{H_t^*}{A_t^*} \beta_t^{H^*} \mathbf{w}_t^{H^*} \right]$$

where:

$$\mathbf{w}_t^{G^*} = (\sigma' \sigma)^{-1} \sigma' \sigma_t^{G^*},$$

$$\mathbf{w}_t^{H^*} = \frac{(\sigma' \sigma)^{-1} \sigma' \sigma_t^{H^*}}{\mathbf{1}' (\sigma' \sigma)^{-1} \sigma' \sigma_t^{H^*}}$$

and $\beta_t^{H^*} = \mathbf{1}' (\sigma' \sigma)^{-1} \sigma' \sigma_t^{H^*}.$

Proof. See appendix A.3.

The optimal strategy (3.5) involves the performance-seeking portfolio, the portfolio replicating a zero-coupon bond maturing at date T , and two hedging portfolios \mathbf{w}^{G^*} and \mathbf{w}^{H^*} . The portfolios \mathbf{w}_t^{PSP} and \mathbf{w}_t^B have the same components as in the problem without income. It can be shown that \mathbf{w}^{G^*} and \mathbf{w}^{H^*} maximise the squared conditional correlation of the wealth process defined in (3.1) with the processes G^* and H^* respectively. H^* is the human capital computed with respect to the minimax pricing kernel M^* . Other pricing kernels may possibly lead to different assessments of the value of future contributions. The quantity $\beta_t^{H^*}$ is the beta of the human capital with respect to the value of the portfolio strategy \mathbf{w}^{H^*} . We note that in the general case, where income risk is not

spanned, both the human capital and the hedging portfolio \mathbf{w}^{H^*} depend on preferences. To compute the optimal weights at a given date t , one needs the financial wealth (which is observed), the human capital H_t^* (which is not directly observable), and the two hedging portfolios (which must be engineered from the dynamics of G^* and H^*). Computing H_t^* and the hedging portfolios is in general a difficult problem, since it requires the knowledge of the "minimax" martingale measure associated with problem (3.3). The derivation of this martingale measure is greatly facilitated if the income payments are replicable. Indeed, in that case, the market is complete, so the minimax pricing kernel coincides with M^0 . We summarise this property in the following corollary.

Corollary 1 Assume that the contributions e_{t_1}, \dots, e_{t_n} are replicable. Then the human capital is independent from the pricing kernel, and:

• The optimal wealth process is:

$$A_t^* = \frac{A_0 + H_0}{G_0 B(0, T)^{1-\frac{1}{\gamma}}} (M_t^0)^{-\frac{1}{\gamma}} B(t, T)^{1-\frac{1}{\gamma}} G_t - H_t,$$

with:

$$G_t = \mathbb{E}_t \left[\left(\frac{M_T^0}{M_t^0 B(t, T)} \right)^{1-\frac{1}{\gamma}} \right]$$

• The optimal portfolio strategy in the presence of income can be written as:

$$\mathbf{w}_t^* = \left(1 + \frac{H_t}{A_t^*} \right) \mathbf{w}_t^0 - \frac{H_t}{A_t^*} \beta_t^H \mathbf{w}_t^H, \quad (3.6)$$

where:

$$H_t = \sum_{t_i > t} \mathbb{E}_t \left[\frac{M_{t_i}^0}{M_t^0} e_{t_i} \right],$$

$$\mathbf{w}_t^H = \frac{\sum_{t_i > t} \mathbb{E}_t \left[\frac{M_{t_i}^0}{M_t^0} e_{t_i} \right] \mathbf{w}_t^i}{\sum_{t_i > t} \mathbb{E}_t \left[\frac{M_{t_i}^0}{M_t^0} e_{t_i} \right] \mathbf{1}' \mathbf{w}_t^i},$$

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$$\beta_t^H = \frac{1}{H_t} \sum_{t_i > t} \mathbb{E}_t \left[\frac{M_{t_i}^0}{M_t^0} e_{t_i} \right] \mathbf{1}' \mathbf{w}_t^i,$$

and \mathbf{w}_t^0 is the optimal portfolio without income, given in proposition 1.

In most practical applications we will assume that income payments are replicable, so the corollary will give us the utility-maximising strategy. In that case, the value of the future contributions is given by the market, rather than assessed by the investor herself. It should be emphasised that the hedging demand against H is not of the same nature as Merton intertemporal hedging demands. First, these hedging demands exist because of the investor's desire to hedge against unexpected changes in the opportunity set, and the opportunity set is the same for every investor. In contrast, the hedging demand against H is motivated by the desire to hedge against the fluctuations in the human capital, which depends on investor's characteristics, through the income process and the horizon. Second, the weights allocated to the portfolios that hedge the state variables in Merton (1973) depend on risk aversion. For instance, proposition 1 shows that logarithmic investors ($\gamma = 1$) do not want to hedge against interest rate or Sharpe ratio risk. But the weight allocated to the portfolio \mathbf{w}^H is independent from risk aversion: all investors, regardless their risk aversion, want to hedge against income risk to the same extent.

3. Introducing Non-Financial Income

4. Specification of Income Stream



4. Specification of Income Stream

In this section, we study different specifications for the income payments. We first describe a deterministic income stream, and we then model the income of a trader and that of an entrepreneur. Finally, we give some indications as to how address the difficult case of unspanned income risk.

4.1 Deterministic Income

We first consider an investment universe that consists of one stock index and one constant-maturity bond, as in the model of section 2. We also assume that the investor receives a deterministic income stream, so her budget constraint is given by (3.1). In order to have a parsimonious specification of the income payments, we will assume that they grow at a constant rate π , so that:

$$e_{t_i} = e_0 e^{\pi t_i},$$

where e_0 is a constant. The only factor of heterogeneity amongst investors is thus the quantity e_0 , which controls the size of the contributions. As a consequence, the contribution e_{t_i} is known as of date 0 and can be replicated by issuing e_{t_i} unit nominal zero-coupon bonds maturing at date t_i . Hence the price of receiving one contribution is:

$$B_e(t, t_i) = \mathbb{E}_t \left[\frac{M_{t_i}^0}{M_t^0} e_{t_i} \right] = B(t, t_i) e_{t_i},$$

$$\mathbf{w}_t^i = \frac{D(t_i - t)}{D(\tau)} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ denotes a portfolio entirely invested in constant-maturity bonds. Summing up the prices of the zero-coupon bonds, we obtain the human capital:

$$H_t = \sum_{t_i > t} B(t, t_i) e_{t_i} = e_0 \sum_{t_i > t} B(t, t_i) e^{\pi t_i}.$$

Since the human capital is subject to interest rate risk only, the portfolio that hedges changes in the human capital is fully invested in bonds:

$$\mathbf{w}_t^H = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and the beta of the human capital with respect to the value of this portfolio is:

$$\begin{aligned} \beta_t^H &= \frac{1}{H_t} \left[\sum_{t_i > t} B(t, t_i) e_{t_i} \frac{D(t_i - t)}{D(\tau)} \right] \\ &= \frac{\sum_{t_i > t} B(t, t_i) e^{\pi t_i} \frac{D(t_i - t)}{D(\tau)}}{\sum_{t_i > t} B(t, t_i) e^{\pi t_i}}, \end{aligned} \quad (4.1)$$

which only depends on the growth rate of the income, not on its initial level. The optimal portfolio strategy can be computed from the beta and the hedging portfolio, using equation (3.6).

So as to analyse the optimal demand for bonds in the presence of a deterministic income, let us temporarily assume that the PSP is invested in stocks only.¹⁰ As a consequence, bonds enter the portfolio without income only through the portfolio replicating the zero-coupon bond of maturity T . Then the optimal weight allocated to nominal bonds is:

$$\begin{aligned} \mathbf{w}_t^{*(1)} &= (\mathbf{w}_t^*)' \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \left(1 + \frac{H_t}{A_t^*} \right) \left(1 - \frac{1}{\gamma} \right) \frac{D(T - t)}{D(\tau)} \\ &\quad - \frac{1}{A_t^*} \left[\sum_{t_i > t} B(t, t_i) e_{t_i} \frac{D(t_i - t)}{D(\tau)} \right]. \end{aligned}$$

As appears from this equation, there are two competing effects that drive the final

10 - For example, this condition holds when $\lambda^r = 0$ and $\rho^S = 0$.

4. Specification of Income Stream

demand for bonds. On the one hand, the position in bonds that would be optimal in the absence of income is magnified since it is multiplied by $1 + H_t/A_t^*$. This tends to cause large long positions in bonds, especially for young investors, for whom the human capital is high compared to financial wealth. On the other hand, the presence of income risk induces a short position in bonds. The size of this short position grows with the ratio of human capital to financial wealth. The excess of the weight of bonds with income over the weight of bonds without income is:

$$\delta_B = \frac{1}{A_t^* D(\tau)} \left[\left(1 - \frac{1}{\gamma}\right) H_t D(T-t) - \sum_{t_i \geq t} B(t, t_i) e_{t_i} D(t_i - t) \right],$$

a quantity which is nonnegative if, and only if, the human capital satisfies:

$$\sum_{t_i \geq t} B(t, t_i) e_{t_i} D(t_i - t) \leq \left(1 - \frac{1}{\gamma}\right) H_t D(T-t). \quad (4.2)$$

It is impossible to predict whether this inequality will be satisfied or not in general. But it is clear that (4.2) holds when γ is infinite, because the duration of the intermediate income payments is lower than the duration of the zero-coupon bond maturing at date T . This means that the infinitely risk-averse investor with income will allocate more to bonds than the equivalent investor with no income. This conclusion is perhaps surprising, given that offsetting the implicit exposure to nominal bonds through the non-financial income requires a short position in bonds. For finite risk aversion levels, no general statement can be made, but (4.2) is more likely to hold for investors with high risk aversion and high time-to-horizon $T - t$.

4.2 Stochastic Replicable Income

In this subsection we consider two specifications for a stochastic income process that maintains market completeness. The first model applies to a worker whose income is indexed on the performance of the stock market, and the second one to an entrepreneur, whose revenues mainly arise from the dividends paid by company-held stocks.

4.2.1 Trader Case

A typical situation is the following: the agent receives a base deterministic income stream plus bonuses indexed on the performance of the stock market. Such a model particularly applies to traders, fund managers, or more generally any worker whose income is related to the performance of the stock market. We thus decompose the contribution as:

$$e_{t_i} = e_{t_i}^d + e_{t_i}^s,$$

where e^d grows at a constant rate π and e^s is a stochastic term that is perfectly correlated to a stock index S . This specification encompasses the case of deterministic income.

