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Introducing a Comprehensive Investment Framework for Goals-Based Wealth Management

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Any remaining errors are ours.

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About the Authors

Romain Deguest is a Senior Research Engineer at EDHEC-Risk Institute. His research on portfolio selection problems and continuous-time asset-pricing models has been published in leading academic journals and presented at numerous seminars and conferences in Europe and North America. He holds master's degrees in Engineering (ENSTA) and Financial Mathematics (Paris VI University), as well as a PhD in Operations Research from Columbia University and Ecole Polytechnique.

Lionel Martellini is Professor of Finance at EDHEC Business School and Scientific Director of EDHEC-Risk Institute. He has graduate degrees in economics, statistics, and mathematics, as well as a PhD in finance from the University of California at Berkeley. Lionel is a member of the editorial board of the *Journal of Portfolio Management* and the *Journal of Alternative Investments*. An expert in quantitative asset management and derivatives valuation, his work has been widely published in academic and practitioner journals and has co-authored textbooks on alternative investment strategies and fixed-income securities.

Vincent Milhau is Deputy Scientific Director of EDHEC-Risk Institute. He holds master's degrees in statistics (ENSAE) and financial mathematics (Université Paris VII), as well as a PhD in finance (Université de Nice-Sophia Antipolis). His research focus is on portfolio selection problems and continuous-time asset-pricing models.

Anil Suri is Managing Director and Head of Portfolio Construction & Investment Analytics at Merrill Lynch Wealth Management. He has been with Merrill Lynch since 2004, where he previously led investment strategy development & analytics in the Alternative Investments area and worked as a Senior Investment Strategist on the Merrill Lynch Research Investment Committee (RIC). Anil earned an M.B.A. with honors from the Wharton School of the University of Pennsylvania, an M.S.E. from Princeton University and a B. Tech. from the Indian Institute of Technology at Delhi.

Hungjen Wang is a Director in the Investment Analytics group at Merrill Lynch Wealth Management. His research focus is on statistical decision making, goals-based wealth management and portfolio management. His work has been published in different journals. He has graduate degrees in electrical engineering and Technology and policy, as well as a PhD in system engineering all from the Massachusetts Institute of Technology.

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From a Product-Centric to an Investor-Centric Approach to Wealth Management

Individual investors' investment problems can be broadly summarised as a combination of various **wealth and/or consumption goals**, subject to a set of **dollar budgets**, defined in terms of initial wealth and future income, as well as **risk budgets** such as maximum drawdown limits for example.

The starting point of an **investor-centric goals-based investment (GBI) approach** consists in recognising that the success or failure of these goals subject to dollar and risk budgets does not critically depend upon the standalone performance of a particular fund nor that of a given asset class. It depends instead upon **how well the investor's portfolio dynamically interacts with the risk factors impacting the present value of the investor's goals as well as the present value of non-tradable assets and future income streams**, if any. In this context, the key challenge for financial advisors is to implement dedicated **investment solutions** aiming to generate the **highest possible probability of achieving investors' goals**, and a reasonably **low expected shortfall** in case adverse market conditions make it unfeasible to achieve those goals. The need to design an asset allocation solution that is a function of the kinds of particular risks to which the investor is exposed, or needs to be exposed to fulfil goals, as opposed to purely focusing on the risks impacting the market as a whole, makes **the use of Modern Portfolio Theory or standard portfolio optimisation techniques mostly inadequate**.

While the efficient management of all risk buckets, versus market risk alone, is a central component of the **Wealth Allocation Framework** (WAF) introduced in Chhabra (2005),[1] the practical implications of this insight have not been fully exploited to date. Most financial advisors still maintain a **sole focus on market risks taken in isolation**, with **investors' preferences crudely summarised in terms of a simple risk-aversion parameter**.

The focus of this paper is to **develop a general operational framework** that can be used by financial advisors to allow individual investors to optimally allocate to categories of risks they face across all life stages and wealth segments so as to **achieve personally meaningful financial goals**. One key feature in developing the risk allocation framework for goals-based wealth management includes the introduction of **systematic rule-based multi-period portfolio construction methodologies**, which is a required element given that risks and goals typically persist across multiple time frames. Academic research has shown that an efficient **use of the three forms of risk management (diversification, hedging and insurance)** is required to develop an **investment solution framework** dedicated to allowing investors to maximise the probabilities of reaching their meaningful goals given their dollar and risk budgets. As a result, the main focus of the framework is on the **efficient management of rewarded risk exposures**.

The framework should not only be thought as a financial engineering device for generating meaningful investment solutions with respect to investors' needs. It should also, and perhaps even more importantly,

1 - Chhabra, A., 2005, Beyond Markowitz: A Comprehensive Wealth Allocation Framework for Individual Investors, *The Journal of Wealth Management*, 7, 5, 8-34.

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encompass a process dedicated to facilitating a meaningful dialogue with the investor. In this context, the reporting dimension of the framework should focus on **updated probabilities of achieving goals** and associated expected shortfalls, as opposed to solely focusing on standard risk and return indicators, which are mostly irrelevant in this context.

Broadly speaking, GBI strategies aim to secure investors' most important goals (labelled as "essential" – see definition below), while also delivering a reasonably high chance of success for achieving other goals, including ambitious ones which cannot be fully funded together with the most essential ones (and which are referred to as "aspirational"). Holding a leverage-constrained exposure to a well-diversified performance-seeking portfolio (PSP) often leads to modest probabilities of achieving such ambitious goals, and individual investors may increase their chances of meeting these goals by holding aspirational assets which generally contain illiquid concentrated risk exposures, for example under the form of equity ownership in a private business.

Introducing a Formal Goals-Based Wealth Management Framework

In a nutshell, the goals-based wealth management framework includes two distinct elements. On the one hand, it involves the **disaggregation of investor preferences into groups of goals that have similar key characteristics**, with **priority ranking** and **term structure of associated liabilities**, and on the other hand it involves the **mapping of these groups to optimised performance or hedging portfolios possessing corresponding risk and return characteristics**, as well as an efficient allocation to such performance and hedging portfolios. More precisely, the framework involves a number of objective and subjective inputs, as well as a number of building block and asset allocation outputs, all of which are articulated within a five-step process.

1. Objective Inputs - Realistic Description of Market Uncertainty

The implementation of the framework requires the use of updated market data (for example yield curve data), as well as the introduction of a **Monte-Carlo simulation model**, which is needed for the estimation of the probabilities of achieving investors' goals. Constructing a Monte-Carlo simulation model involves **realistic stochastic processes as well as a dynamically calibrated set of parameter values** that are chosen so as to minimise the model pricing errors, that is, the distance between market prices and model-implied prices for a set of reference instruments.[2]

2. Subjective Inputs - Detailed Description of Investor Situation

The implementation of the framework requires a number of inputs from the investor, including on the one hand the **investor's existing assets and liabilities**, as well as an **estimate of future consumption and revenues streams**, and on the other hand **a list of the meaningful goals** that should be integrated in the wealth management process. Investors' goals can be classified in 3 groups: (i) **essential goals** (EG), which are affordable and secured goals, (ii) **important goals** (IG), which are affordable but non-secured goals,[3] and (iii) **aspirational goals** (AG), which are

2 - Goals-based investing strategies are based on observable quantities, and their implementation is therefore not subject to model or parameter risk. The specifications of a model, and the associated parameter values, are only needed to compute probabilities to achieve various goals, which is an important ingredient in the dialogue with private investors.

3 - The reason an investor may decide not to secure otherwise affordable important goals is to generate more upside potential and, as a result, increase the probability of achieving aspirational goals.

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non-affordable (and non-secured) goals.[4] If a goal originally perceived as essential by an investor is not affordable (or generally, if securing it involves too high an opportunity cost), the investor is invited to secure a lower level of consumption or wealth. The **classification of goals is intrinsically subject to interactions between the investor and the financial advisor.** This interaction is needed to allow the investor to **measure affordable goals against versus non-affordable ones**, and **what the opportunity costs associated with securing affordable goals are.** This interaction also involves periodic (say, annual) revisions. Indeed, the funding status of the goal (i.e. its affordability or non-affordability) depends on the present value of the goal, thus on market conditions and notably on interest rates, and the investor's current wealth as well as future income. Moreover, the investor's priorities may vary over time.

3. Building Block Outputs - Goal-Hedging and Performance-Seeking Portfolios

The first output of the framework consists in designing a **goal-hedging portfolio** (GHP in short) for each essential goal. The general objective assigned to this portfolio is to **secure the goal with certainty**, and to do so **at the cheapest cost.** Its exact nature depends on the type of goal under consideration. In the simple case of a consumption-based goal for example, the GHP is a dedicated bond portfolio (a real bond portfolio if consumption cash-flows are inflation-linked) with coupon payments matching the consumption cash-flows or (as a first order approximation) with duration matching the duration of the goal cash-flows. For more complex goals, such as multiple-horizon wealth goals in the presence of income streams, the GHP can be a dedicated portfolio of exchange options, which can be replicated accurately or approximately through a suitable dynamic portfolio strategy.[5]

In addition to financing hedging portfolios associated with all essential goals, the investor also needs to generate performance so as to reach important and aspirational goals with a non-zero probability. In this context, investors should allocate some fraction of their assets to a **well-diversified PSP** in an attempt to **harvest risk premia on risky assets across financial markets**. An efficient GBI process will focus on utilising low cost access to rewarded risk factors (beta exposures) to achieve this objective. A consensus is emerging regarding the **inadequacy of market cap-weighted indices as investment benchmarks**, and a new paradigm known as **smart beta investing** is emerging, starting from the equity space, with a focus on the **efficient harvesting of multiple risk premia** in the equity universe. These smart beta benchmarks **blur the traditional clear-cut split between active versus passive portfolios** (see Amenc et al. (2014)[6]) and offer a set of **cost-efficient and attractive investment vehicles** in wealth management.

4. Asset Allocation Outputs - Dynamic Split Between Risky and Safe Building Blocks

One natural benchmark strategy consists in securing all essential goals, and investing the available liquid wealth in one or several performance portfolios allowing for the most efficient harvesting of market risk premia. This strategy, which is appealing

4 - A formal mathematical definition as well as operational verification criteria can be given for the concept of affordability. In the presence of income cash-flows, verification procedures are more complex because of the competition between current wealth versus future income in securing goals. The key insight is that *future income should be favoured over initial wealth* when securing a goal. Intuitively, this is because this principle allows investors to use the maximum possible amount of current wealth to generate performance through efficient and well-rewarded investments in rewarded risk factors.