Since the stock index and the constant-maturity bond are traded, each contribution is perfectly replicable, and the human capital is:

$$H_t = \sum_{t_i > t} \left[B(t, t_i) e_{t_i}^d + \mathbb{E}_t \left[\frac{M_{t_i}^0}{M_t^0} e_{t_i}^s \right] \right].$$

As in subsection 4.1, we assume that the deterministic part of the income grows at a constant rate π :

$$e_{t_i}^d = e_0^d e^{\pi t_i}.$$

4. Specification of Income Stream

We also assume that the stochastic part is equal to a constant proportion of the growth rate of the stock index over the period $[0, t_i]$:

$$e_{t_i}^s = e_0^s \frac{S_{t_i}}{S_0}.$$

This reduced-form model means that the stochastic income is proportional to the performance of the stock market since the portfolio has started. Using these equalities, we can rewrite the human capital as:

$$H_t = e_0^d \sum_{t_i > t} B(t, t_i) e^{\pi t_i} + e_0^s \frac{S_t}{S_0} N_t, \quad (4.3)$$

where N_t is the number of indices i such that $t < t_i \leq T$. An application of Ito's lemma shows that the hedging portfolio against H and the beta of H with respect to this portfolio are given by:

$$\mathbf{w}_t^H = \frac{1}{H_t \beta_t^H} \left[e_0^d \sum_{t_i > t} B(t, t_i) e^{\pi t_i} \frac{D(t_i - t)}{D(\tau)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e_0^s \frac{S_t}{S_0} N_t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \quad (4.4)$$

$$\beta_t^H = \frac{1}{H_t} \left[e_0^d \sum_{t_i > t} B(t, t_i) e^{\pi t_i} \frac{D(t_i - t)}{D(\tau)} + e_0^s \frac{S_t}{S_0} N_t \right]. \quad (4.5)$$

In general, \mathbf{w}^H will be a mixture of bonds and stocks. It is only when the deterministic part is zero that \mathbf{w}^H is invested in stocks only, and only when the stochastic part is zero that it is invested in bonds only. As in the deterministic case, the optimal portfolio policy is given by equation (3.6). The expression for the weights involves the hedging portfolio \mathbf{w}^H , the beta of the human capital with respect to the hedging portfolio \mathbf{w}^H , denoted by β_t^H , and the strategy that is optimal in the absence of income (see (2.5)).

4.2.2 Entrepreneur Case

We now consider a model with three assets: a constant-maturity bond B , a stock index S , and the stock S^0 of the company the

entrepreneur started. Then, we assume that the entrepreneur holds one share of S^0 , and receives an income equal to the dividends paid by S^0 . These dividends are paid at the dates $t_1, \dots, t_n = T$, and the amount of dividends paid at time t_i is equal to a percentage q of the stock price S^0 . For simplicity, we take q to be constant. The income received by the entrepreneur at date t_i is therefore given by:

$$e_{t_i} = q S_{t_i-}^0. \quad (4.6)$$

The stock price S^0 is a right-continuous process that exhibits discontinuities at the payment dates:

$$dS_t^0 = S_t^0 \left[r_t + \sigma^{S^0} \lambda^{S^0} \right] dt + S_t^0 \sigma^{S^0} dz_t^{S^0} - S_{t-}^0 \sum_{i=1}^n q \mathbf{1}_{\{t_i=t\}},$$

where the last term reflects the decrease in value after a dividend has been paid out. The value of the index with dividends reinvested, \tilde{S}^0 , is a continuous process:

$$d\tilde{S}_t^0 = \tilde{S}_t^0 \left[r_t + \sigma^{S^0} \lambda^{S^0} \right] dt + \tilde{S}_t^0 \sigma^{S^0} dz_t^{S^0}.$$

The stock index S , the nominal short-term rate and the constant-maturity bond evolve as in the model of section 2. The vector form of the dynamic equations of the assets is:

$$\begin{aligned} \frac{dS_t}{S_t} &= [r_t + \sigma^S \lambda_t^S] dt + (\boldsymbol{\sigma}^S)' dz_t, \\ \frac{dB_t}{B_t} &= [r_t - D(\tau) \sigma^r \lambda^r] + [\boldsymbol{\sigma}^B(\tau)]' dz_t, \\ \frac{d\tilde{S}_t^0}{\tilde{S}_t^0} &= [r_t + \sigma^{S^0} \lambda^{S^0}] dt + (\boldsymbol{\sigma}^{S^0})' dz_t. \end{aligned}$$

and the vector form of the dynamic equations of the state variables is:

$$\begin{aligned} dr_t &= a(b - r_t) dt + (\boldsymbol{\sigma}^r)' dz_t, \\ d\lambda_t^S &= \kappa(\bar{\lambda} - \lambda_t^S) dt + (\boldsymbol{\sigma}^\lambda)' dz_t. \end{aligned}$$

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In these equations, z is a 3-dimensional Brownian motion, and all volatility vectors are 3-dimensional column vectors (we remind that there are only three sources of risk since interest rate risk is spanned by the bond and Sharpe ratio risk is spanned by the stock index itself).

The volatility vector and the duration of the bond are given by:

$$\sigma^B(\tau) = -D(\tau)\sigma^r, \quad D(\tau) = \frac{1 - e^{-a\tau}}{a}.$$

We also let σ denote the constant volatility matrix of the three locally risky assets:

$$\sigma = \begin{pmatrix} \sigma^B(\tau) & \sigma^S & \sigma^{S^0} \end{pmatrix},$$

and with λ_t the market price of risk vector:

$$\lambda_t = \sigma (\sigma' \sigma)^{-1} \begin{pmatrix} -D(\tau)\sigma^r \lambda^r \\ \sigma^S \lambda^S \\ \sigma^{S^0} \lambda^{S^0} \end{pmatrix}.$$

In addition to the fixed position in S^0 , the entrepreneur also holds a financial portfolio containing the stock index S , the bond B , the stock S^0 and the cash. All the assets contained in the financial portfolio are continuously and frictionless tradable. In particular, income risk is spanned, hence the market is complete. In addition to the income stream (4.6), the entrepreneur also receives the dividends paid by the shares of S^0 that she holds in her financial portfolio, and we assume that these dividends are reinvested in S^0 . Hence a portfolio strategy is described by a 3-dimensional vector w_t containing the weights allocated to S , B and \tilde{S}^0 respectively. The budget constraint can be written as:

$$dA_t = A_t [r_t + w_t' \sigma' \lambda_t] dt + A_t w_t' \sigma' dz_t + e_{t_i} \mathbb{1}_{\{t=t_i\}}$$

Since the market is complete, there is a unique price for the stream of contributions:

$$\begin{aligned} H_t &= \sum_{t_i > t} \mathbb{E}_t \left[\frac{M_{t_i}^0}{M_t^0} e_{t_i} \right] \\ &= q S_t^0 \sum_{t_i > t} \left[\prod_{t < t_j \leq t_i} (1 - q) \right] \\ &= \frac{q}{1 - q} S_t^0 \sum_{t_i > t} (1 - q)^{N(j; t < t_j \leq t_i)}, \end{aligned} \quad (4.7)$$

where $N(j; t < t_j \leq t_i)$ is the number of dividend payments falling within the interval $]t; t_i]$. It is clear that the only source of risk that affects H is z^{S^0} , because we have assumed a constant dividend rate. Hence the portfolio that perfectly replicates the unexpected changes in H is:

$$w_t^H = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and the coefficient β_t^H is 1.

Since income risk is spanned, the optimal allocation strategy for the entrepreneur is given by equation (3.6), which we rewrite here for convenience:

$$w_t^* = \left(1 + \frac{H_t}{A_t^*}\right) w_t^0 - \frac{H_t}{A_t^*} \beta_t^H w_t^H.$$

It remains to compute the optimal wealth process A^* and the optimal strategy in the absence of income, w^0 . In fact, we know by corollary 1 that the optimal wealth with income is related to the optimal wealth without income through:

$$A_t^* = \left(1 + \frac{H_0}{A_0}\right) A_t^0 - H_t.$$

Hence we need only to compute A_t^0 and w_t^0 . In fact, the model under consideration is very similar to the one studied in section 2. The only difference is the presence of a third

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asset, namely the stock S^0 . In spite of this difference, the derivation of the optimal wealth and optimal strategy without income are exactly the same as in proposition 1. We thus give their expressions in the following without providing any proof. First, the optimal wealth is:

$$A_t^0 = \frac{A_0}{g_1(0, \lambda_0^S) B(0, T)^{1-\frac{1}{\gamma}}} (M_t^0)^{-\frac{1}{\gamma}} g_1(t, \lambda_t^S),$$

where

$$g_1(t, \lambda_t^S) = \exp \left[\frac{1-\gamma}{\gamma} \left[D_1(T-t) + D_2(T-t) \lambda_t^S + \frac{1}{2} D_3(T-t) (\lambda_t^S)^2 \right] \right], \tag{4.8}$$

and D_1, D_2 and D_3 satisfy the same system of ordinary differential equations as the functions C_1, C_2 and C_3 of proposition 1, with the same terminal conditions $D_1(0) = D_2(0) = D_3(0)$, and the following expressions for the vectors Λ_1 and Λ_2 :

$$\Lambda_1 = \sigma(\sigma'\sigma)^{-1} \begin{pmatrix} -D(\tau)\sigma^r\lambda^r \\ 0 \\ \sigma^{S^0}\lambda^{S^0} \end{pmatrix},$$

$$\Lambda_2 = \sigma(\sigma'\sigma)^{-1} \begin{pmatrix} 0 \\ \sigma^S \\ 0 \end{pmatrix}.$$

Second, the optimal portfolio weights without income are:

$$\mathbf{w}_t^0 = \frac{1}{\gamma} \frac{\lambda_t^{PSP}}{\sigma_t^{PSP}} \mathbf{w}_t^{PSP} + \left(1 - \frac{1}{\gamma}\right) \frac{D(T-t)}{D(\tau)} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left(1 - \frac{1}{\gamma}\right) [D_2(T-t) + D_3(T-t)\lambda_t^S] \frac{\sigma^\lambda}{\sigma^S} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{4.9}$$

where the PSP is computed over S, B and S^0 .

4.3 Stochastic Non-Replicable Income

If the income stream is not replicable, the minimax pricing kernel in the utility maximisation problem with income does not necessarily coincide with the M^0 defined in (2.2). As mentioned in the introduction, computing the minimax pricing kernel is generally a difficult problem, even for simple specifications of the model, and the solutions that have been produced in the literature are mostly based on the numerical solution of the Hamilton-Jacobi-Bellman equation.

We take a different approach here. Instead of numerically solving a partial differential equation, we attempt to approximate the optimal strategy by using a pricing kernel that is different from the minimax one. An approximation to the actual minimax pricing kernel in the presence of income is the *minimal* pricing kernel, that puts a zero loading on non-traded risks, and is the M^0 defined in equation (2.2).¹¹ Were income risk spanned, the minimal pricing kernel would coincide with the minimax one. The human capital computed with the minimal pricing kernel is:

$$H_t = \sum_{t_i > t} \mathbb{E}_t \left[\frac{M_{t_i}^0}{M_t^0} e_{t_i} \right]$$

We need therefore to compute $\mathbb{E}_t [M_{t_i}^0 e_{t_i}]$ for $t_i \geq t$. For tractability purposes, we assume that the income payments are the values at the dates t_i of an income rate that follows a Geometric Brownian motion:

$$de_t = e_t [\mu^e dt + (\sigma^e)'\ dz_t]$$

The next proposition describes the human capital computed with the minimal pricing kernel with no income, and the hedging portfolio \mathbf{w}^H .

11 - See Föllmer and Schweizer (1989) and Schweizer (1990) for examples of use of the minimal pricing kernel in problems of option replication in incomplete markets.