5 - Note that investors often hold assets such as cash reserves or residence ownerships that serve the purpose of hedging implicit safety goals.

6 - Amenc, N., F. Goltz, A. Lodh, and L. Martellini, 2014, Towards Smart Equity Factor Indices: Harvesting Risk Premia Without Taking Unrewarded Risks, *Journal of Portfolio Management*, 40, 4, 106-122.

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since it secures essential goals with probability 1 and generates some upside potential required for the achievement of important and aspirational goals, is in fact a specific case of a wider class of (in general) **dynamic GBI strategies**. These strategies advocate that the allocation to PSPs versus GHPs should be taken as some **function of the current wealth level and the present value of the fraction of essential goals that is not financed by future cash inflows**, with the key property that this function (whose parameters in general depend on market conditions) should converge to zero when wealth converges to levels required for securing essential goals.[7] **The simplest** example of a dynamic strategy satisfying this property is one that takes the **investment in the PSP equal to a multiple of the margin for error** (corresponding to the function being taken as a linear function), with a unit multiplier value leading to the benchmark buy-and-hold strategy. In implementation, the multiplier is taken as a suitable function of market conditions, thus allowing the opportunity cost of downside protection to be decreased by activating the insurance component only when most needed.

This class of strategies, which are **reminiscent of constant proportion portfolio insurance strategies extended to an integrated goals-based wealth management process**, can be shown to be optimal in the sense that they are the solution to an expected utility maximisation problem with (implicit) goals for a leverage-constrained myopic investor. Such base case strategies have to be further **extended to encompass a number of practically important dimensions**, including **the presence of taxes** or **MULTIPLE essential goals**, including those that potentially apply to different wealth processes.

5. Reporting Outputs – Updated Probabilities of Reaching Goals

The framework is meant to be used both **for generating meaningful portfolio advice** as well as **for facilitating the dialogue with the investors**, and provides a set of subjective outputs (probability of reaching goals and associated expected shortfall) as well as objective outputs (allocation recommendations at all points in time).[8] For a given allocation strategy (e.g. a fixed-mix rebalancing towards the investor's current allocation or a more complex and more optimal GBI strategy), a number of indicators are reported, including the success probability for a strategy to achieve any particular goal as well as the associated expected shortfall.

Paradigm Changes in Wealth Management

The wealth management industry is about to experience **a profound paradigm change**. It is expected that the next generation of financial advisors will focus on building a modern approach to wealth management that will **depart from a product-centric search for performance** to focus on **satisfying the clients' needs through a dedicated investor-centric goals-based investment solution approach** (Ellis (2014)).[9]

Any investment process should start with a **thorough understanding of the investor problem**. Individual investors do not need investment *products* with alleged superior performance; **they need investment *solutions* that could help them meet**

7 - This condition can be regarded as a necessary and sufficient condition for ensuring the protection of essential goals with probability 1.
8 - In practice, it is likely more operationally effective to envision having two separate processes, supported by distinct IT tools, an asset-liability management tool meant to facilitate the relationship with the investor and the associated reporting requirements, and an asset management tool, dedicated to the execution of portfolio recommendations.
9 - Ellis, C., 2014, The Rise and Fall of Performance Investing, *Financial Analysts' Journal*, 70, 4, 14-23.

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their goals subject to prevailing dollar and risk budget constraints.

This paper introduces a **general operational framework**, which formalises the **goals-based risk allocation approach to wealth management proposed in Chhabra (2005)**, and which allows individual investors to optimally allocate to categories of risks they face across all life stages and wealth segments so as to **achieve personally meaningful financial goals**.

Through **a number of realistic case study examples**, we document the benefits of the approach, which respects the individual investor's essential goals with the highest degree of probability, while allowing for substantial upside potential that leads to a reasonably high probability of achieving ambitious aspirational goals.

In addition to developing and analysing optimal portfolio construction methodologies, this paper also introduces **robust heuristics**, which can be thought of as reasonable approximations for optimal strategies that can **accommodate a variety of implementation constraints**, including the presence of **taxes**, the presence of **short-sale constraints**, the presence of **parameter estimation risk**, as well as **limited customisation** constraints.

1. Introduction

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Individual investors' investment problems can be broadly summarised as a combination of various wealth and/or consumption goals, subject to a set of dollar budgets, defined in terms of initial wealth and future income, as well as risk budgets such as maximum drawdown limits.[10] It is important to note that the success or failure to satisfy these goals subject to dollar and risk budgets does not critically depend upon the standalone performance of a particular fund nor that of a given asset class. It depends instead upon how well the performance on the investor's portfolio dynamically interacts with the risk factors impacting the present value of the investor's goals. In this context, it becomes clear that the key challenge for financial advisors is to implement dedicated investment solutions aiming to generate the highest possible probability of achieving investors' goals, and a reasonably low expected shortfall in case adverse market conditions make it unfeasible to achieve those goals. In other words, what will prove to be the decisive factor is the ability to design an asset allocation solution that is a function of the kinds of particular risks to which the investor is exposed, or needs to be exposed to fulfil goals, as opposed to purely focusing on the risks impacting the market as a whole. These simple insights have far reaching implications, including on regulatory requirements such as the "prudent man rule", which is the requirement that investment managers or any fiduciary agents must only invest funds entrusted to them with prudence. This prudent approach might actually become counter-productive if it is cast in an isolated context, that is, with a sole focus on market risks without a proper integration of the investor's goals. For example, a seemingly safe short-term investment strategy such as the roll-over of money market debt can prove to be very risky from the perspective of meeting long-term consumption needs.

From the academic standpoint, the recognition of the critical importance of investors' personal risks, in addition to market risks, was first emphasised in the seminal work by Merton (1971, 1973), and subsequent papers on dynamic asset allocation decisions in the presence of income and consumption risks (see the literature review in Section 3.2). In this paradigm, which extends the standard efficient frontier paradigm introduced by Markowitz (1952) to an intertemporal setting, the optimal allocation strategy is shown to involve, in addition to the standard mean-variance efficient PSP, dedicated hedging portfolios that are designed to hedge investors against unfavourable changes in the risk factors impacting their income and consumption streams. While this framework serves as the foundation for most of modern dynamic asset pricing theory, the key implications of this paradigm for the wealth management industry have not been recognised until recently, with financial advisors mostly maintaining a focus on the management of market risks in isolation. The need for financial advisors to focus on the proper management of personal and aspirational risks in addition to the management of market risk was clarified in the Goals-Based Wealth Allocation Framework proposed in Chhabra (2005). In a nutshell, the goals-based wealth management framework involves the hierarchical disaggregation of investor preferences into groups of goals that have similar key characteristics, and

10 - See Section 2 for a detailed classification and analysis of investors' goals.
11 - See Section 3.2 for a detailed review of the related literature.

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the implementation on an investment strategy designed to enhance the probability for the investor of achieving these goals. While Chhabra (2005) provides a thorough analysis at the conceptual level of the challenges required to implement a goals-based investment process, it does not present, however, a fully operational framework for implementing this process. Subsequent work has provided additional useful insights into goals-based investing. In particular Das et al. (2010) propose to integrate Markowitz's modern portfolio theory (Markowitz (1952)) and Shefrin and Statman's behavioral portfolio theory (Shefrin and Statman (2000)) into a new mental account framework where risk is defined as the probability of failing to reach the threshold level in each mental account. In a related effort, an early paper by Nevins (2004) and subsequent work by Brunel (2011) both suggest that goals-based investing is naturally suited to address investment needs of individuals who frame their preferences and risk-aversion in terms of probabilities of achieving or failing a number of meaningful goals (see also Brunel (2002) for an early discussion of goals-based investing). All of these approaches recommend that sub-portfolios are separately managed to optimize the probability of meeting each one of the client's goals (see also Wang et al. (2011) for analytical solutions in a one-period model and Gaussian returns), which is seemingly at odds with some of the key prescriptions of intertemporal portfolio selection models (Merton (1969, 1971)) which recommend that an investor's portfolio includes a single performance-seeking component in addition to a variety of hedging components. Overall, a series of questions remain outstanding regarding what extension of existing financial engineering techniques, if any, is required to formally establish the goals-based allocation framework. The main objective of this paper is to introduce a general operational framework[12] that can be firmly grounded in dynamic asset pricing theory and used by a financial advisor to allow individual investors to optimally allocate to categories of risks they face across all life stages and wealth segments so as to achieve personally meaningful goals. One key feature in developing the risk allocation framework for goals-based investing (GBI) strategies includes the introduction of multi-period portfolio construction methodologies, which is a required element given that risks and goals typically persist across multiple time frames.

Broadly speaking, the framework will encompass two main kinds of ingredients, namely the identification of the suitable building blocks on the one hand, and the identification of suitable decisions in terms of allocation to these building blocks on the other hand. We note at this stage that the framework only involves rule-based strategies, based either on observable quantities or on estimated parameter values. This notably excludes the use of stochastic optimisation techniques, which are typically well-suited for the analysis of optimal decisions under uncertainty when the number of possible future states is limited, but suffer from a black-box aspect, and cannot easily accommodate a realistically rich description of uncertainty. It is also important to emphasise that the framework should not only be thought of as a financial engineering device for generating meaningful investment solutions with respect to investors' needs. It

12 - Our paper is also related to the literature on commitment-directed investing (see Mindlin (2013) and CDI Advisors Research (2014)).

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should also encompass a process dedicated to facilitating a meaningful dialogue with the investor. In this context, the reporting dimension of the framework should focus on updated probabilities of achieving goals, as opposed to solely focusing on standard risk and return indicators, which are not necessarily relevant in this context.

From the academic standpoint, one contribution of our paper is to extend the seminal analysis of Dybvig and Huang (1988) to the presence of non-portfolio income. It is well-known that the existence of a state-price deflator, or equivalently of an equivalent martingale measure, is not sufficient to avoid arbitrage opportunities. The classical counterexample is the "doubling strategy" of Harrisson and Kreps (1979), which generates a riskless gain from nothing in a finite time frame, but does so at the cost of potentially unlimited losses. One possible remedy to the presence of arbitrage opportunities in dynamically complete continuous-time markets is the introduction of an integrability condition on the strategy weights (see Harrison and Pliska (1981)), but this mathematical restriction lacks economic interpretation. As an alternative, Dybvig and Huang (1988) propose imposing a nonnegativity constraint (or in fact any negative lower bound) on wealth, which admits a natural interpretation as a credit constraint. They show that this condition rules out arbitrage opportunities and that it is equivalent to an integrability condition. While it allows for consumption, their framework, however, does not include non-portfolio income. It turns out that adding this ingredient is not a straightforward extension of their work because it modifies the definition of feasible consumption-investment plans. In particular, nonnegative wealth has to be required at all dates, not only at the final date as in their paper, in order to prevent investors from borrowing against their future income. Our paper examines in detail the question of financing a given consumption plan in the presence of income, when financial wealth only, as opposed to financial wealth plus human capital, is restricted to be nonnegative. An important result we obtain is a general "affordability criterion", which characterises feasible plans and extends the well-known criterion stating that a consumption plan is financed if the investor's initial wealth exceeds the present value of the consumption payments. We also show that the introduction of forward contracts leads, in general, to a further decrease in the minimum wealth required to secure a given consumption plan.

The rest of the paper is organised as follows. In Section 2, we introduce a formal risk allocation framework for GBI strategies and we present a set of theoretical optimality results regarding affordable goals and the relationship with the efficient design of building blocks involved in such strategies. Section 3 examines the question of how to efficiently allocate across the risk buckets defined in the preceding section and discusses the implementation challenges in a real-world setting. Section 4 presents an application of the framework to three different case studies, representing three possible types of investors clustered in different groups, defined in terms of affluence and life stage. Section 5 presents a number of conclusions, and technical details and proofs of the main results are relegated to a dedicated appendix.

2. Efficient Allocation Within Risk Buckets

2. Efficient Allocation Within Risk Buckets

In this section, we introduce a formal continuous-time framework for the goals-based investing problem. We then present the various types of goals that will be considered in the paper, we define the notion of the affordability of a goal, which corresponds to attainability, and we present formal necessary and sufficient conditions of affordability, which can be regarded as verification criteria to characterise affordable goals in practice. We next describe efficient composition of the building blocks, also known as risk buckets, which will be involved in GBI strategies, distinguishing between a safety risk bucket, a performance-seeking risk bucket and a risk bucket containing all non-tradable and illiquid assets, if any, that an investor may hold for wealth mobility purposes.

2.1 Notation and Assumptions

Following the seminal work of Merton (1971), we cast the intertemporal portfolio choice problem within a continuous-time framework. Uncertainty in the economy is represented by a filtered probability space $(X, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is a sigma-algebra on X, and $\mathbb{P}$ is a probability measure that represents investor's beliefs. Unless otherwise stated, the investment horizon is a finite quantity T (which for example can be time of retirement or time of death depending on the context), and the initial date is 0, so the time span is $[0,T]$. The probability space supports a d-dimensional Brownian motion $\underline{z}$, d being the number of independent sources of risk in the economy, and is equipped with the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ generated by this Brownian motion: $\mathcal{F}_t$ is a sigma-algebra on X, which represents the information accumulated by the investor up to date t. All stochastic processes introduced below are implicitly assumed to be progressively measurable with respect to this filtration. This technical assumption is not very restrictive, as it is satisfied as soon as all stochastic processes are adapted (i.e. the value of the stochastic process on a date t is a $\mathcal{F}_t$-measurable random variable) and right-continuous (see Karatzas and Shreve (2000)). These two conditions will be verified for all the processes that we consider in this paper.

2.1.1 Asset Prices

The nominal short-term interest rate on date t, for lending or borrowing on the infinitesimal horizon dt, is denoted by r_t. The investment universe is assumed to contain a locally risk-free asset, whose price S_{0t} is the continuously compounded short-term rate:

$$S_{0t} = \exp\left[\int_0^t r_s ds\right].$$

There are also n "locally risky" assets, whose prices $S_1,\ldots,S_n$ follow diffusion processes as in Merton (1971):

$$\frac{dS_{it}}{S_{it}} = [r_t + \sigma_{it}\lambda_{it}]dt + \underline{\sigma}_{it}' d\underline{z}_t, \quad (1)$$

where λ_{it} is the Sharpe ratio, $\underline{\sigma}_{it}$ is the $d\times 1$ volatility vector, and σ_{it} is the scalar volatility.[13] At this stage, no particular restriction is imposed on the risk and return parameters, beyond the progressive measurability. In particular, Sharpe ratios and volatilities can be stochastic. More restrictive assumptions will have to be made in order to derive utility-maximising portfolio strategies, but these will be specified later (see Section 3.2.2).

We let $\underline{\sigma}_t$, $\underline{\Sigma}_t$ and $\underline{\mu}_t$ denote the $d\times n$ volatility matrix, the $n\times n$ covariance matrix

13 - We use underbars to denote vectors and matrices. For instance, the scalar σ_{it} is the norm of the vector $\underline{\sigma}_{it}$.

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and the $n \times 1$ vector of expected excess returns of the risky assets:

$$\underline{\sigma}_t = (\underline{\sigma}_{1t} \quad \cdots \quad \underline{\sigma}_{nt}),$$

$$\underline{\Sigma}_t = \underline{\sigma}_t' \underline{\sigma}_t,$$

$$\underline{\mu}_t = (\sigma_{1t}\lambda_{1t} \quad \cdots \quad \sigma_{nt}\lambda_{nt})'.$$

All these moments are instantaneous; because of non-trivial term structure effects, the moments evaluated over a non-infinitesimal horizon may be different from the above ones.

A critically useful notion is that of "state-price deflator", which will be used to find the present value of claims with uncertain payoffs at later dates. Formally, a positive process $(M_t)_{0 \leq t \leq T}$ is said to be a state-price deflator if for $i = 1, \ldots, n$, the deflated price $M_t S_{it}$ follows a martingale. The existence of one such price deflator is ensured by the condition of absence of arbitrage opportunities (see for example Duffie (2001)). As shown by He and Pearson (1991), there exist infinitely many state-price deflators if markets are dynamically incomplete, that is, if the number of sources of risk (d) exceeds the number of risky assets (n). Among these, one is of particular interest, namely the state price deflator associated with the "spanned price of risk vector":

$$\underline{\lambda}_t = \underline{\sigma}_t \underline{\Sigma}_t^{-1} \underline{\mu}_t,$$

$$M_t = \exp\left[-\int_0^t \left(r_s + \frac{\|\underline{\lambda}_s\|^2}{2}\right) ds - \int_0^t \underline{\lambda}_s' d\underline{z}_s\right],$$

where $\|\underline{\lambda}_s\|$ denotes Euclidian norm of the vector $\underline{\lambda}_s$. The vector $\underline{\lambda}_t$ has size $d \times 1$, and is said to be spanned because it falls in the span of the volatility matrix.

To each pricing kernel is associated an "equivalent martingale measure" (in short, EMM), which is defined by its Radon-Nikodym density with respect to $\mathbb{P}$:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left[\int_0^T r_s \, ds\right] \times M_T.$$

Since there is a one-to-one correspondence between state-price deflators and EMMs, the EMM is unique, if, and only if, the state-price deflator is unique, that is, if the market is dynamically complete. The price at date t of a payoff P_T paid on date T can be obtained by two equivalent formulas:

$$\mathbb{E}_t\left[\frac{M_T}{M_t} P_T\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s \, ds\right) P_T\right].$$

In this equality (which follows from Bayes' formula), $\mathbb{E}_t$ denotes the expectation operator conditional on the information available to date t under the probability $\mathbb{P}$, and $\mathbb{E}_t^{\mathbb{Q}}$ is the expectation under $\mathbb{Q}$.

2.1.2 Portfolio Strategies

The investor is endowed with a positive initial capital A_0, which he/she invests in the n risky assets and the risk-free one. The portfolio is said to be *self-financing* or *self-financed* if no cash is infused in or withdrawn from the portfolio. We let $\underline{N}_t = (N_{1t}, \ldots, N_{nt})'$ be the $n \times 1$ vector of numbers of shares of the risky assets held on date t. The number of shares of the risk-free asset is thus:

$$N_{0t} = \frac{A_t - \sum_{i=1}^n N_{it} S_{it}}{S_{0t}}.$$

The budget constraint describes the evolution of liquid wealth. It reads:

$$dA_t = \sum_{i=0}^n N_{it} dS_{it} = \sum_{i=1}^n N_{it} dS_{it} + \left[A_t - \sum_{i=1}^n N_{it} S_{it}\right] r_t dt. \quad (2)$$

Equivalently, the strategy can be described in terms of the sums invested in the risky

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assets. Let $\underline{q}_t = (q_{1t}, \ldots, q_{nt})'$ be the $n \times 1$ vector of these sums. They are related to the numbers of shares through:

$$q_{it} = N_{it} S_{it}, \text{ for } i = 1, \ldots, n.$$

Finally, when wealth is positive, one can compute the weights allocated to the various risky assets:

$$w_{it} = \frac{q_{it}}{A_t}, \text{ for } i = 1, \ldots, n.$$

The sum of the weights is not necessarily equal to 1, and the balance $[1 - \sum_{i=1}^{n} w_{it}]$ is invested in cash. Thus, the third representation of a self-financing portfolio strategy is the weight vector process $\underline{w}$, where the value of the process on date t is the vector of weights $\underline{w}_t = (w_{1t}, \ldots, w_{nt})'$.

For the purpose of computing optimal portfolio strategies according to various criteria, it is useful to express the expected return and the volatility of the portfolio in terms of the portfolio composition. To do this, it suffices to substitute the dynamics of the risky assets (Equation (1) back into (2)). This gives the following two equivalent expressions:

$$dA_t = \left[r_t A_t + \underline{q}_t' \underline{\mu}_t\right] dt + \underline{q}_t' \underline{\sigma}_t' d\underline{z}_t,$$

$$dA_t = A_t \left[r_t + \underline{w}_t' \underline{\mu}_t\right] dt + A_t \underline{w}_t' \underline{\sigma}_t' d\underline{z}_t. \quad (3)$$

Thus, the portfolio expected excess return and volatility have the familiar expressions respectively given by a linear function and by the square root of a quadratic function of the weights:

$$\mu_{Pt} = \underline{w}_t' \underline{\mu}_t, \quad \sigma_{Pt} = \sqrt{\underline{w}_t' \Sigma_t \underline{w}_t}.$$

Ruling out negative wealth may be desirable for several reasons. First, negative wealth for an individual investor means bankruptcy, which is a situation that investors generally seek to avoid at all costs. The second reason is of a theoretical nature. It is known that in continuous-time models, the existence of a state-price deflator (or equivalently, of an equivalent martingale measure) does not alone imply the absence of arbitrage opportunities.[14] One recovers the implication by imposing a nonnegative wealth constraint (see Dybvig and Huang (1988)). It is important to have this condition in mind when computing a utility-maximising payoff (see Section 3.2).