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Proposition 4 *The human capital computed with the minimal pricing kernel can be represented as:*

$$H_t = e_t \sum_{t_i > t} h(t_i, t, r_t, \lambda_t^S)$$

where:

$$h(t_i, t, r_t, \lambda_t^S) = \exp [E_1(t_i - t) - D(t_i - t)r_t + E_3(t_i - t)\lambda_t^S]$$

$$E_3(u) = \frac{\mathbf{u}'_1 (\boldsymbol{\sigma}'\boldsymbol{\sigma})^{-1} \boldsymbol{\sigma}'\boldsymbol{\sigma}^e}{\sigma^\lambda - \kappa} [1 - e^{(\sigma^\lambda - \kappa)u}]$$

and E_1 solves an ordinary differential equations (ODE) given in appendix A.4 (see equation (A.7)).

The portfolio that maximises the squared correlation with the innovations in the human capital is:

$$\begin{aligned} \mathbf{w}_t^H &= \mathbf{w}^e + \frac{e_t}{H_t} \frac{1}{D(\tau)} \left[\sum_{t_i > t} D(t_i - t)h \right] \mathbf{u}_2 \\ &\quad - \frac{e_t}{H_t} \frac{\sigma^\lambda}{\sigma^S} \left[\sum_{t_i > t} E_3(t_i - t)h \right] \mathbf{u}_1 \end{aligned}$$

where $\mathbf{w}^e = (\boldsymbol{\sigma}'\boldsymbol{\sigma})^{-1} \boldsymbol{\sigma}'\boldsymbol{\sigma}^e$ is a portfolio achieving the maximum squared correlation with the income rate process.

Proof. See appendix A.4.

The assumption that the minimax pricing kernel with income is the same as the one with no income simplifies the computation of the hedging demand against human capital that appears in the optimal allocation (see the last term in (3.5)). It also simplifies the computation of the other hedging demands, because they depend on the minimax pricing kernel. In the end, our proxy for the utility-maximising strategy is:

$$\hat{\mathbf{w}}_t = \left(1 + \frac{H_t}{A_t}\right) \mathbf{w}_t^0 - \frac{H_t}{A_t} \mathbf{w}_t^H$$

where the human capital H and the portfolio \mathbf{w}_t^H are computed using the minimal pricing kernel (see proposition 4) and \mathbf{w}_t^0 is the weight vector that would be optimal in the absence of income (see proposition 1).

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Our objective in this section is two-fold. First, we show that completely ignoring the presence of non-financial income in the design of a portfolio strategy yields large utility costs. Second, we argue that while it is impossible to provide each investor with a dedicated and fully customized fund, a valuable approximation of the true optimal strategy can be constructed. The possibility to approximate the optimal strategy without incurring prohibitive utility costs is crucial for the design of investment solutions that are applicable to a large class of investors.

In this section, we consider the same three specifications of the income process as in section 4: a deterministic income, the income of a trader and the income of an entrepreneur.

5.1 Market Parameter Values

The base case set of parameter values is summarised in table 1. For the interest rate dynamics, we have chosen a volatility equal to 1.5%, which is close to the value obtained by calibrating the model from data on 3-month T-Bills over the post-1953 period. The long-term mean is taken equal to 3%, and the speed of mean reversion equal to 20%: those values have been chosen so as to make the probability of occurrence of negative rates acceptably low.¹² Finally, the market price of interest rate risk is taken equal to -20% , which is close to the calibrated value (around -25%). These values imply an expected excess return of 1.09% per year for a 10-year bond index over the cash. The short-term rate is initialized at 2% rather than $b = 3\%$ in order to reflect the low current level of interest rates.

Regarding the parameters of the Sharpe ratio process, there is no consensus in the literature, since the Sharpe ratio is not observable. Munk *et al.* (2004) estimate a value of 6.08% for the speed of mean-reversion of the Sharpe ratio, but the high variance of the estimator shows that this parameter is quite difficult to calibrate. We set it to 15%. The long-term mean of the Sharpe ratio is taken equal to 50%, which is only slightly higher than the 44.1% of Munk *et al.* (2004), but substantially higher than the historical Sharpe ratio of the S&P500. Finally, the volatility of the Sharpe ratio is set to 10%. Note that to achieve higher robustness in our simulations, we will maintain the simulated Sharpe ratio process between 25 % and 75 %.

The correlation parameters are set as follows. We take the correlation between unexpected stock returns and innovations to the Sharpe ratio to be -1 . This choice makes Sharpe ratio risk spanned by the stock itself, and is supported by different calibrations on US market data (Campbell and Viceira (1999) report a correlation of -96% between excess returns on the CRSP value-weighted index and the dividend yield; Xia (2001) finds a value of -93% ; and the authors' own calibration yields a value of -98.3%). The other correlations are set to zero.

5.2 Monetary Utility Losses

We need a quantitative measure in order to compare sub-optimal strategies to the optimal one. The criterion that is coherent with expected utility is the Monetary Utility Loss (MUL). For a given initial capital A_0 , the MUL x is defined as the capital such that the investor is indifferent between

¹² - The long-term distribution of the short-term rate is a Gaussian distribution with mean b and variance $(\sigma^*)^2 = (2a)$.

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the following two options: (a) follow the sub-optimal strategy starting with the initial capital A_0 ; (b) follow the optimal strategy starting with a lower initial wealth $A_0 - x$. Formally, this definition can be written as:

Expected Utility (Optimal, Initial Wealth = $A_0 - x$) = Expected Utility (Sub-Optimal, Initial Wealth = A_0):

The practical computation of the MUL is made easier by the following proposition, that gives the indirect utility for an investor who faces the opportunity set described in section 2 and the budget constraint (3.1).

Proposition 5 *The indirect utility for an investor who faces the opportunity set of section 2 and receives a replicable income stream reads:*

$$J(t, A_t + H_t, r_t, \lambda_t^S) = \frac{1}{1 - \gamma} \left(\frac{A_t + H_t}{B(t, T)} \right)^{1 - \gamma} g(t, \lambda_t^S)^\gamma,$$

where the function g is given in proposition 1.

Proof. See appendix A.5.

A similar expression can be written if the investor faces an extended opportunity set, as it is the case of the entrepreneur, subsection 4.2.2. In this case, we obtain:

$$J_1(t, A_t + H_t, r_t, \lambda_t^S) = \frac{1}{1 - \gamma} \left(\frac{A_t + H_t}{B(t, T)} \right)^{1 - \gamma} g_1(t, \lambda_t^S)^\gamma,$$

where g_1 is defined in equation (4.8).

Using the analytical expressions for the indirect utility function, we can write a more explicit representation of the MUL. Let $EU(A_0)$ denote the expected utility from terminal wealth when A_0 is invested at date 0 in the sub-optimal strategy. Using the definition of x , we have:

$$x = A_0 + H_0 - B(0, T) \left[\frac{(1 - \gamma)EU(A_0)}{g(0, \lambda_0^S)^\gamma} \right]^{\frac{1}{1 - \gamma}}$$

in the case where the opportunity set is that of section 2. When the opportunity set is that of subsection 4.2.2, the function g must be replaced by g_1 .

5.3 Cost of Ignoring Labour Income

If there was no income, the optimal portfolio rule would be given by the vector w_t^0 of equation (2.5), but in the presence of income, it is given by w_t^* defined in equation (3.6). In order to measure the utility cost of ignoring labour income, we implement the two strategies, subject to the budget constraint (3.1), and compute the Monetary Utility Losses (MULs) of both strategies. The difference between these two MULs is a measure of the cost to pay if one decides to ignore the income risk in the strategy. Our numerical experiments are based on 10,000 Monte Carlo simulations using the market parameters given in table 1. We assume in the following numerical illustrations that the contributions take place each quarter, and that the strategies are also rebalanced on a quarterly basis. By definition, the MUL depends on the risk aversion, which is a subjective, non-observable parameter. In order to avoid choosing an arbitrary value for this parameter, we will compute all MULs for three different attitudes towards risk. The three values for γ are calibrated in such a way that the average allocation to stocks, computed over all scenarios and all dates, be equal to a predefined target. We take this target to be 40% for the aggressive investor, 30% for the moderate, and 20% for the defensive.

The results for an investor who receives a deterministic income, growing at an annual rate of $\pi = 2\%$, are displayed in

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table 2. Panel (a) contains the MULs for a strategy that incorporates the human capital and the short position in w^H , as is recommended by the theory (see corollary 1). These MULs are positive because the portfolio is rebalanced on a discrete-time basis (they would be zero if rebalancing were done continuously). Panel (b) contains the MULs for a strategy that does not recognise the presence of non-financial income, and therefore chooses the weights w_t^0 of proposition 1. We observe that as risk aversion decreases, the cost of ignoring the income, as measured by the excess of the MULs in panel (b) over the MULs in panel (a), increases significantly. It may reach up to 10% of the initial total wealth for aggressive investors and a human capital larger than or equal to the financial wealth at the outset. This cost is smaller for defensive investors. This can be explained by the fact that for a large risk aversion, the portfolios w_t^* and w_t^0 have similar compositions. This property follows from the fact that w_t^0 is mostly invested in constant-maturity bonds, while w_t^H is fully invested in bonds. As a consequence, the two terms $H_t/A_t^* w_t^0$ and $H_t/A_t^* \beta_t^H w_t^H$ in (3.6) tend to cancel each other out. Nonetheless, a comparison of panels (a) and (b) shows that if income is taken into account, then the utility loss is divided by 2 at least for all levels of risk aversion. These results clearly show that taking into account the existence of non-financial income leads to substantial welfare gains. Unsurprisingly, these gains are increasing in the level of income: ignoring income is relatively innocuous if income is low. However, such a simplification is harmful for investors who have large income and are ready to take risk, since it leads to underinvesting in stocks: for an aggressive investor with

H_0 twice as large as A_0 , the MUL is more than 4.7 times higher if income is ignored.

Table 3 refers to the case of a trader whose income is equally spread between base deterministic income and stochastic bonus ($\omega = 50\%$). Again, we observe that the cost of ignoring non-financial income increases when the human capital represents a larger fraction of the investor's total wealth at the initial date.

Finally, tables 4 to 6 display the MULs for the entrepreneur using different parameter choices for his stochastic income. In table 4, we have considered the following set of parameter values: $\lambda^{S^0} = 30\%$, $\rho^{rS^0} = 0\%$, $q = 1.5\%$ (quarterly dividend rate), and $\sigma^{S^0} = 30\%$, $\rho^{SS^0} = 75\%$. As in the deterministic case, the results illustrate that when the income increases, the cost of ignoring income, as measured by the excess of the MULs in panel (b) over the MULs in panel (a), becomes more and more significant, reaching 40% for the defensive investor with a large income ($H_0 = 2A_0$). In contrast with the deterministic case, the welfare loss from ignoring income is larger for defensive investors than for those with low risk aversion. Although this property is difficult to justify formally, it is likely to be an effect of the stock-like nature of the entrepreneur's human capital. The portfolio w_t^0 is heavily shifted towards stocks if the risk aversion is low, so that the wealth generated by w_t^0 is more stock-like than bond-like, just like the human capital. For high levels of risk aversion, the portfolio w_t^0 is mostly invested in bonds, while w_t^H is made of stocks only, so the portfolios w_t^* and w_t^0 are very different from each other, and the cost of choosing w_t^0 rather than w_t^* becomes higher. In table 5, we have

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considered the case of a lower correlation between the stock S^0 and the index S , leading to significant lower MUL values for the optimal strategy w_t^* . This illustrates that the cost of ignoring income increases. This can be explained by the fact that the stock index S is less correlated to stock S^0 , leading to a very poor hedge of the income risk in strategy w_t^0 , which only contains the stock index S . The same conclusion holds for a higher volatility σ^{S^0} in table 6. Nonetheless, we observe in this case that the cost of ignoring income comes from an increase of the MUL of strategy w_t^0 . This is not surprising since the income is more volatile and should therefore be taken into consideration in the design of the strategy. Finally, we observe that MUL values are, on average, lower for low correlations between S and S^0 (table 5) than for high correlations (tables 4 and 6). This can be explained by the fact that for high correlations, the PSP exploits the possible arbitrages between assets S and S^0 , by shorting the one that has the lowest risk premium in order to buy the other one. Nonetheless, as we consider a quarterly rebalancing frequency (known to be a good trade-off between transaction costs, and utility losses), it is hard to capture these arbitrages in a discrete-time setting. This explains why the discrete-time optimal strategy goes further away from the continuous-time optimal strategy.