14 - This is because "doubling strategies" are possible (see Harrisson and Kreps (1979) and Duffie (2001)).

In the presence of consumption, the budget Equation (1) has to be modified. In the literature on dynamic portfolio choice, consumption is traditionally represented as a continuous-time process, but for the sake of realism, we model it here as a discrete process: the consumption dates are denoted as $T_1 < \ldots < T_p$, and are comprised between 0 and T. The consumption of date t_j is denoted as c_{T_j}, and is assumed to be a $\mathcal{F}_{T_j}$-measurable random variable, nonnegative with probability 1.

Because of the presence of a consumption stream, the investor's portfolio is no longer self-financed, and because of the discrete nature of consumption, it is not even continuous, as was the case before with (1). We define A_t as the wealth of date t, after the consumption expenditure has been made: thus, A is a right-continuous process with left limits. The left limit at date t, denoted by A_{t-}, is the value of wealth just before the consumption payment. Of course, jumps in wealth occur only on the consumption dates; wealth is continuous between two consecutive dates. To write the new budget equation, let us introduce a family of n Heaviside functions $\left(J_{T_1}, \ldots, J_{T_p}\right)$, each of them being an indicator function

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(which is also right-continuous):

$$J_{T_j,t} = \begin{cases} 0 & \text{if } t < T_j \\ 1 & \text{if } t \geq T_j \end{cases}.$$

With these notations, the budget constraint becomes:

$$dA_t = \sum_{i=1}^{n} N_{it} dS_{it} + \left[A_t - \sum_{i=1}^{n} N_{it} S_{it} \right] r_t dt - \sum_{j=1}^{p} c_{T_j} dJ_{T_j,t}. \quad (4)$$

Heuristically, the differential element $dJ_{T_j,t}$ can be thought of as a quantity equal to 1 when t equals T_j, and 0 otherwise.[15]

15 - The rigorous mathematical definition of $dJ_{T_j,t}$ is as a Dirac measure: this measure assigns a mass of 0 to time intervals that do not contain T_j, and a mass of 1 to any interval that contains this date.

A last extension consists in the introduction of income in the budget constraint. This is in fact formally equivalent to having negative consumption. In order to alleviate the notational burden, we assume that the income dates coincide with the consumption dates (this entails no loss of generality, as it suffices to introduce additional payments equal to zero to satisfy this condition).

As usual, we take the income payments $y_{T_1},..., y_{T_p}$ to be measurable with respect to the filtration $(\mathcal{F}_t)_t$. The most general form of the budget constraint reads:

$$dA_t = \sum_{i=1}^{n} N_{it} dS_{it} + \left[A_t - \sum_{i=1}^{n} N_{it} S_{it} \right] r_t dt + \sum_{j=1}^{p} y_{T_j} dJ_{jt} - \sum_{j=1}^{p} c_{T_j} dJ_{jt}. \quad (5)$$

Again, A_{t-} denotes the value of wealth just before the income or consumption payment, and A_t is the value of wealth just after the cash flow.

2.2 Defining Affordable Goals

A key concept in goals-based wealth management is that of *affordable* goals, which are intuitively defined as goals that an investor can secure with full certainty with some suitably designed hedging strategy given available wealth and future income. The notion of affordability will subsequently be used in the classification of goals within three distinct groups, namely *essential* goals, *important* goals and *aspirational* goals (see Section 2.4). In this section, we first provide a formal definition for this concept, which is relatively straightforward in the absence of income, but becomes more involved in the presence of income cash-flows because of the possible competition between current wealth and future savings in the process of securing the target consumption or wealth goals. The general definition of affordability which we derive, however, is not very operational. For this reason, we also introduce a number of sufficient and necessary conditions for affordability which can be used as verification criteria in practice.

2.2.1 Affordability of Wealth-Based Goals

A first distinction exists between wealth-based goals and consumption-based goals. A wealth-based goal is expressed as a minimum level of wealth that the investor wants to reach at a certain horizon, and a consumption-based goal is a target (possibly inflation-linked) payment that the investor wants to make. The main difference between the two types is that a consumption-based goal impacts the investor's wealth, while a wealth-based one has no effect on the budget constraint.

2.2.1.1 Wealth-Based Goal with Single Horizon

The first type of goal that we consider is a simple wealth-based goal expressed as

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follows: assume that the investor has a horizon T, and would like his wealth at date T to be above a level G_T. Having such a wealth-based goal is analogous to imposing a floor on terminal wealth. The payoff G_T can be a constant (e.g. a target nominal amount), or it can be uncertain at date 0, in case the investor requires a wealth level contingent on the economic conditions prevailing on date T. For instance, the goal may be expressed as a target real, as opposed to nominal, wealth level, the objective being to protect the purchasing power against erosion due to inflation. In the language of the literature on option pricing and hedging, the wealth-based goal is represented by a $\mathcal{F}_T$-measurable and nonnegative payoff G_T. In what follows, we adopt the following formal definition for the affordability of a wealth-based goal at horizon T.

Definition 1 (Affordability of a Wealth-Based Goal with a Single Horizon).
A wealth-based goal characterised by the horizon T and the nonnegative minimum wealth level G_T is said to be affordable if there exists a portfolio strategy $\underline{w}$ such that, starting from the investor's initial capital, the wealth satisfies the budget constraint (3) and the inequality $A_T \geq G_T$ with probability 1. Such a strategy is said to secure the goal.

The question of the affordability of the goal encompasses in fact two independent questions:
Q1 Is the payoff G_T replicable with the available risky assets?
Q2 If the answer to Q1 is positive, is the investor's initial wealth sufficient to attain the payoff?

The answer to Q1 is related to the risk factors that impact goal value. In somewhat informal terms, if the goal value depends on "unspanned" risk factors, that is, risk factors that cannot be hedged with available securities, the payoff will not be replicable. This is the case, for instance, if the goal value is indexed on inflation, and the investment universe only contains stocks and nominal bonds. Because no portfolio of these assets can exactly replicate realised inflation, the payoff is not replicable. In such a situation, the markets are dynamically incomplete in the sense of Duffie (2001). Since the goal value cannot be replicated exactly, one can at best form a strategy that approximates the value in the sense of a hedging criterion. Various criteria have been proposed in the literature, some of which will be reviewed below (see Section 3.2.1). In order to avoid the technicalities associated with non-replicable payoffs, we assume in what follows that all wealth-based goals are *replicable*, in the sense that there exists an initial capital $\tilde{G}_0$, and a dynamic portfolio strategy $\underline{w}_G$, referred to as the goal-hedging portfolio (GHP), such that G_T is the value at date T of the solution to the stochastic differential equation:

$$\frac{d\tilde{G}_t}{\tilde{G}_t} = \left[r_t + \underline{w}_{Gt}'\underline{\mu}_t\right]dt + \underline{w}_{Gt}'\underline{\sigma}_t' d\underline{z}_t,$$

with the initial condition $\tilde{G}_0$. In other words, G_T is the value at date T of some portfolio invested in the available risky assets and cash. In the previous equation, $\tilde{G}_t$ represents the present value of the goal, which is also the value of the GHP. An assumption that guarantees the existence of the GHP is the dynamic completeness of the markets: Duffie (2001) shows that if the number of risky assets (n) equals the number of independent risk factors (d), then any payoff is attainable.

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This assumption is only sufficient, however, but not necessary to guarantee the affordability of $\boldsymbol{G_T}$.

The question Q2 is irrelevant if the goal is not replicable. A non-replicable goal is by definition not affordable. On the other hand, if $\boldsymbol{G_T}$ is replicable, then there exists a unique no-arbitrage price for this payoff, and this price is $\tilde{\boldsymbol{G}}_{\boldsymbol{0}}$. Then, by absence of arbitrage opportunities, having $A_T \geq G_T$ with probability one implies that the initial values satisfy $A_0 \geq \tilde{\boldsymbol{G}}_{\boldsymbol{0}}$. Conversely, if $A_0 \geq \tilde{\boldsymbol{G}}_{\boldsymbol{0}}$, then investing A_0 in the GHP delivers the wealth $\frac{A_0}{\tilde{G}_0}\boldsymbol{G_T}$ at date T, a payoff which is clearly greater than or equal to $\boldsymbol{G_T}$ in any state of the world. Thus, a simple test for Q2 is the comparison between the investor's initial capital and the present value of the goal. If $A_0 \geq \tilde{\boldsymbol{G}}_{\boldsymbol{0}}$, then the goal is affordable; if $A_0 < \tilde{\boldsymbol{G}}_{\boldsymbol{0}}$, it is not. The following proposition summarises this simple but important result.

Proposition 1 (Affordability Criterion of a Replicable Wealth-Based Goal with a Single Horizon).
Consider a wealth-based goal represented by the nonnegative minimum wealth level $\boldsymbol{G_T}$, and assume that this goal is replicable. It is affordable if, and only if, the initial wealth and the present value of the payoff $\boldsymbol{G_T}$ satisfy $A_0 \geq \tilde{\boldsymbol{G}}_{\boldsymbol{0}}$. If this condition is satisfied, then the goal is secured by the strategy that invests A_0 in the goal-hedging portfolio.

Two comments are in order. First, it is not because a goal is affordable that it will be secured with any strategy. The success in achieving a goal depends of course on the initial wealth, but also on how wealth is invested: the notion of affordability corresponds to the existence of at least one strategy that attains the goal with probability 1, but not all strategies will have this property. Second, the non-affordability does not imply that the goal will never be attained: it can still be attained in some states of the world, but the probability of having $A_T \geq G_T$ will be less than 1.

2.2.1.2 Wealth-Based Goal with Multiple Horizons

It can happen that a wealth-based goal is not expressed as a minimum level of wealth at a given horizon T, but as a series of minimum wealth levels at different dates. Formally, a wealth-based goal is defined by a set of horizons $\mathcal{T}$ and a family of random variables $(\boldsymbol{G_t})_{\boldsymbol{t}\in\mathcal{T}}$, such that G_t is $\mathcal{F}_t$-measurable and nonnegative. Note that this definition extends that of a wealth-based goal with a single horizon: for such a goal, $\mathcal{T}$ has a single element. On the other hand, in Case Study 1 below (see Section 4.1), the investor is concerned with achieving a fixed minimum level of real wealth at the end of each year over the next 35 years; in this particular example, $\mathcal{T}$ has 35 elements.

The definition of affordability in the case of multiple horizons is a natural extension of Definition 1.

Definition 2 (Affordability of a Wealth-Based Goal with Multiple Horizons).
A wealth-based goal represented by the set of horizons $\mathcal{T}$ and the nonnegative minimum wealth levels $(\boldsymbol{G_t})_{\boldsymbol{t}\in\mathcal{T}}$ is said to be affordable if there exists a portfolio strategy $\underline{\boldsymbol{w}}$ such that, starting from the investor's initial capital, the wealth satisfies the budget constraint (2) and the inequality

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$A_t \geq G_t$ with probability 1 for all $t \in \mathcal{T}$. Such a strategy is said to secure the goal.

It is clear that this definition is more general than Definition 1: a wealth-based goal with a single horizon is merely a special case of a goal with multiple horizons, with a set of horizons reduced to a single element.

The first case study (see Section 4.1) provides two examples of such goals. First, the investor wants to protect a minimum level of real wealth at the end of each year: G_t is then equal to the initial wealth multiplied by realised inflation. Second, the investor wants, at the end of each year, to have at least 85% of the maximum wealth attained at previous year ends. This second goal is referred to as the drawdown (DD) goal. It is expressed as:

$$A_{T_j} \geq m_{DD} \times \overline{A}_{T_{j-1}}, \text{ for all } j = 2,\ldots, p,$$

where $\overline{A}_t$ is the maximum to date of wealth, given by:

$$\overline{A}_{T_{j-1}} = \max\left[A_{T_1}, \ldots, A_{T_{j-1}}\right],$$

and m_{DD} is the maximum drawdown tolerated, for which typical values are 10% and 15%.[16]

16 - Note that in the academic literature on drawdown constraints, the running maximum is taken over the continuum of dates that precede the current date.

We are now interested in finding an affordability criterion similar to that of Proposition 1, which is based on the comparison between initial wealth and the present value of the goal. This extension is not straightforward, as there is no clear notion of "present value" for a sequence of payoffs occurring at different dates, even if each payoff G_t itself has a well-defined present value. Moreover, the case of the max drawdown goal (in short, max DD goal) must be treated separately. Indeed, the definition of the minimum wealth levels and investment decisions are intertwined in this case, so that the present value of the payoffs G_t depends on portfolio weights. We thus make a distinction between two types of wealth-based goals:

- A goal where the minimum wealth levels G_t are exogenous, that is, they are not affected by investment decisions;
- A goal such as the max DD goal, where the minimum wealth levels G_t are endogenous, that is, they depend upon investment decisions.

The easier case is that of the DD goal. It turns out that this goal is affordable whichever the initial wealth, provided the short-term rate remains nonnegative. Indeed, under this condition, the locally risk-free asset has only nonnegative returns, so that its drawdown along each path is zero.

Proposition 2 (Affordability of DD Goal). *For any choice of m_{DD} between 0 and 1 and any initial wealth, the DD goal associated with the threshold m_{DD} is affordable if the short-term interest rate remains nonnegative. This goal is secured by investing the entire wealth in cash.*

We now turn to an exogenous goal. To get a feel for the expression of the minimum capital to invest in order to secure a goal with multiple horizons, consider an example with two dates $T_1 < T_2$, and take a strategy that secures the goal. The value of the strategy satisfies (with probability 1):

$$A_{T_2} \geq G_{T_2},$$

hence, by absence of arbitrage opportunities, we have at date T_1:

$$A_{T_1} \geq \mathbb{E}_{T_1}\left[\frac{M_{T_2}}{M_{T_1}} G_{T_2}\right].$$

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As a result, the wealth of date T_1 satisfies:

$$A_{T_1} \geq \max\left[G_{T_1}, \mathbb{E}_{T_1}\left[\frac{M_{T_2}}{M_{T_1}} G_{T_2}\right]\right].$$

Using again the absence of arbitrage opportunities, we obtain that the initial wealth must be such that:

$$A_0 \geq \mathbb{E}\left[M_{T_1} \max\left[G_{T_1}, \mathbb{E}_{T_1}\left[\frac{M_{T_2}}{M_{T_1}} G_{T_2}\right]\right]\right].$$

The right-hand side of this equation is the price of an exchange option that pays the maximum of the first goal value and the present value of the second goal value at date T_1. Similarly, for a larger number of goals, the minimum capital to invest involves the prices of embedded exchange options. The following proposition gives a necessary and sufficient affordability criterion for a wealth-based goal with multiple horizons. In order to ensure that all exchange option payoffs are attainable, we maintain the assumption of complete markets, an assumption that implies the uniqueness of the state-price deflator.[17]

Proposition 3 (Affordability Criterion of a Wealth-Based Goal with Multiple Horizons in Complete Markets).
Assume that markets are complete and consider a wealth-based goal represented by the exogenous and nonnegative minimum wealth levels $(G_{T_1},\dots, G_{T_p})$ *on dates* $T_1 <\dots< T_p$. *Let* $T_0 = 0$ *and* $G_{T_0} = 0$, *and define the recursive sequence of payoffs* $\left(K_{T_j}\right)_{j=0,\dots,p}$ *and their prices:*

$$K_{T_p} = G_{T_p};$$

$$\widetilde{K}_{T_{p-j},T_{p-j+1}} = \mathbb{E}_{T_{p-j}}\left[\frac{M_{T_{p-j+1}}}{M_{T_{p-j}}} K_{T_{p-j+1}}\right], \text{ for } j = 1, \dots, p;$$

$$K_{T_{p-j}} = \max\left[G_{T_{p-j}}, \widetilde{K}_{T_{p-j},T_{p-j+1}}\right], \text{ for } j = 1, \dots, p.$$

Then, the wealth-based goal is affordable if, and only if, the initial wealth satisfies $A_0 \geq K_0$. *If this condition is satisfied, then the goal is secured by investing* A_0 *in a roll-over of exchange options expiring on dates* $T_1,\dots,T_p$ *with payoffs* $K_{T_1},\dots, K_{T_p}$.

Proof. See Appendix 6.1.1.
A first important result contained in Proposition 3 is the formula for the minimum capital to invest in order to secure the goal, which is equal to K_0. More generally, Appendix 6.1.1 shows that if a strategy secures the goal, then the wealth at each date T_j satisfies $A_{T_j} \geq K_{T_j}$ for $j = 1,\dots, p$. Taking the present value of both sides of the inequality, we obtain:

$$A_t \geq \widetilde{K}_{T_j,T_{j+1}},$$

for $T_j < t \leq T_{j+1}$ and $j = 0,\dots, p-1$,

hence the minimum capital to invest at date t is $\widetilde{K}_{k+1,T_k}$. This property justifies the following definition of the "present value" for a goal with multiple horizons.

Definition 3 (Present Value of Wealth-Based Goal with Multiple Horizons).
Consider an exogenous wealth-based goal with multiple horizons and the option payoffs defined in Proposition 3. The present value of the goal is defined as:

$$\widetilde{G}_t = \mathbb{E}_t\left[\frac{M_{T_{j+1}}}{M_t} K_{T_{j+1}}\right],$$

for $T_j < t \leq T_{j+1}$ and $j = 0,\dots, p-1$,

$$\widetilde{G}_0 = K_0.$$

The present value of the goal at date t is thus defined as the minimum capital to invest on this date to secure the goal.[18] It should be noted that unlike for a goal with a single horizon, this present value is not

17 - We recall that the maximum of two replicable payoffs is not necessarily a replicable payoff (see the case of a European call option written on an underlying asset with stochastic volatility, when volatility risk is not spanned by the underlying itself).
18 - We assume here the absence of non-portfolio income, an ingredient which will be discussed in Section 2.2.4 below.

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always continuous. It is left-continuous, but jumps may occur on the goal horizons. In what follows, we denote the right limit of the present value at date T_j with $\tilde{G}_{T_j+}$. Moreover, the product $M_t\tilde{G}_t$ does not follow a martingale.

A second contribution from Proposition 3 is to show that one can construct a GHP by rolling over the exchange options paying $\tilde{G}_{T_j}$ at date T_j, starting from the initial capital $\tilde{G}_0$. The value of the GHP is thus:

$$GHP_t = \left[\prod_{k=1}^{j} \frac{\tilde{G}_{T_k}}{\tilde{G}_{T_k+}}\right] \times \tilde{G}_t$$

$$\text{for } T_j < t \leq T_{j+1} \text{ and } j = 0, \ldots, p-1, \quad (6)$$

$$GHP_0 = \tilde{G}_0.$$

(It is continuous on the dates $T_1,\ldots,T_p$.) Because $\tilde{G}_{T_k} = K_{T_k}$ is the maximum of G_{T_k} and $\tilde{G}_{T_k+}$ is equal to $\tilde{K}_{k+1,T_k}$, the product within the brackets is greater than or equal to 1. Hence:

$$GHP_{T_j} \geq \tilde{G}_{T_j},$$

that is, the value of the GHP is greater than or equal to the present value of the goal. In particular, at horizon T_j, the value of the GHP satisfies:

$$GHP_{T_j} \geq \tilde{G}_{T_j} = \max\left[G_{T_j}, \tilde{G}_{T_j+}\right].$$

This property means that the GHP actually protects a wealth level that is higher than the minimum G_{T_j}. This apparent "overprotection" is due to the presence of other goals after date T_j. This property marks a difference with respect to the case of the single horizon. With a single horizon, the value of the GHP coincides with the present value of the goal, while the two concepts are distinct in the presence of multiple horizons.

In order to compute the GHP in practice, it is of interest to have analytical expressions for the option prices. For two horizons, this is a tractable task, at least under convenient parameter assumptions. Indeed, K_{T_1} is the maximum of G_{T_1} and the price on date T_1 of the payoff G_{T_2}, so that K_0 is the price of an option which pays the maximum of two payoffs. If G_{T_1} and G_{T_2} are log-normally distributed, and the interest rate and risk premia are constant, then the exchange option can be priced analytically. When the number of horizon dates exceeds two, the pricing exercise becomes more complex. To see this, it suffices to note that with three horizons, K_{T_1} involves the price of an option whose payoff is the maximum of G_{T_2} and the price of another option. Overall, for more than two horizons, it proves impossible in general to derive an analytical expression for the minimum initial capital requirement, K_0, and for the associated replicating portfolio strategy. Furthermore, a Monte-Carlo pricing is difficult to implement because the conditional expectation within the definition of $K_{T_{p-j}}$ must itself be estimated by Monte-Carlo.