5.4 Partition of the Set of Investors

We now turn to the partition of investors. The idea is to develop a series of funds that are not perfectly customized, but are "close" to the ideal, utility-maximising, portfolio strategy. Closeness will be measured by a difference of MULs: if the discretized optimal and the approximated funds

have similar MULs with respect to the continuous-time optimal strategy, then the approximation is satisfactory. All the approximated funds will retain the general form of the optimal strategy (see equation (3.6)), but they are constructed in such a way that their implementation does not require the use of the various parameters that govern the income process of the investor. For this reason, we call these strategies "robustified".

5.4.1 Principles

There are three reasons why the optimal strategy (3.6) depends on the income profile of the investor. First, the human capital depends on the forthcoming contributions to be made by the investor in the future. Second, the financial wealth depends on the contributions that have been made in the past. Third, both the coefficient β^H and the portfolio w^H depend on the risk factors that impact the income. In fact, what eventually matters for the weights is not the current wealth or the human capital per se, but the ratio of these two quantities. In order to make the strategy robust with the level of income, we will replace the actual ratio H_t / A_t with an approximated value. The approximation must retain the property that the ratio shrinks to zero as horizon approaches. The simplest way of accounting for this essential feature is to consider a deterministic scheme of decrease for the ratio. We thus take the following approach: the approximated ratio H_t / A_t is defined as:

$$\hat{R}_t = \frac{H_0}{A_0} \left[1 - \frac{t}{T} \right]. \quad (5.1)$$

Hence \hat{R}_0 coincides with the actual initial ratio H_0 / A_0 , and \hat{R}_T is equal to zero, as it is the case for the final ratio H_T / A_T .

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The process \widehat{R} is parameterized by the initial ratio H_0 / A_0 , which is specific to each investor because H_0 depends on the level of the contributions that the investor is willing to make during the life of the fund. In order to have a unique vector \widehat{w}_t for a class of investors, we replace the actual ratio H_0 / A_0 in (5.1) by a proxy. This proxy is chosen as the closest element to the actual H_0 / A_0 in a pre-defined set of ratios. In proceeding this way, we obtain a partition of investors.

For the scalar β^H and the portfolio w^H , no approximation rule can be designed at this level of generality. For the time being, we denote the proxies as $\widehat{\beta}^H$ and \widehat{w}^H , so that the weights of the robustified strategy are given by:

$$\widehat{w}_t = (1 + \widehat{R}_t) w_t^0 - \widehat{R}_t \widehat{\beta}_t^H \widehat{w}_t^H.$$

5.4.2 Deterministic Income

The model with deterministic income was described in subsection 4.1. From the expression of β^H given by equation (4.1), we note that it is independent from e_0 . Hence this parameter is not specific to the investor, which means that the proxy $\widehat{\beta}^H$ can be taken equal to the actual β^H . Moreover, the endowment hedging portfolio w^H is fully invested in bonds. Nevertheless, heterogeneity arises from the level of income, which is captured by the human capital. Hence, when all investors have deterministic income, only the state space of the ratios H_0 / A_0 needs to be partitioned. Finally, the weights of the robustified strategy are:

$$\widehat{w}_t = (1 + \widehat{R}_t) w_t^0 - \widehat{R}_t \beta_t^H \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (5.2)$$

where w_t^0 is the portfolio of stock index and constant-maturity bonds written in (2.5). The vector of weights (5.2) is therefore parameterized by the ratio H_0 / A_0 . The sets of proxies for this ratio are defined as follows:

- Coarse Partition: [0 2]
- Medium Partition: [0 1 2]
- Fine Partition: [0 0.5 1 1.5 2]

Panel (a) of Table 7 shows Average Monetary Utility Losses (AMULs), computed for a "continuum" of 9 investors with ratio H_0 / A_0 ranging from 0 to 200% and using the suboptimal strategy (5.2) in which the actual ratio H_0 / A_0 is replaced by the closest value in each partition.¹³ The Average Monetary Utility Losses are computed from MUL as follows:

$$AMUL = \frac{1}{9} \sum_{k=1}^9 MUL((H_0/A_0)^{(k)}, \widehat{R}_0^{(k)})$$

where $MUL((H_0/A_0)^{(k)}, \widehat{R}_0^{(k)})$ denotes the Monetary Utility Losses computed for an investor having a true ratio equal to $(H_0 / A_0)^{(k)}$ but whose allocation has been computed with $\widehat{R}_0^{(k)}$ instead. The AMULs are also computed in the case where the discretized optimal strategy is used (row w_t^*) and in the case where non-financial income is ignored (row w_t^0). We observe that the AMULs for strategies involving partitions are always higher than the AMULs for the strategy w_t^* . This result comes as no surprise, given that the use of the actual ratio H_t / A_t , the actual coefficient β_t^H and the portfolio w_t^H that is most correlated with the human capital of the investor leads to a better approximation of the optimal continuous-time strategy. The strategies based on partitions use proxies for these quantities, which inevitably leads to a utility loss. The interesting point is that the

13 - The ratios H_0 / A_0 for the nine investors are equal to the five elements of the fine partition, plus the four midpoints.

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marginal welfare loss induced by the partition is not that large. Of course, it becomes larger and larger as the mesh increases. But for the fine and even for the medium partitions, the AMULs are smaller than for the strategy that just ignores the presence of income. Hence a classification of the set of investors in three categories after their level of income leads to higher welfare than the simplistic assumption that the income is zero.

5.4.3 Stochastic Income: Trader Case

The optimal allocation for the trader (see subsection 4.2.1) has the form (3.6). The expressions for the human capital H , the hedging portfolio \mathbf{w}^H , and the coefficient β^H are given in equations (4.3), (4.4) and (4.5). As noted above, \mathbf{w}^H contains in general both stocks and bonds, and their respective weights are determined by the income structure. The higher the deterministic part, the higher the weight allocated to bonds; the higher the stochastic part, the higher the weight allocated to stocks. But an implementable investment solution cannot be based on the income parameters of a particular investor. Therefore, we will make the strategy (3.6) robust to these parameters by computing the fraction of the initial human capital that comes from the deterministic part, and the fraction that is explained by the stochastic part. Formally, we have, from equation (4.3):

$$H_t^d = e_0^d \sum_{t_i > t} B(t, t_i) e^{\pi t_i}$$

$$H_t^s = e_0^s \frac{S_t}{S_0} N_t.$$

In what follows, we denote with ω the weight of the deterministic stream in the human capital at date 0:

$$\omega = \frac{H_0^d}{H_0}, \quad 1 - \omega = \frac{H_0^s}{H_0}.$$

Hence $\omega = 1$ means that the income is purely deterministic, and $\omega = 0$ that it is purely stochastic. We then approximate the hedging portfolio \mathbf{w}^H by:

$$\hat{\mathbf{w}}_t^H = \begin{pmatrix} \omega \\ 1 - \omega \end{pmatrix}$$

and the beta by:

$$\hat{\beta}_t^H = \omega \beta_t^{H,d} + (1 - \omega) \beta_t^{H,s},$$

where $\beta^{H,d}$ and $\beta^{H,s}$ are respectively the beta when income is purely deterministic and when it is purely stochastic. Equation (4.5) then shows that:

$$\hat{\beta}_t^H = \omega \frac{\sum_{t_i > t} B(t, t_i) e^{\pi t_i} \frac{D(t_i - t)}{D(\tau)}}{\sum_{t_i > t} B(t, t_i) e^{\pi t_i}} + 1 - \omega.$$

Panel (b) of Table 7 shows Average Monetary Utility Loss (AMUL) computed for investors with different risk-aversion, and for which the allocation $\hat{\mathbf{w}}_t$ has been computed with two risky assets: the bond, and the stock index. The sources of sub-optimality are the deterministic approximation of the actual ratio H_t/A_t^* by the deterministic quantity \hat{R}_t , the use of a proxy in substitution for the actual ratio H_0/A_0 in the computation of \hat{R}_t , and the use of the approximated $\hat{\mathbf{w}}_t^H$ and $\hat{\beta}_t^H$. The three partitions used in the numerical experiment are defined as follows:

- Coarse Partition: [0 2]
- Medium Partition: [0 1 2]
- Fine Partition: [0 0.5 1 1.5 2]

We consider a continuum of investor with ratio H_0/A_0 ranging from 0 to 2, and implement for each of them a sub-optimal strategy $\hat{\mathbf{w}}_t$ that uses an elements of the

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desired partition, chosen to be the closest to the actual ratio H_0 / A_0 of the investor.

In panel (b) of table 7, we report Average Monetary Utility Losses for the three partitions and various risk-aversion parameters. As in the deterministic case, the medium and the fine partitions perform better on average than a strategy that ignores the presence of income. The results even strengthen the case for partitions, since for the aggressive and the moderately risk-averse investor, a classification of the investors into two categories is already sufficient to improve on the strategy w^0 .

5.4.4 Stochastic Income: Entrepreneur Case

The optimal portfolio rule for the entrepreneur (see subsection 4.2.2) is written in equation (3.6). The human capital is given by (4.7), the portfolio w^H is fully invested in S^0 , and the beta of the human capital with respect to the hedging portfolio is 1. As in the case of deterministic income, we also approximate the ratio H_t / A_t^* by the quantity \hat{R}_t defined in (5.1). The weights of the robustified strategy that approximates (3.6) are thus given by:

$$\hat{w}_t = (1 + \hat{R}_t) w_t^0 - \hat{R}_t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (5.3)$$

where w_t^0 is the portfolio of stock index, constant-maturity bonds and stock S^0 written in equation (4.9). The sources of sub-optimality are the deterministic approximation of the actual ratio H_t / A_t^* by the deterministic quantity \hat{R}_t , and the use of a proxy in substitution for the actual ratio H_0 / A_0 in the computation of \hat{R}_t . The three partitions used in the numerical

experiment are defined as follows:

- Coarse Partition: [0 1]
- Medium Partition: [0 0.5 1]
- Fine Partition: [0 0.25 0.5 0.75 1.0]

We consider a continuum of investors with ratio H_0 / A_0 ranging from 0 to 1, and implement for each of them the sub-optimal strategy (5.3) that uses an element of the desired partition, chosen as the closest to the actual ratio H_0 / A_0 . The range [0; 1] is smaller in this study in order to avoid negative values for the wealth A due to leveraged short positions in the risky-assets S and S^0 through the ratio H_T / A_T .