These technical issues can be avoided in some cases, where the maximum operators can be eliminated from the expressions of the payoffs K_{T_j}. This is possible for instance when the goal values satisfy a certain "monotony condition", as discussed in the following corollary.

Corollary 1 (Affordability Criterion of a Wealth-Based Goal with Multiple Horizons in Complete Markets).
Let the assumptions of Proposition 3 be satisfied, and assume in addition that the goal values satisfy the following monotony condition:

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$$\mathbb{E}_{T_j}\left[\frac{M_{T_{j+1}}}{M_{T_j}}G_{T_{j+1}}\right] \leq G_{T_j},$$

for $j = 1,\ldots, p-1$. (7)

Then, the payoffs K_{T_j} defined in Proposition 3 are given by:

$K_{T_j} = G_{T_j}$, for $j = 1,\ldots, p$,

and the present value of the goal is:

$$\tilde{G}_t = \mathbb{E}_t\left[\frac{M_{T_{j+1}}}{M_t}G_{T_{j+1}}\right],$$

for $T_j < t \leq T_{j+1}$ and $j = 0,\ldots, p-1$,

$$\tilde{G}_0 = \mathbb{E}[M_{T_1}G_{T_1}].$$

Proof. See Appendix 6.1.2.
Under the assumptions of the corollary, the computation of the exchange option payoffs is greatly facilitated. The minimum capital to invest in order to secure the goal is simply the present value of the first goal value, and a strategy that secures the goal is to roll-over zero-coupon bonds which pay the goal values at the goal horizons.

A first example of a situation where condition (7) is satisfied is when each goal value is the present value of the next one (i.e. the inequality in (7) is an equality). It is then equivalent to secure the goal values on the intermediate horizons and to secure only the last goal value. This problem has been studied by Deguest, Martellini and Milhau (2014).

A second example of a situation where condition (7) holds is when all goal values are equal to each other $G_{T_j} = G_0$, where G_0 is some constant), and for all $j = 1,\ldots, p-1$, the nominal zero-coupon rate of maturity $h_j = T_{j+1} - T_j$ prevailing at date T_j is nonnegative.

Indeed, we have:

$$\mathbb{E}_{T_j}\left[\frac{M_{T_{j+1}}}{M_{T_j}}G_{T_{j+1}}\right] = G_0 \times \mathbb{E}_{T_j}\left[\frac{M_{T_{j+1}}}{M_{T_j}}\right],$$

and the conditional expectation in the right-hand side is equal to $\exp\left[-x^n_{T_j,h_j}\right]$, where $x^n_{T_j,h_j}$ is the nominal zero-coupon rate of maturity h_j. If this rate is nonnegative, then (7) is satisfied.

A third situation where the assumptions of the corollary are satisfied is when all goal values are equal to a constant G_0 times realised inflation, and the real zero-coupon rate of maturity h_j on date T_j is nonnegative. To see this, let Φ_t denote the price index on date t, so that Φ_{T_j}/Φ_0 is the realised inflation between dates 0 and T_j. We have:

$$\mathbb{E}_{T_j}\left[\frac{M_{T_{j+1}}}{M_{T_j}}G_{T_{j+1}}\right] = G_0 \times \frac{\Phi_{T_j}}{\Phi_0} \times \mathbb{E}_{T_j}\left[\frac{M_{T_{j+1}}\Phi_{T_{j+1}}}{M_{T_j}\Phi_{T_j}}\right]$$
$$= G_0 \times \frac{\Phi_{T_j}}{\Phi_0} \times \exp\left[-x^r_{T_j,h_j}\right],$$

where $x^r_{T_j,h_j}$ is the real rate. Again, (7) is clearly verified if this rate is nonnegative. This third example corresponds to the first application of our framework (see Section 4.1 below).

2.2.2 Affordability of Consumption-Based Goals

A consumption-based goal is not expressed as a minimum wealth level to attain at some horizon, since it consists of ensuring that a given consumption stream can be financed with the investor's portfolio strategy. As in Section 2.1.2, we represent a consumption-based goal by the discrete-time process $(c_{T_1},\ldots, c_{T_p})$. Dybvig and Huang (1988) define a "financed consumption plan" as a plan that leaves the final wealth nonnegative, the rationale being that the agent should not be allowed to end up at terminal date

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with a net a positive amount of debt. In the following definition, we adopt an apparently stronger criterion, which is that wealth should remain nonnegative at all dates. In fact, as shown in Appendix 6.1.3, this criterion is equivalent to the definition of Dybvig and Huang (1988).

Definition 4 (Affordability of a Consumption-Based Goal).
A consumption-based goal represented by the consumption dates $T_1,\dots, T_p$ and the nonnegative consumption stream $(c_{T_1},\dots, c_{T_p})$ is said to be affordable if there exists a portfolio strategy $\underline{\boldsymbol{w}}$ such that, starting from the investor's initial capital, the wealth satisfies the budget constraint (4) and the inequality $A_t \geq 0$ for all t in [0, T] with probability 1. Such a strategy is said to secure the goal.

As shown in Appendix 6.1.3, the condition "$A_t \geq 0$ for all $\boldsymbol{t} \in [0, T]$ with probability 1" in Definition 4 can be replaced by "$A_T \geq 0$ with probability 1".

As for a wealth-based goal, two main questions arise:
Q1 Are the payoffs $c_{T_1},\dots, c_{T_p}$ replicable with the available risky assets?
Q2 If the answer to Q1 is positive, is the investor's initial wealth sufficient to finance the consumption stream?

In what follows, we will assume that the answer to Q1 is positive, which amounts to assuming that there exists a set of p securities (e.g. inflation-linked pure discount bonds) maturing on the consumption dates with payoffs equal to the consumption payments. The price of the j^{th} replicating security is:

$$\mathbb{E}_t\left[\frac{M_{T_j}}{M_t} c_{T_j}\right].$$

By taking a buy-and-hold position in these replicating securities, one can synthesise a portfolio with payoffs that match all consumption expenditures. This portfolio is the goal-hedging portfolio, and its price is:

$$\tilde{G}_t = \mathbb{E}_t\left[\sum_{\substack{j=1 \\ T_j > t}}^{p} \frac{M_{T_j}}{M_t} c_{T_j}\right].$$

By definition, $\tilde{\boldsymbol{G}}_t$ is the price after the payment at date t and it is therefore right-continuous, and the initial price of the consumption stream is thus $\tilde{\boldsymbol{G}}_0$. It is shown in Appendix 6.1.3 that wealth satisfies:

$$A_t = \mathbb{E}_t\left[\frac{M_T}{M_t} A_T\right] + \tilde{G}_t,$$

for all t in [0, T].

Hence, if the goal is affordable, that is, if A_T is nonnegative, we have $A_t \geq \tilde{\boldsymbol{G}}_t$ for all t, and in particular, the initial wealth satisfies $A_0 \geq \tilde{\boldsymbol{G}}_0$.

The reciprocal implication sounds even more obvious: if the initial wealth satisfies $A_0 \geq \tilde{\boldsymbol{G}}_0$, then one can purchase at least one share of the GHP, so that the series of payoffs of the portfolio will cover the consumption outflows. A mathematical proof of such an intuitive result may seem somewhat unnecessary, but we nevertheless give it in Appendix 6.1.3 because it provides an explicit expression for the wealth generated by this strategy. In detail, let $\hat{\boldsymbol{G}}_t$ denote the total return index for the GHP, which is the value of the portfolio strategy with payoffs re-invested in it. Because the payoff dates are $T_1,\dots,T_p$, we have:

$$\hat{G}_t = \tilde{G}_t \prod_{\substack{j=1 \\ T_j \leq t}}^{p} \left(1 + \frac{c_{T_j}}{\tilde{G}_{T_j}}\right).$$

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The wealth achieved with the previous strategy is:

$$A_t = \tilde{G}_t + \left(\frac{A_0}{\tilde{G}_0} - 1\right)\hat{G}_t.$$

In particular, the terminal wealth is 0, so the goal is attained. The following proposition summarises this discussion in the form of an affordability criterion.

Proposition 4 (Affordability Criterion of a Replicable Consumption-Based Goal).
Consider a consumption-based goal represented by the nonnegative payment stream $(c_{T_1}, \ldots, c_{T_p})$, and assume that each payment is replicable. The goal is affordable if, and only if, the initial wealth and the present value of the consumption stream satisfy $A_0 \geq \tilde{G}_0$. If this condition is satisfied, then the goal is secured by the strategy that invests A_0 in the portfolio which payoff cash-flows match the consumption needs.

2.2.3 Joint Affordability of Multiple Goals

The previous section has introduced affordability criteria for wealth-based and consumption-based goals taken in isolation. But in real-world situations, investors have in general multiple goals (see examples in the case studies in Section 4), and we therefore extend our affordability criteria to the case of multiple goals.

2.2.3.1 Multiple Wealth-Based Goals

We consider the most general case, where the goals have multiple horizons (a goal with a single horizon fits into this category). The definition of the joint affordability of multiple goals involves no subtlety, although it is more demanding in terms of notations than the similar definitions for individual goals.

Definition 5 (Joint Affordability of Multiple Wealth-Based Goals).
Assume that the investor has L wealth-based goals, that the l^{th} goal is characterised by the set of horizons $\mathcal{T}_l = (T_1^l, \ldots, T_{p_l}^l)$ and the nonnegative minimum wealth levels $\left(G^l_{T_1^l}, \ldots, G^l_{T^l_{p_l}}\right)$. The L goals are said to be jointly affordable if there exists a portfolio strategy $\underline{\boldsymbol{w}}$ such that, starting from the investor's initial capital, the wealth satisfies the budget constraint (2) and the inequalities:

$$A_{T_j} \geq G^l_{T_j^l},$$

for all $j = 1, \ldots, p_l$ for all $l = 1, \ldots, L$.

Such a strategy is said to secure the goals. It is in fact clear that such multiple goals can be expressed as a single goal. Formally, let $\mathcal{T}$ denote the union of the sets of horizons, and for each t in $\mathcal{T}$ and each $l = 1, \ldots, L$, let G_t^l be zero if t is not an element of $\mathcal{T}_l$. Then, define the payoffs:

$$G_t = \max(G_t^1, \ldots, G_t^L), \text{ for } t \in \mathcal{T}.$$

Then, the joint affordability of the L goals is equivalent to the affordability of the single goal characterised by the set of horizons $\mathcal{T}$ and the minimum wealth levels $(G_t)_{t \in \mathcal{T}}$. In other words, the multiple goal case is not different from the case of a single wealth-based goal with multiple horizons.

If all goals are exogenous, this remark in particular enables the minimum capital required to afford the L goals as in Proposition 3 to be computed. It would be of interest to have an expression for the present value of the single goal as a function of those of the L goals. This, however, is not a straightforward task, given that the value of the single goal is the maximum of the L goal values. In particular, computing its present value requires the

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pricing of exchange options between the various goals, which will involve in the best case a pricing equation with unobservable parameters such as volatilities as inputs. An easier objective is to provide a lower bound for the present value. This is the content of the following proposition.

Proposition 5 (Necessary Affordability Condition of Multiple Wealth-Based Goals).
Consider L exogenous wealth-based goals, and let $\tilde{G}_0^l$ *be the present value of the* l^{th} *goal in the sense of the Definition 3 and* $\tilde{G}_0$ *be the present value of the multiple goals. Then:*

$$\tilde{G}_0 \geq \max(\tilde{G}_0^1, \dots, \tilde{G}_0^L).$$

Proof. See Appendix 6.1.5.
This proposition confirms the obvious property that multiple goals cannot be jointly secured if at least one of them is not affordable. The converse implication is not true. It is not because all goals are affordable separately (which implies that $A_0 \geq \max(\tilde{G}_0^1, \dots, \tilde{G}_0^L)$ that they are jointly affordable (which requires the stronger condition $A_0 \geq \tilde{G}_0$. However, if equality in Proposition 5 holds, then the individual affordability of each goal implies that all goals are jointly affordable.

2.2.3.2 Multiple Consumption-Based Goals
The notations here are similar to those of the previous section. There are L goals, each of them being described by its own set of consumption dates, $\mathcal{T}_l$, and its set of consumption payments. The definition of joint affordability is again a straightforward extension of the definition of affordability for a single goal.

Definition 6 (Joint Affordability of Multiple Consumption-Based Goals).
Assume that the investor has L consumption-based goals and the l^{th} *goal is characterised by the set of consumption dates* $\mathcal{T}_l = (T_1^l, \dots, T_{p_l}^l)$ *and the consumption expenses* $\left(c_{T_1}^l, \dots, c_{T_{p_l}}^l\right)$. *The L goals are said to be jointly affordable if there exists a portfolio strategy* $\underline{w}$ *such that, starting from the investor's initial capital, the wealth satisfies the budget constraint (4) and the inequality* $A_T \geq 0$ *with probability 1. Such a strategy is said to secure the goals.*

As for wealth-based goals, one can represent the L goals as a single goal, by merging the sets of consumption dates in a single set $\mathcal{T}$, and by letting c_t^l be zero if t is not an element of $\mathcal{T}_l$ for each $l = 1, \dots, L$. Then, the total consumption of date t is:

$$c_t = \sum_{l=1}^{L} c_t^l, \text{ for } t \in \mathcal{T}.$$

Thus, the L goals have been replaced by a single goal, which is the sum of the individual goals. It is then equivalent to be able to afford the L goals or the single aggregate goal. Because it is simpler to price the sum of several payoffs than their maximum, we obtain that the minimum capital required to afford the multiple goals is a simple function of the individual minimum capital requirements, as shown in the following proposition.

Proposition 6 (Affordability Criterion of Multiple Consumption-Based Goals).
Consider L consumption-based goals, and let $\tilde{G}_0^l$ *be the present value of the* l^{th} *stream. Then, the goals are jointly affordable if, and only if, the initial wealth satisfies*

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$$A_0 \geq \sum_{l=1}^{L} \tilde{G}_0^l.$$

Proof. The verification is immediate from the definition of the aggregate goal, since the present value of the sum of the L consumption payments is the sum of the L present values.

2.2.3.3 Wealth-Based and Consumption-Based Goals

From what precedes, multiple wealth-based goals can be re-expressed as a single wealth-based goal, and a similar operation can be performed for consumption-based goals. The only difference relates to the aggregation operation: while consumption-based goals simply add up, achieving several wealth-based goals is equivalent to achieving the highest goal. Hence, the most general situation, where the investor has various goals of both types, can be described in terms of two goals only: a wealth-based goal and a consumption-based goal.

The definition of the joint affordability in this context raises no particular problem, except that one has to specify how to assess the achievement of the wealth-based goal on a particular date if a consumption payment takes place on this very date. Indeed, it may happen that the wealth before consumption be larger than the minimum wealth level, while the consumption expense causes portfolio value to fall below the minimum. The choice that we make in this paper is to consider wealth before consumption. This option is after all arbitrary, and it leads to higher success probabilities (in the sense defined in Appendix 6.6.4) than the alternative choice, which would be to measure wealth after consumption. It should be noted that this distinction only matters if there is some overlap between consumption and horizon dates. Otherwise, both options are equivalent.

Mathematically, the definition reads as follows.

Definition 7 (Joint Affordability of a Wealth-Based and a Consumption-Based Goal).
Consider a wealth-based goal represented by the horizons $\mathcal{T}^W$ the minimum wealth levels $(G_t)_{t \epsilon \mathcal{T}^W}$, and a consumption-based goal characterised by the consumption dates $\mathcal{T}^C$ and the expenses $(c_t)_{t \epsilon \mathcal{T}^C}$. The two goals are said to be jointly affordable if there exists a portfolio strategy $\underline{\boldsymbol{w}}$ such that, starting from the investor's initial capital, the wealth satisfies the budget constraint (4) and the inequality:

$$A_{t-} \geq G_t,$$
$$\text{for all } t \,\epsilon\, \mathcal{T}^W, \; A_T \geq 0.$$

Such a strategy is said to secure the goals.

This definition does not give an operational affordability criterion because it requires finding a strategy that secures the goal. Thus, it would be useful to have an expression for the minimum capital at date 0 to invest in order to secure both goals. This question is addressed in the following proposition, which involves a sequence of compound option payoffs and prices, as with Proposition 3.

Proposition 7 (Joint Affordability Criterion for a Wealth-Based and a Consumption-Based Goal).
Consider a wealth-based goal represented by the minimum wealth levels $G_{T_1}, \ldots, G_{T_p}$, and a consumption-based goal characterised by expenses $C_{T_1}, \ldots, C_{T_p}$. Define the recursive

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sequence of payoffs:

$$K_{T_p} = G_{T_p};$$

$$K_{T_j} = \max\left(G_{T_j}, \tilde{K}_{T_j,T_{j+1}} + \tilde{c}_{T_j,T_{j+1}}\right), \quad \text{for } j = 0, \dots, p-1;$$

$$\tilde{K}_{T_j,T_{j+1}} = \mathbb{E}_{T_j}\left[\frac{M_{T_{j+1}}}{M_{T_j}} K_{T_{j+1}}\right];$$

$$\tilde{c}_{T_j,T_{j+1}} = \mathbb{E}_{T_j}\left[\frac{M_{T_{j+1}}}{M_{T_j}} c_{T_{j+1}}\right]$$

Then, the two goals are jointly affordable if, and only if, the initial wealth satisfies $A_0 \geq K_0$.

Proof. The proof of this result is similar to that of Proposition 3 and is omitted from the paper. It is available from authors upon request.

This result has the merit of being general, but as with Proposition 3, it is hard to apply in practice due to the complex structure of the payoffs. Nevertheless, it is not difficult to find a sufficient condition of affordability: if A_0 is larger than the sum of the present values of the two goals, then the goals are jointly affordable.

2.2.4 Non-Portfolio Income and Affordability

At this point, the various definitions of affordability are based on budget constraints which either assume a self-financing portfolio or a portfolio with consumption outflows. One significant additional source of complexity is related to the presence of income cash-flows, since the definition of affordability needs to be extended to account for the fact that future consumption goals can be financed and secured either from current wealth or from future savings. It turns out that the general definition that was given before (Definition 4) still applies in the presence of income, subject to a modification to the budget constraint, which now incorporates the non-portfolio income stream.

For brevity, and with no real loss of generality, we will focus the discussion on consumption-based goals, which are arguably of the most critical practical relevance, for example in the context of financing a retirement goal. Wealth-based goals could be handled in a similar way. In this context, the general definition of affordability can be written as follows.

Definition 8 (Affordability of a Consumption-Based Goal in the Presence of Income).
A consumption-based goal is said to be affordable if there exists a portfolio strategy $\underline{\mathbf{w}}$ *such that, starting from the investor's initial capital, the wealth satisfies the budget constraint (5) and the inequality* $A_t \geq 0$ *for all t in [0, T] with probability 1.*

This definition, however, does not provide an empirically testable criterion of affordability. The main focus of the remainder of this section is to provide necessary and sufficient conditions of affordability which can be applied in the presence of income streams. The second focus of this section is to discuss the corresponding goal-hedging portfolio strategies, with an explicit analysis of the competition between current wealth and future income in the composition of the goal-hedging portfolio.

Before introducing the general results, and in an attempt to ease the intuition, we first look at simple examples with a limited number of dates.

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2.2.4.1 One Income Cash Flow and One Consumption Cash Flow

Let us first consider a highly stylised example with a consumption goal of G_2 = \$100 in year 2, an income y_1= \$40 in year 1, and an initial liquid wealth A_0 =\$100. In this situation the key question is not to assess whether the goal is affordable or not; the goal is clearly affordable since current wealth alone, without future income, is already sufficient to secure the consumption goal. The outstanding question in this situation is rather to determine what the "best" way to secure the goal is. In this context, it is useful to introduce the intuitive definition of the *cheapest goal-hedging portfolio* as the portfolio strategy that allows an investor to secure a given consumption goal with the lowest amount of initial wealth. Intuitively, such a strategy should be preferred to other strategies that would also lead to 100% probability of achieving the goal but would require a higher amount of initial wealth, since the cheapest portfolio strategy is by definition the one that allows for the maximum access to risk premia harvested on performance-seeking assets (see Section 3 for more details on the optimal use of the remaining wealth, if any, that is left after all essential goals have been secured).