Panel (c) of table 7 displays Average Monetary Utility Losses for the three partitions, and three risk-aversion parameters. Again, both the medium and the fine partitions appear to generate smaller average utility losses than a strategy ignoring the presence of income. The main difference from the previous two cases is that the marginal benefits from using the medium partition are rather small. Indeed, panel (d) of table 7 shows that, for lower values of the correlation ρ^{S,S^0} between the stock index and the entrepreneur's income, the medium partition can lead to higher MULs than the strategy ignoring the stochastic income. In that case, more substantial welfare gains are obtained by choosing a more accurate classification scheme.

5.5 Partition of the Set of Market Conditions

Since the equity Sharpe ratio λ^S is not directly observable, we rely, in practice, on estimation methods in order to estimate this process at all time. Since Merton (1980), it has been recognised that statistics is

5. Numerical Illustrations

of little help in estimating equity risk premium.¹⁴ Therefore, we follow Martellini and Milhau (2010) and propose to discretize the state space of this process. Since this discretization introduces an additional loss of optimality, we will quantify its impact by computing Monetary Utility Losses (MULs).

that the implementation of the optimal strategy with a parsimonious partition of the market conditions is a good proxy for the optimal strategy.

Let us now describe the three-point partition of the Sharpe ratio process that we have considered. We recall that the equity Sharpe ratio follows the mean-reverting process (2.1), and that in order to increase the robustness of our numerical experiments, we have truncated the process such that it lies in the interval [25%,75%]. Therefore, a natural three-point partition involves the three following standard values: $\lambda_{\text{inf}}^S = 25\%$, $\lambda_{\text{mid}}^S = 50\%$, and $\lambda_{\text{sup}}^S = 75\%$. This means that we can distinguish between low, moderate and high risk premium levels $\widehat{\lambda}_t^S$ at all time t :

$$\begin{aligned} \lambda_t^S \in \left(-\infty, \frac{\lambda_{\text{inf}}^S + \lambda_{\text{mid}}^S}{2} \right], & \Rightarrow \widehat{\lambda}_t^S = \lambda_{\text{inf}}^S, \\ \lambda_t^S \in \left(\frac{\lambda_{\text{inf}}^S + \lambda_{\text{mid}}^S}{2}, \frac{\lambda_{\text{mid}}^S + \lambda_{\text{sup}}^S}{2} \right], & \Rightarrow \widehat{\lambda}_t^S = \lambda_{\text{mid}}^S, \\ \lambda_t^S \in \left(\frac{\lambda_{\text{mid}}^S + \lambda_{\text{sup}}^S}{2}, \infty \right), & \Rightarrow \widehat{\lambda}_t^S = \lambda_{\text{sup}}^S. \end{aligned}$$

In order to take this new source of sub-optimality into account in our numerical experiments, we have reproduced the Average Monetary Utility Losses (AMULs) of table 7, where the equity Sharpe ratio λ_t^S is replaced by $\widehat{\lambda}_t^S$. These results are gathered in table 8, and show that the additional loss of optimality is very small compared to the sub-optimality introduced by partitioning the set of investors (see section 5.4). This result is in line with the numerical experiments led in Martellini and Milhau (2010), in which the authors found

14 - Even if one assumes a constant expected return the task is challenging. As pointed by Merton (1980), the precision of the naive estimator (the average of past log-returns) is completely insensitive to the sampling frequency. The only way to decrease it is to take longer samples, over which the assumption of a constant expected return becomes less and less realistic.

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Conclusion



Conclusion

While being the ideal solution for ultra high-net worth clients and large family offices, a fully customized approach cannot be implemented for all high-net worth individuals. In this context, it appears more than appropriate for the asset management industry to work towards the design of *life-cycle* funds that can allow for the incorporation of a class of private investors' horizon, objectives, attitude towards risk and non-financial income. Our paper focuses on the latter aspect. Taking labour income into account is not trivial, because its level income is specific to each investor, and the risk factors that impact its evolution depend on its origin (fixed salary and/or bonuses). We show that in spite of this high degree of heterogeneity, it is possible to group investors according to similar income profiles and implement a unique investment strategy for all members of a same class. This unique portfolio rule is obtained by approximating a utility maximising portfolio strategy with a strategy that assumes a deterministic decrease scheme for the ratio of human capital to financial wealth. This ratio, which is increasing in the horizon and in the level of income, determines the amounts respectively allocated to the strategy that would be optimal without income, and to the portfolio that hedges away the fluctuations in the human capital. It is important to point out that despite using a deterministic approximation for the ratio, our portfolio strategies are still state-dependent. Indeed, they depend on equity risk premium, which is stochastic. Our proxy for the utility-maximising strategy is parameterized by the initial ratio of human capital to financial wealth, the portfolio replicating the human capital and the beta of the human capital with respect to this hedging portfolio.

Since all these parameters are investor-specific, we replace them by a unique value within each class. Proceeding this way, we end up with a strategy that retains the general form of the utility-maximising one, but is robust to investor's characteristics.

Our results show that beyond this formal resemblance, the robustified strategy is also close to the optimal one in terms of Monetary Utility Loss. In other words, implementing the former strategy instead of the optimal one does not incur prohibitive utility costs. Moreover, we showed that a parsimonious partition of the equity risk premium state-space does not further degrade the value of Monetary Utility Losses, which illustrates that our strategies are robust to small measurement errors in the estimation of λ^S . Finally, and more importantly, the utility loss is smaller than if income were completely ignored. It is only if the approximation is very rough (e.g. because the number of classes investors are assigned to is too small), that the utility loss becomes substantial and can exceed the losses incurred by a strategy that assumes away the presence of non-financial income.

Overall, our research has significant potential implications for the design of stochastic, state-dependent, asset allocation policies, which stand in contrast to the deterministically time-dependent allocation strategies currently implemented in the context of target date funds. We strongly believe that there is certainly ample room for added value between one-(allocation)-size-fits-all (private investors with same age) solutions and do-it-yourself approaches to long-term investment decisions.

Appendix, Tables and Figures



Appendix

A.1 Proof of Proposition 1

The Lagrangian of the static problem reads:

$$\mathcal{L} = \mathbb{E} [U(A_T) - \eta (M_T^0 A_T - A_0)],$$

so the first-order optimality condition gives:

$$A_T^* = (U')^{-1} (\eta M_T^0) = (\eta M_T^0)^{-\frac{1}{\gamma}}.$$

The optimal wealth at an intermediate date t is thus:

$$A_t^* = \mathbb{E}_t \left[\frac{M_T^0}{M_t^0} A_T^* \right] = \eta^{-\frac{1}{\gamma}} (M_t^0)^{-\frac{1}{\gamma}} B(t, T)^{1-\frac{1}{\gamma}} G_t,$$

where we have introduced $G_t = \mathbb{E}_t \left[\left(\frac{M_T^0}{M_t^0 B(t, T)} \right)^{1-\frac{1}{\gamma}} \right]$. The budget constraint implies that:

$$\eta^{-\frac{1}{\gamma}} = \frac{A_0}{B(0, T)^{1-\frac{1}{\gamma}} G_0}.$$

Let us now apply Ito's lemma in order to find the diffusion term in the dynamics of A^* :

$$\frac{dA_t^*}{A_t^*} = (\dots) dt + \left[\frac{1}{\gamma} \boldsymbol{\lambda}_t + \left(1 - \frac{1}{\gamma} \right) \boldsymbol{\sigma}_t^B + \boldsymbol{\sigma}_t^G \right]' dz_t.$$

The volatility vector of A^* is also equal to $\boldsymbol{\sigma}_t \mathbf{w}_t^*$, as it is clear from equation (2.3). Hence:

$$\mathbf{w}_t^* = (\boldsymbol{\sigma}_t' \boldsymbol{\sigma}_t)^{-1} \boldsymbol{\sigma}_t' \left[\frac{1}{\gamma} \boldsymbol{\lambda}_t + \left(1 - \frac{1}{\gamma} \right) \boldsymbol{\sigma}^B(T-t) + \boldsymbol{\sigma}_t^G \right]$$

We must therefore compute $\boldsymbol{\sigma}_t^G$. Let us conjecture that G_t is a function of time and Sharpe ratio:

$$G_t = g(t, \lambda_t^S) \tag{A.1}$$

The process $\left((M_t^0 B(t, T))^{1-\frac{1}{\gamma}} g(t, \lambda_t^S) \right)_t$ is a martingale, so that its drift term is zero.

This drift term can be obtained by applying Ito's lemma, which leads to:

$$0 = \frac{g\lambda}{g} \left[\kappa (\bar{\lambda} - \lambda_t^S) + \frac{1-\gamma}{\gamma} (\boldsymbol{\sigma}^\lambda)' \boldsymbol{\lambda}_t \right] + \frac{g\lambda\lambda}{2g} (\boldsymbol{\sigma}^\lambda)^2 + \frac{1-\gamma}{2\gamma^2} \|\boldsymbol{\lambda}_t\|^2.$$

Using the decomposition $\boldsymbol{\lambda}_t = \boldsymbol{\Lambda}_1 + \lambda_t^S \boldsymbol{\Lambda}_2$, one can write a partial differential equation for g :

$$0 = \frac{g\lambda}{g} \left[\kappa (\bar{\lambda} - \lambda) + \frac{1-\gamma}{\gamma} (\boldsymbol{\sigma}^\lambda)' [\boldsymbol{\Lambda}_1 + \lambda \boldsymbol{\Lambda}_2] \right] + \frac{g\lambda\lambda}{2g} (\boldsymbol{\sigma}^\lambda)^2 + \frac{1-\gamma}{2\gamma^2} \|\boldsymbol{\Lambda}_1 + \lambda \boldsymbol{\Lambda}_2\|^2. \tag{A.2}$$

The terminal condition is $g(T, \lambda) = 1$. Conversely, if some function g solves equation (A.2) with the same terminal condition, then $\left((M_t^0)^{-\frac{1}{\gamma}} B(t, T)^{1-\frac{1}{\gamma}} g(t, \lambda_t^S) \right)_t$ is a martingale,

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which shows that:

$$g(t, \lambda_t^S) = \mathbb{E}_t \left[\left(\frac{M_T^0 B(T, T)}{M_t^0 B(t, T)} \right)^{1-\frac{1}{\gamma}} \right] = G_t.$$

This equality confirms that G_t is indeed of the form (A.1). We conjecture that the solution to (A.2) is of the form:

$$g(t, \lambda) = \exp \left[\frac{1-\gamma}{\gamma} \left[C_1(T-t) + C_2(T-t)\lambda + \frac{1}{2}C_3(T-t)\lambda^2 \right] \right],$$

with $C_i(0) = 0$ for $i = 1, 2, 3$. Plugging back the relevant derivatives of g into the right-hand side of (A.2) yields a quadratic function of λ with time-dependent coefficients. This function must be identically zero, so all the coefficients must be equal to zero. These conditions are equivalent to the system of coupled ordinary differential equations written in the proposition. It remains to compute the hedging portfolio \mathbf{w}_t^G . To do this, we apply Ito's lemma to $g(t, \lambda_t^S)$ so as to obtain the volatility vector of G :

$$\sigma_t^G = \frac{1-\gamma}{\gamma} \frac{g_\lambda}{g} \sigma^\lambda = \left(1 - \frac{1}{\gamma} \right) [C_2(T-t) + C_3(T-t)\lambda_t^S] \frac{\sigma^\lambda}{\sigma^S},$$

where $\mathbf{w}_t^{\text{HD}} = (0, 1)'$.

A.2 Proof of Proposition 2

Since the zero-coupon paying e_{t_i} at date t_i is replicable, its price follows:

$$dB_e(t, t_i) = B_e(t, t_i) [(r_t + \sigma_{z_c, e}(t, t_i)' \boldsymbol{\lambda}_t) dt + \sigma_{z_c, e}(t, t_i)' dz_t],$$

with $\sigma_{z_c, e}(t, t_i) = \sigma_t \mathbf{w}_t^i$ for some portfolio strategy \mathbf{w}^i .

On the other hand, the dynamics of H is given by:

$$\begin{aligned} dH_t &= \sum_{t_i > t+dt} B_e(t+dt, t_i) - \sum_{t_i > t} B_e(t, t_i) \\ &= \sum_{t_i > t} dB_e(t, t_i) - \sum_{i=1}^n e_{t_i} \mathbb{1}\{t \leq t_i < t+dt\} \\ &= r_t H_t dt + \sum_{t_i > t} B_e(t, t_i) \sigma_{z_c, e}(t, t_i)' (\boldsymbol{\lambda}_t dt + dz_t) - \sum_{i=1}^n e_{t_i} \mathbb{1}\{t \leq t_i < t+dt\}. \end{aligned}$$

For a vanishing dt , we thus obtain that:

$$dH_t = \left[r_t H_t + \left(\sum_{t_i > t} B_e(t, t_i) \sigma_{z_c, e}(t, t_i) \right)' \boldsymbol{\lambda}_t \right] dt + \left[\sum_{t_i > t} B_e(t, t_i) \sigma_{z_c, e}(t, t_i) \right]' dz_t - \sum_{i=1}^n e_{t_i} \mathbb{1}\{t = t_i\}.$$

Hence the volatility vector of H is:

$$\sigma_t^H = \frac{1}{H_t} \sum_{t_i > t} B_e(t, t_i) \sigma_{z_c, e}(t, t_i) = \beta_t^H \sigma_t \mathbf{w}_t^H,$$

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where:

$$\mathbf{w}_t^H = \frac{1}{H_t \beta_t^H} \sum_{t_i > t} B_e(t, t_i) \mathbf{w}_t^i$$

$$\beta_t^H = \frac{1}{H_t} \sum_{t_i > t} B_e(t, t_i) \mathbf{1}' \mathbf{w}_t^i.$$

Summing up the dynamics of A and H , we obtain that:

$$\frac{d(A_t + H_t)}{A_t + H_t} = \left[r_t + \left(\frac{A_t}{A_t + H_t} \mathbf{w}_t + \frac{H_t}{A_t + H_t} \beta_t^H \mathbf{w}_t^H \right)' \boldsymbol{\sigma}_t' \boldsymbol{\lambda}_t \right] dt + \left(\frac{A_t}{A_t + H_t} \mathbf{w}_t + \frac{H_t}{A_t + H_t} \beta_t^H \mathbf{w}_t^H \right)' \boldsymbol{\sigma}_t' dz_t,$$

which shows that $A+H$ is indeed the value of a self-financing trading strategy in stocks, bonds and cash, with a weight vector equal to

$$\mathbf{w}_t^{A+H} = \frac{A_t}{A_t + H_t} \mathbf{w}_t + \frac{H_t}{A_t + H_t} \beta_t^H \mathbf{w}_t^H.$$

A.3 Proof of Proposition 3

If M^* denotes the minimax pricing kernel, the first-order optimality condition in (3.3) gives:

$$A_T^* = \eta^{1-\frac{1}{\gamma}} (M_T^*)^{-\frac{1}{\gamma}},$$

where the Lagrange multiplier η can be derived from the budget constraint $\mathbb{E}[M_T^* A_T^*] = A_0 + H_0^*$:

$$\eta^{1-\frac{1}{\gamma}} = \frac{A_0 + H_0^*}{G_0^*}.$$

More generally, the optimal wealth at time $t \leq T$ is:

$$A_t^* = \mathbb{E}_t \left[\frac{M_T^*}{M_t^*} A_T^* \right] - H_t^* = \frac{A_0 + H_0^*}{G_0^*} G_t^* (M_t^*)^{-\frac{1}{\gamma}} B(t, T)^{1-\frac{1}{\gamma}} - H_t^*,$$

where:

$$G_t^* = \mathbb{E}_t \left[\left(\frac{M_T^*}{M_t^* B(t, T)} \right)^{1-\frac{1}{\gamma}} \right].$$

Applying Ito's lemma on both sides and matching the diffusion terms, we get that:

$$A_t^* \boldsymbol{\sigma}_t \mathbf{w}_t^* = \frac{A_0 + H_0^*}{G_0^*} G_t^* (M_t^*)^{-\frac{1}{\gamma}} \left[\frac{1}{\gamma} (\boldsymbol{\lambda}_t + \boldsymbol{\nu}_t^*) + \boldsymbol{\sigma}_t^{G^*} + \left(1 - \frac{1}{\gamma} \boldsymbol{\sigma}_t^B \right) \right] - H_t^* \boldsymbol{\sigma}_t^{H^*},$$

where $\boldsymbol{\lambda}_t + \boldsymbol{\nu}_t^*$ denotes the minimax market price of risk vector, and $\boldsymbol{\sigma}_t^{G^*}$ and $\boldsymbol{\sigma}_t^{H^*}$ are the volatility vectors of G^* and H^* . Multiplying both sides of this equation by $(\boldsymbol{\sigma}_t' \boldsymbol{\sigma}_t)^{-1} \boldsymbol{\sigma}_t'$, we obtain the optimal portfolio \mathbf{w}_t^* .

When income payments are replicable, the market is complete, so that the minimax pricing kernel agrees with the minimal one, $M_t^* = M_t^0$. Hence the optimal wealth with income can be written in terms of the optimal wealth without income and of the human capital:

$$A_t^* = \frac{A_0 + H_0}{A_0} A_t^0 - H_t,$$

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where H_t is the human capital computed with the pricing kernel M^0 . Hence:

$$A_t^* \sigma_t \mathbf{w}_t = \frac{A_0 + H_0}{A_0} A_t^0 \mathbf{w}_t^0 - H_t \beta_t^H \mathbf{w}_t^H = \frac{A_t^* + H_t}{A_t^*} \mathbf{w}_t^0 - H_t \beta_t^H \mathbf{w}_t^H.$$

A.4 Proof of Proposition 4

The human capital can be rewritten as:

$$H_t = e_t \sum_{t_i > t} \mathbb{E}_t \left[\frac{M_{t_i} e_{t_i}}{M_t e_t} \right]$$

Let us assume that the price of the i^{th} zero-coupon admits the following parametric representation:

$$\mathbb{E}_t \left[\frac{M_{t_i} e_{t_i}}{M_t e_t} \right] = h(t_i, t, \lambda_t^S, r_t), \quad t \leq t_i$$

Writing that $M_t e_t g$ must be a martingale, we obtain that:

$$\begin{aligned} \mu_e - r + \frac{h_t}{h} + \frac{h_\lambda}{h} \kappa (\bar{\lambda} - \lambda) + \frac{h_r}{h} a(b - r) + \frac{h_{\lambda\lambda}}{2h} (\sigma^\lambda)^2 + \frac{h_{rr}}{2h} (\sigma^r)^2 + \frac{h_{\lambda r}}{h} (\sigma^\lambda)' \sigma^r \\ - (\Lambda_1 + \lambda \Lambda_2 - \sigma^e)' \left[\frac{h_\lambda}{h} \sigma^\lambda + \frac{h_r}{h} \sigma^r \right] - (\Lambda_1 + \lambda \Lambda_2)' \sigma^e = 0 \end{aligned} \quad (\text{A.3})$$

with the initial condition $g(t_i, t_i, \lambda, r) = 1$. The linearity of the PDE and of the dynamics of the state variables suggests the following representation of g :

$$h(t_i, t, \lambda, r) = \exp \left[E_1(t_i - t) + E_2(t_i - t)r + E_3(t_i - t)\lambda + \frac{1}{2} E_4(t_i - t)\lambda^2 \right]$$

with $E_j(0) = 0$, for $j = 1, \dots, 4$. Inserting the partial derivatives of h into the left-hand side of (A.3) yields a polynomial function in (r, λ, λ^2) . All the coefficients of this function must be zero, which leads to a system of ODEs satisfied by the functions E_j . Cancelling the term in r yields:

$$E_2' + aE_2 = -1$$

which, together with the initial condition $E_2(0) = 0$, implies that $E_2(u) = \frac{e^{-au} - 1}{a} = -D(u)$. Canceling the term in λ^2 yields:

$$E_4'(t_i - t) = \frac{(\sigma^\lambda)^2}{2} E_4^2(t_i - t) - 2(\kappa - \Lambda_2) E_4(t_i - t) \quad (\text{A.4})$$

and cancelling the term in λ gives:

$$\begin{aligned} E_3'(t_i - t) = \left[\kappa \bar{\lambda} - (\Lambda_1 - \sigma^e)' \sigma^\lambda \right] E_4(t_i - t) - \left(\kappa + \Lambda_2' \sigma^\lambda \right) E_3(t_i - t) + \Lambda_2' \sigma^r D(t_i - t) \\ + (\sigma^\lambda)^2 E_3(t_i - t) E_4(t_i - t) - (\sigma^\lambda)' \sigma^r D(t_i - t) E_4(t_i - t) - \Lambda_2' \sigma^e \end{aligned} \quad (\text{A.5})$$

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Equation (A.4) with the initial condition $E_4(0) = 0$ implies that E_4 is identically zero, so that (A.5) is equivalent to:

$$E_3'(u) = -\left(\kappa + \Lambda_2' \sigma^\lambda\right) E_3(u) + \Lambda_2' \sigma^r D(u) - \Lambda_2' \sigma^e \quad (\text{A.6})$$

But we have $\Lambda_2' \sigma^r = 0$, since $\Lambda_2 = \sigma(\sigma' \sigma)^{-1}(1, 0)'$ and $\sigma^r = \sigma \mathbf{u}_1$. Moreover, we have $\sigma^\lambda = -\sigma^S$, so that $\Lambda_2' \sigma^\lambda = -\sigma^\lambda$. Hence we have that:

$$E_3(u) = \frac{\Lambda_2' \sigma^e}{\sigma^\lambda - \kappa} \left[1 - e^{(\sigma^\lambda - \kappa)u} \right]$$

Finally, cancelling the constant term yields:

$$\begin{aligned} E_1'(t_i - t) &= \frac{(\sigma^\lambda)^2}{2} [E_3^2(t_i - t) + E_4(t_i - t)] + \frac{(\sigma^r)^2}{2} D^2(t_i - t) - (\sigma^\lambda)' \sigma^r D(t_i - t) E_3(t_i - t) \\ &\quad + [\kappa \bar{\lambda} - (\Lambda_1 - \sigma^e)' \sigma^\lambda] E_3(t_i - t) - [ab - (\Lambda_1 - \sigma^e)' \sigma^r] D(t_i - t) + \mu^e - \Lambda_1' \sigma^e \end{aligned}$$

Since E_4 is zero, this is equivalent to:

$$\begin{aligned} E_1'(u) &= \frac{\sigma_\lambda^2}{2} E_3^2(u) + \frac{(\sigma^r)^2}{2} D^2(u) - (\sigma^\lambda)' \sigma^r D(u) E_3(u) + [\kappa \bar{\lambda} - (\Lambda_1 - \sigma^e)' \sigma^\lambda] E_3(u) \\ &\quad - [ab - (\Lambda_1 - \sigma^e)' \sigma^r] D(u) + \mu^e - \Lambda_1' \sigma^e \end{aligned} \quad (\text{A.7})$$

Applying Ito's lemma to:

$$H_t = e_t \sum_{t_i > t} h(t_i, t, r_t, \lambda_t^S)$$

we obtain that the volatility vector of H is:

$$\sigma_t^H = \sigma^e + \frac{e_t}{H_t} \sum_{t_i > t} h(t_i, t, r_t, \lambda_t^S) \left[-D(t_i - t) \sigma^r + E_3(t_i - t) \sigma^\lambda \right]$$

The hedging portfolio w^H is defined by $w_t^H = (\sigma' \sigma)^{-1} \sigma' \sigma_t^H$. Hence the expression for w_t^H given in the proposition.

A.5 Proof of Proposition 5

The indirect utility is defined as the expected utility from the optimal terminal wealth:

$$J_t = \frac{1}{1 - \gamma} \mathbb{E}_t \left[(A_T^*)^{1 - \gamma} \right]$$

where, from corollary 1:

$$A_T^* = \frac{A_0 + H_0}{G_0 B(0, T)^{1 - \frac{1}{\gamma}}} (M_T^0)^{-\frac{1}{\gamma}}$$

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Hence:

$$\begin{aligned}
 J_t &= \frac{1}{1-\gamma} \left(\frac{A_0 + H_0}{G_0 B(0, T)^{1-\frac{1}{\gamma}}} \right)^{1-\gamma} \mathbb{E}_t \left[(M_T^0)^{1-\frac{1}{\gamma}} \right] \\
 &= \frac{1}{1-\gamma} \left(\frac{A_0 + H_0}{G_0 B(0, T)^{1-\frac{1}{\gamma}}} (M_t^0)^{-\frac{1}{\gamma}} B(t, T)^{-\frac{1}{\gamma}} \right)^{1-\gamma} G_t \\
 &= \frac{1}{1-\gamma} \left(\frac{A_t^* + H_t}{B(t, T)} \right)^{1-\gamma} G_t^\gamma.
 \end{aligned}$$

In particular, the indirect utility is a function of time, total wealth, interest rate and Sharpe ratio.

Tables and Figures

B Tables and Figures

Table 1: Base case parameter values

Nominal Short-Term Rate	
b	0.03
a	0.2
σ^r	0.015
λ^r	-0.2
Stock Index	
σ^S	0.2
Sharpe Ratio	
κ	0.15
σ^λ	0.1
$\bar{\lambda}$	0.5
Deterministic Income	
π	0.02
Stochastic Income	
σ^{S^0}	[0.3; 0.4]
λ^{S^0}	0.3
q	0.015
Correlations	
$\rho^{S\lambda}$	-1
ρ^{Sr}	0
ρ^{SS^0}	[0.50; 0.75]
ρ^{rS^0}	0
Initial Values	
S_0	100
λ_0	$\bar{\lambda}$
r_0	0.02

This table displays the base case values used for the parameters introduced in section 4. Note that the initial values of the deterministic and stochastic income are calibrated by setting an initial value for H_0 .

Table 2: Monetary Utility Losses (MULs) of Strategies for an Investor with Deterministic Income.

(a) Strategies with Deterministic Income w_t^*

	Defensive	Moderate	Aggressive
$H_0/A_0 = 0$	0.87	1.26	1.54
$H_0/A_0 = 0.5$	1.68	1.96	2.19
$H_0/A_0 = 1$	2.26	2.40	2.58
$H_0/A_0 = 1.5$	2.68	2.70	2.83
$H_0/A_0 = 2$	3.00	2.92	3.01

(b) Strategies Ignoring Income w_t^0

	Defensive	Moderate	Aggressive
$H_0/A_0 = 0$	0.87	1.26	1.54
$H_0/A_0 = 0.5$	3.88	5.33	6.44
$H_0/A_0 = 1$	6.13	8.30	9.97
$H_0/A_0 = 1.5$	7.73	10.40	12.46
$H_0/A_0 = 2$	8.91	11.93	14.28

Panel (a) displays Monetary Utility Losses (MUL) of quarterly rebalanced strategies implemented with the discretized optimal allocation with deterministic income. Panel (b) displays Monetary Utility Losses (MUL) of quarterly rebalanced strategies implemented with the discretized optimal allocation without income. We compute the MUL for several values of risk-aversion and several proportions of income versus initial capital. All MULs are computed w.r.t. the optimal strategy in continuous time and are expressed in percentage of the total wealth.

Tables and Figures

Table 3: Monetary Utility Losses (MULs) of Strategies for a Trader with Stochastic Income.

(a) Strategies with Stochastic Income w_t^*			
	Defensive	Moderate	Aggressive
$H_0/A_0 = 0$	0.87	1.26	1.54
$H_0/A_0 = 0.5$	1.23	1.59	1.85
$H_0/A_0 = 1$	1.44	1.77	2.02
$H_0/A_0 = 1.5$	1.58	1.88	2.12
$H_0/A_0 = 2$	1.68	1.96	2.19

(b) Strategies Ignoring Income w_t^0			
	Defensive	Moderate	Aggressive
$H_0/A_0 = 0$	0.87	1.26	1.54
$H_0/A_0 = 0.5$	2.79	3.12	3.66
$H_0/A_0 = 1$	4.86	4.89	5.41
$H_0/A_0 = 1.5$	6.36	6.16	6.65
$H_0/A_0 = 2$	7.44	7.09	7.55

Panel (a) displays Monetary Utility Losses (MUL) of quarterly rebalanced strategies implemented with the discretized optimal allocation with stochastic income. Panel (b) displays Monetary Utility Losses (MUL) of quarterly rebalanced strategies implemented with the discretized optimal allocation without income. In both panels, we have assumed that the initial human capital H_0 is equally spread among the deterministic and stochastic components. We compute the MUL for several values of risk-aversion and several proportions of income versus initial capital. All MULs are computed w.r.t. the optimal strategy in continuous time and are expressed in percentage of the initial total wealth.

Table 4: Monetary Utility Losses (MULs) of Strategies for an Entrepreneur with Stochastic Income (Case $\sigma^{S_0} = 30\%$, $\rho^{S_0} = 75\%$).

(a) Strategies with Stochastic Income w_t^*			
	Defensive	Moderate	Aggressive
$H_0/A_0 = 0$	6.24	9.04	11.55
$H_0/A_0 = 0.5$	5.83	8.65	11.18
$H_0/A_0 = 1$	5.63	8.45	10.99
$H_0/A_0 = 1.5$	5.50	8.34	10.88
$H_0/A_0 = 2$	5.42	8.26	10.80

(b) Strategies Ignoring Income w_t^0			
	Defensive	Moderate	Aggressive
$H_0/A_0 = 0$	6.24	9.04	11.55
$H_0/A_0 = 0.5$	19.90	17.61	15.85
$H_0/A_0 = 1$	32.50	29.73	27.05
$H_0/A_0 = 1.5$	40.30	37.45	34.55
$H_0/A_0 = 2$	45.58	42.73	39.76

Panel (a) displays Monetary Utility Losses (MUL) of quarterly rebalanced strategies implemented with the discretized optimal allocation with stochastic income. Panel (b) displays Monetary Utility Losses (MUL) of quarterly rebalanced strategies implemented with the discretized optimal allocation without income. We compute the MUL for several values of risk-aversion and several proportions of income versus initial capital. All MULs are computed w.r.t. the optimal strategy in continuous time and are expressed in percentage of the initial total wealth.

Tables and Figures

Table 5: Monetary Utility Losses (MULs) of Strategies for an Entrepreneur with Stochastic Income (Case $\sigma^{S_0} = 30\%$, $\rho^{SS_0} = 50\%$)

(a) Strategies with Stochastic Income w_t^*			
	Defensive	Moderate	Aggressive
$H_0/A_0 = 0$	1.28	1.84	2.33
$H_0/A_0 = 0.5$	0.82	1.38	1.88
$H_0/A_0 = 1$	0.59	1.15	1.65
$H_0/A_0 = 1.5$	0.45	1.01	1.51
$H_0/A_0 = 2$	0.36	0.92	1.42

(b) Strategies Ignoring Income w_t^0			
	Defensive	Moderate	Aggressive
$H_0/A_0 = 0$	1.28	1.84	2.33
$H_0/A_0 = 0.5$	18.39	16.75	15.47
$H_0/A_0 = 1$	30.35	27.68	25.67
$H_0/A_0 = 1.5$	38.75	35.08	32.51
$H_0/A_0 = 2$	45.06	40.62	37.51

Panel (a) displays Monetary Utility Losses (MUL) of quarterly rebalanced strategies implemented with the discretized optimal allocation with stochastic income. Panel (b) displays Monetary Utility Losses (MUL) of quarterly rebalanced strategies implemented with the discretized optimal allocation without income. We compute the MUL for several values of risk-aversion and several proportions of income versus initial capital. All MULs are computed w.r.t. the optimal strategy in continuous time and are expressed in percentage of the initial total wealth.

Table 6: Monetary Utility Losses (MULs) of Strategies for an Entrepreneur with Stochastic Income (Case $\sigma^{S_0} = 40\%$; $\rho^{SS_0} = 75\%$).

(a) Strategies with Stochastic Income w_t^*			
	Defensive	Moderate	Aggressive
$H_0/A_0 = 0$	6.17	8.94	11.42
$H_0/A_0 = 0.5$	5.78	8.56	11.06
$H_0/A_0 = 1$	5.58	8.37	10.88
$H_0/A_0 = 1.5$	5.46	8.26	10.76
$H_0/A_0 = 2$	5.38	8.18	10.69

(b) Strategies Ignoring Income w_t^0			
	Defensive	Moderate	Aggressive
$H_0/A_0 = 0$	6.17	8.94	11.42
$H_0/A_0 = 0.5$	23.09	20.96	19.19
$H_0/A_0 = 1$	37.19	34.74	32.28
$H_0/A_0 = 1.5$	45.84	43.44	40.88
$H_0/A_0 = 2$	51.68	49.35	46.82

Panel (a) displays Monetary Utility Losses (MUL) of quarterly rebalanced strategies implemented with the discretized optimal allocation with stochastic income. Panel (b) displays Monetary Utility Losses (MUL) of quarterly rebalanced strategies implemented with the discretized optimal allocation without income. We compute the MUL for several values of risk-aversion and several proportions of income versus initial capital. All MULs are computed w.r.t. the optimal strategy in continuous time and are expressed in percentage of the initial total wealth.

Tables and Figures

Table 7: Average MULs of Strategies Implemented with Three Partitions for the Ratio H_0/A_0

(a) Investor with Deterministic Income

	Defensive	Moderate	Aggressive
Strategy w^*	2.13	2.28	2.46
Strategy w^0	5.60	7.58	9.11
Fine Partition	3.93	5.10	6.56
Medium Partition	4.22	5.56	7.18
Coarse Partition	5.34	7.30	9.50

(b) Trader with Deterministic and Stochastic Income

	Defensive	Moderate	Aggressive
Strategy w^*	1.38	1.71	1.96
Strategy w^0	4.49	4.54	5.02
Fine Partition	3.72	3.29	3.44
Medium Partition	3.93	3.45	3.62
Coarse Partition	4.70	4.04	4.28

(c) Entrepreneur with Stochastic Income (Case $\sigma^{S_0} = 30\%$, $\rho^{SS_0} = 75\%$)

	Defensive	Moderate	Aggressive
Strategy w^*	5.87	8.69	11.22
Strategy w^0	18.97	17.66	16.80
Fine Partition	16.28	14.66	15.58
Medium Partition	17.96	15.75	16.21
Coarse Partition	24.41	19.82	18.88

(d) Entrepreneur with Stochastic Income (Case $\sigma^{S_0} = 30\%$, $\rho^{SS_0} = 50\%$)

	Defensive	Moderate	Aggressive
Strategy w^*	0.87	1.43	1.92
Strategy w^0	17.28	15.68	14.60
Fine Partition	17.15	13.70	11.82
Medium Partition	19.04	15.48	13.39
Coarse Partition	25.60	20.98	18.46

Panels (a), (b), (c), and (d) display the Average Monetary Utility Losses (AMULs) of quarterly rebalanced strategies w^* and w^0 , and sub-optimal strategies \hat{w}^* implemented with an approximated ratio \hat{R}_t instead of H_t/A_t , and using three different partitions for the initial value \hat{R}_0 : the fine partition denotes the grid $[0; 0.5; 1; 1.5; 2]$, the medium partition the grid $[0; 1; 2]$, and the coarse partition the grid $[0; 2]$ for the investor with deterministic income and the trader. The fine partition denotes $[0; 0.25; 0.5; 0.75; 1]$, the medium one $[0; 0.5; 1]$, and the coarse one $[0; 1]$ for the entrepreneur with stochastic income. We compute MULs for nine initial ratios \hat{R}_0 ranging from 0 to 200%. All the MULs entering the computation of the average are computed with respect to the optimal continuous-time strategy, and are expressed as proportions of the initial total wealth $A_0 + H_0$.

Tables and Figures

Table 8: Average MULs of Strategies Implemented with Three Partitions for the Ratio H_0 / A_0 and a Three-Point Partition of the Equity Sharpe Ratio State-Space.

(a) Investor with Deterministic Income

	Defensive	Moderate	Aggressive
Strategy w^*	2.59	2.93	3.28
Strategy w^0	5.84	7.95	9.58
Fine Partition	4.57	6.07	7.79
Medium Partition	4.86	6.52	8.40
Coarse Partition	5.97	8.26	10.70

(b) Trader with Deterministic and Stochastic Income

	Defensive	Moderate	Aggressive
Strategy w^*	1.79	2.31	2.75
Strategy w^0	4.76	4.96	5.54
Fine Partition	4.20	4.05	4.48
Medium Partition	4.40	4.21	4.65
Coarse Partition	5.12	4.76	5.28

(c) Entrepreneur with Stochastic Income (Case $\sigma^{S_0} = 30\%$, $\rho^{SS_0} = 75\%$)

	Defensive	Moderate	Aggressive
Strategy w^*	7.62	11.25	14.56
Strategy w^0	19.35	18.39	17.93
Fine Partition	16.78	16.11	17.89
Medium Partition	18.35	17.01	18.33
Coarse Partition	24.46	20.60	20.49

(d) Entrepreneur with Stochastic Income (Case $\sigma^{S_0} = 30\%$, $\rho^{SS_0} = 50\%$)

	Defensive	Moderate	Aggressive
Strategy w^*	1.39	2.20	2.95
Strategy w^0	17.19	15.87	14.92
Fine Partition	17.26	14.13	12.49
Medium Partition	19.13	15.84	14.00
Coarse Partition	25.50	21.13	18.92

Panels (a), (b), (c), and (d) display the Average Monetary Utility Losses (AMULs) of quarterly rebalanced strategies w^* and w^0 , and sub-optimal strategies \hat{w}^* implemented with an approximated ratio \hat{R}_t instead of H_t / A_t , and using three different partitions for the initial value \hat{R}_0 : the fine partition denotes the grid $[0; 0.5; 1; 1.5; 2]$, the medium partition the grid $[0; 1; 2]$, and the coarse partition the grid $[0; 2]$ for the investor with deterministic income and the trader. The fine partition denotes $[0; 0.25; 0.5; 0.75; 1]$, the medium one $[0; 0.5; 1]$, and the coarse one $[0; 1]$ for the entrepreneur with stochastic income. We compute MULs for nine initial ratios \hat{R}_0 ranging from 0 to 200%. All the MULs entering the computation of the average are computed with respect to the optimal continuous-time strategy, and are expressed as proportions of the initial total wealth $A_0 + H_0$.

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About EDHEC-Risk Institute



About EDHEC-Risk Institute

Founded in 1906, EDHEC is one of the foremost French business schools. Accredited by the three main international academic organisations, EQUIS, AACSB, and Association of MBAs, EDHEC has for a number of years been pursuing a strategy for international excellence that led it to set up EDHEC-Risk in 2001.

With sixty-six professors, research engineers, and research associates, EDHEC-Risk has the largest asset management research team in Europe.

The Choice of Asset Allocation and Risk Management

EDHEC-Risk structures all of its research work around asset allocation and risk management. This issue corresponds to a genuine expectation from the market.

On the one hand, the prevailing stock market situation in recent years has shown the limitations of diversification alone as a risk management technique and the usefulness of approaches based on dynamic portfolio allocation.

On the other, the appearance of new asset classes (hedge funds, private equity, real assets), with risk profiles that are very different from those of the traditional investment universe, constitutes a new opportunity and challenge for the implementation of allocation in an asset management or asset-liability management context.

This strategic choice is applied to all of the Institute's research programmes, whether they involve proposing new methods of strategic allocation, which integrate the alternative class; taking extreme risks into account in portfolio construction; studying the usefulness of derivatives in implementing asset-liability management approaches; or orienting the concept of dynamic "core-satellite" investment management in the framework of absolute return or target-date funds.

An Applied Research Approach

In an attempt to ensure that the research it carries out is truly applicable, EDHEC has implemented a dual validation system for the work of EDHEC-Risk. All research work must be part of a research

programme, the relevance and goals of which have been validated from both an academic and a business viewpoint by the Institute's advisory board. This board is made up of internationally recognised researchers, the Institute's business partners, and representatives of major international institutional investors. Management of the research programmes respects a rigorous validation process, which guarantees the scientific quality and the operational usefulness of the programmes.

Six research programmes have been conducted by the centre to date:

- Asset allocation and alternative diversification
- Style and performance analysis
- Indices and benchmarking
- Operational risks and performance
- Asset allocation and derivative instruments
- ALM and asset management

These programmes receive the support of a large number of financial companies. The results of the research programmes are disseminated through the EDHEC-Risk locations in London, Nice, and Singapore.

In addition, EDHEC-Risk has developed a close partnership with a small number of sponsors within the framework of research chairs or major research projects:

- Regulation and Institutional Investment, *in partnership with AXA Investment Managers*
- Asset-Liability Management and Institutional Investment Management, *in partnership with BNP Paribas Investment Partners*
- Risk and Regulation in the European Fund Management Industry, *in partnership with CACEIS*

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- **The Benefits of Volatility Derivatives in Equity Portfolio Management,**
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The philosophy of the Institute is to validate its work by publication in international journals, as well as to make it available to the sector through its position papers, published studies, and conferences.

Each year, EDHEC-Risk organises a major international conference for institutional investors and investment management professionals with a view to presenting the results of its research: EDHEC-Risk Institutional Days.

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EDHEC-Risk Institute: Key Figures, 2009-2010

Nbr of permanent staff	66
Nbr of research associates	18
Nbr of affiliate professors	6
Overall budget	€9,600,000
External financing	€6,345,000
Nbr of conference delegates	2,300
Nbr of participants at EDHEC-Risk Indices & Benchmarks seminars	582
Nbr of participants at EDHEC-Risk Institute Risk Management seminars	512
Nbr of participants at EDHEC-Risk Institute Executive Education seminars	247

About EDHEC-Risk Institute

Research for Business

The Institute's activities have also given rise to executive education and research service offshoots. EDHEC-Risk's executive education programmes help investment professionals to upgrade their skills with advanced risk and asset management training across traditional and alternative classes.

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www.edhec-risk.com/Aleducation/PhD_Finance

The EDHEC-Risk Institute PhD in Finance is designed for professionals who aspire to higher intellectual levels and aim to redefine the investment banking and asset management industries. It is offered in two tracks: a residential track for high-potential graduate students, who hold part-time positions at EDHEC, and an executive track for practitioners who keep their full-time jobs. Drawing its faculty from the world's best universities and enjoying the support of the research centre with the greatest impact on the financial industry, the EDHEC-Risk Institute PhD in Finance creates an extraordinary platform for professional development and industry innovation.

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www.edhec-risk.com/indexes/efficient

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improved risk/reward efficiency compared to cap-weighted indices. The weighting of the portfolio of constituents achieves the highest possible return-to-risk efficiency by maximising the Sharpe ratio (the reward of an investment per unit of risk). These indices provide investors with an enhanced risk-adjusted strategy in comparison to cap-weighted indices, which have been the subject of numerous critiques, both theoretical and practical, over the last few years. The index series is based on all constituent securities in the FTSE All-World Index Series. Constituents are weighted in accordance with EDHEC-Risk's portfolio optimisation, reflecting their ability to maximise the reward-to-risk ratio for a broad market index. The index series is rebalanced quarterly at the same time as the review of the underlying FTSE All-World Index Series. The performances of the EDHEC-Risk Efficient Indices are published monthly on www.edhec-risk.com.

EDHEC-Risk Alternative Indexes

www.edhec-risk.com/indexes/pure_style

The different hedge fund indexes available on the market are computed from different data, according to diverse fund selection criteria and index construction methods; they unsurprisingly tell very different stories. Challenged by this heterogeneity, investors cannot rely on competing hedge fund indexes to obtain a "true and fair" view of performance and are at a loss when selecting benchmarks. To address this issue, EDHEC Risk was the first to launch composite hedge fund strategy indexes as early as 2003. The thirteen EDHEC-Risk Alternative Indexes are published monthly on www.edhec-risk.com and are freely available to managers and investors.

About La Française AM



About La Française AM

As a multi specialist asset manager, La Française AM has a management approach based on unwavering convictions. Both investor interest and satisfaction are made a priority.

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EDHEC-Risk Institute Publications (2008–2011)

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