In the example, one GHP strategy (call it strategy LIQ) would consist of purchasing a pure discount bond which pays \$100 on date 2, at a price that is strictly less than \$100 provided that nominal rates are nonnegative. This strategy secures the goal, but it does not use income at all, and as a result has a high opportunity cost in terms of usage of current wealth.

Another goal-hedging strategy would be based on the recognition that the \$40 received at date 1 can be used to finance a fraction of the goal. Since the one-year rate prevailing at date 1 is not known at date 0, we do not know what exactly this fraction will be, but we know that income will generate at least \$40 at date 2 (still to the extent that nominal rates are nonnegative). In this context, there remain \$60 to finance with current wealth, which can be done by purchasing a pure discount bond that pays \$60 at date 2. We refer to this strategy as INC-ZER because it assumes a zero re-investment rate for future income. According to the aforementioned definition, it is less expensive than strategy LIQ since it requires the use of a lower amount of initial wealth (the present value of \$60, as opposed to the present value of \$100).

It turns out that an even cheaper portfolio strategy exists. To see this, consider a strategy (call it strategy INC-CMP – this name will be justified in Section 2.2.4.2) that invests in a bond option that will pay $(100 \times b_{1,2} - 40)^+$ at date 1, where $b_{1,2}$ is the price at date 1 of a pure discount bond with \$1 face value and maturity two years. In this particular example, for most reasonable values, we have that $100 \times b_{1,2} - 40$ almost surely (price at date 1 of a one year pure discount bond paying \$100 will be more than \$40), so that the option will pay off $100 \times b_{1,2} - 40$, an amount to which will be added \$40 worth of year income, so that the net cash-flow is $100 \times b_{1,2}$, which is exactly the minimum amount of money needed to generate, after being invested in the one-year pure discount bond at date 1, \$100 at date 2. It can easily be seen that the required amount of initial wealth

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for this strategy, that is, the price of the bond option paying $(100 \times b_{1,2} - 40)^+$, is strictly lower than the present value of \$60, so strategy INC-CMP is cheaper than strategy INC-ZER. In the next section, we will provide a general result showing that this strategy (suitably extended to a context with multiple income and consumption dates as a roll-over of compounded options) is actually the cheapest replicating strategy, in the sense that there is no strategy that can replicate the goal starting with a lower amount of initial wealth.

In fact, the last statement only holds in the absence of forward contracts. If we assume that forward contracts exist, a cheaper replicating strategy is available. This strategy (strategy INC-FWD) can be described as follows. Enter at date 0 (at no upfront cost) in a forward contract that will set the one year rate in one year from now equal to the current one year forward rate denoted by $f_{1,1} = \frac{b_{0,1}}{b_{0,2}} - 1$, and invest the present value of (the positive part) of $100 - 40 \times (1 + f_{1,1})$ in the pure discount bond with maturity two years. It is obvious that strategy INC-FWD is cheaper than strategy INC-ZER, but it also turns out to be cheaper than strategy INC-CMP. Indeed, the cost of protection at date 0 with strategy INC-CMP is:

$$\mathbb{E}\left[M_2 \times [100 - 40 \times (1 + f_{1,1})]^+\right] = [100 \times b_{0,2} - 40 \times b_{0,1}]^+,$$

where M_2 denotes the state-price deflator at date 2.

With strategy INC-CMP, the cost is the price of the call option which pays $(100 \times b_{1,2} - 40)^+$ on date 1, and this price is larger than the intrinsic value of the call, which is precisely the right-hand side of the previous equality.

In brief, this simple example has allowed us to obtain a first understanding of various strategies that can be used to secure a goal in the presence of income, with a key focus on the desire to use the lowest amount of initial wealth to reach the objective, thanks to the best possible use of future income. The analysis can be easily extended to a set-up with two income cash-flows and one consumption cash-flow.

Two Income Cash Flows and One Consumption Cash Flow

We now consider an example with a consumption goal G_3 =\$100 in year 3, with A_0 =\$100 and income streams y_1=\$40, y_2 =\$10 in years 1 and 2, respectively. The goal is again clearly affordable since it can be secured with current wealth only. To do this, the investor may simply invest the present value of \$100 in a pure discount bond with maturity 3, which is a replicating strategy (strategy LIQ) that does not rely at all on future income, and as such is very expensive in terms of use of current wealth. Another strategy (strategy INC-ZER) consists at date 0 in investing the present value of 100-(40+10)=\$50 in a 3-year pure discount bond, at date 1 investing the present value of 100-10-50=\$40 in a 2-year pure discount bond, and at date 2 investing the present value of 100-40-50=\$10 in a 1-year pure discount bond. This strategy will clearly replicate the goal, and is less expensive than strategy LIQ since the present value of \$50 is lower than the present value of \$100.

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We now turn to the cheapest replicating strategy, at least in the absence of forward contracts. This strategy involves income and a compound option, and is thus referred to as INC-CMP. At date 0, purchase the compound option which pays $(P_1 - 40)^+$ at date 1, where P_1 is the price at date 1 of the bond option which pays $(100 \times b_{2,3} - 10)^+$ at date 2. At date 1, use the compound option payoff, which will be equal to $(P_1 - 40)$ if the option expires in the money (this will be the case here for reasonable parameter values), to which will be added year 1 income to generate $P_1 - 40 + 40 = P_1$, to purchase the one-year bond option. At date 2, use the bond option payoff, which will be equal to the quantity $(100 \times b_{2,3} - 10)$, almost surely positive for reasonable parameter values, to which will be added year 2 income to generate $100 \times b_{2,3} - 10 + 10 = 100 \times b_{2,3}$, which is exactly the lowest amount of money to use at date 2 to invest in a one-year pure discount bond so as to secure the \$100 consumption goal at date 3. As in the previous example, it can be shown that this is the cheapest replicating strategy in the absence of forward contracts.

The replicating strategy with forward contracts looks as follows: at date 0 enter (at no upfront cost) into forward contracts that will set the two year rate in one year from now and the one year rate in two years from now equal to the current corresponding forward rates and invest from current wealth the present value of $100 - 40 \times (1 + f_{1,1})(1 + f_{2,1}) - 10 \times (1 + f_{2,1})$ in a the pure discount bond with maturity date 3 years. Here we have the forward rates defined respectively by $f_{1,1} = \frac{b_{0,1}}{b_{0,2}} - 1, f_{2,1} = \frac{b_{0,2}}{b_{0,3}} - 1$. This strategy is obviously less expensive than strategy INC-ZER since $100 - 40 \times (1 + f_{1,1})(1 + f_{2,1}) - 10 \times (1 + f_{2,1}) < 50$, and it can be shown that it is also less expensive than strategy INC-CMP in situations when the forward contracts exist (see Section 2.2.4.6 below for a justification).

2.2.4.3 An Example with Five Cash Flows

In the previous two examples, the income dates precede the consumption date. This corresponds to the retirement goal, which we will study in more detail below (see Section 2.2.4.6). Let us now consider a schedule with alternating periods of income and consumption. There are five cash flow dates:

- At date 1, income is \$50;
- At date 2, consumption is \$20 and at date 3, it is \$50;
- At date 4, income is \$20;
- At date 5, consumption is \$100.

A first obvious strategy (strategy LIQ) that secures the goal is to purchase a bond that will pay at each date the excess, if any, of consumption over income. This bond will have irregularly spaced cash flows: it will pay \$20 at date 2, \$50 at date 3, \$100 at date 5, and nothing at dates 1 and 4. Its price at date 0 is:

$$C_1 = 20 \times b_{0,2} + 50 \times b_{0,3} + 100 \times b_{0,5}.$$

But this policy ignores the possibility of carrying forward the unspent fraction of income from one date to the other. For instance, a cash flow of \$20 at date 2 is not necessary if one has secured the \$50 received at date 1.

This remark leads to the definition of the following strategy, which assumes that excess income is invested at a zero rate. It is referred to as INC-ZER. Suppose that the

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$50 of date 1 are invested in a one-year zero-coupon at date 1, so that they become $50 at date 2 in the worst case (the case where the one-year rate at date 1 was zero). The $50 will finance the $20 of consumption, and there will be a surplus of $30 left. Investing these $30 in a new one-year zero-coupon bond leads to an income of $30 (at least) at date 3. These $30 do not fully cover the consumption expenditure, which is $50. To make up for the gap, the investor has to purchase at date 0 a three-year zero-coupon that will pay $20 at date 3. Similarly, if the investor secures the $20 received at date 4 by investing them at the one-year rate, he will be left with a deficit of $80 at date 5: to compensate for this deficit, he needs to purchase at date 0 a zero-coupon that will pay $80 at date 5. Thus, the cost of the protection as seen from date 0 is:

$$C_2 = 20 \times b_{0,3} + 80 \times b_{0,5}.$$

We clearly have $C_1 > C_2$, so the strategy INC-ZER is less expensive than the one that uses liquid wealth only.

2.2.4.4 The General Case: Affordability Conditions and the Cheapest Goal-Hedging Portfolio

The three examples discussed above allow us to emphasise that future income should be preferred to current wealth when it comes to securing goals. Intuitively, this is because doing so leaves the highest amount of current wealth available for investment in performance-seeking assets, which in turn is critically needed to achieve some non-affordable goals with a positive probability (see Section 2.3 for the classification of non-affordable goals). Note that this discussion is based upon the implicit assumption that future income is obtained with certainty; if there is uncertainty about future income, an investor may prefer to use liquid wealth to secure the goals with probability 1.

We now provide a series of general results that extend the intuitions gained in the simple examples to a general setting with multiple income and consumption dates. We start with a necessary and sufficient affordability condition of a consumption-based goal in the presence of income, which corresponds to a generalised version of the strategy denoted by strategy INC-CMP in the analysis of the simple examples.

Proposition 8 (Affordability Criterion of a Consumption-Based Goal in the Presence of Income).
Assume that markets are complete, let wealth evolve according to the budget constraint (5), and consider a consumption-based goal with the same payment dates as the income dates. Let $T_0 = 0$ and $y_{T_0} = 0$, and define the recursive sequence of payoffs $\left(V_{T_j}\right)_{j=0,\dots,p}$ and their prices:

$$V_{T_p} = \left(c_{T_p} - y_{T_p}\right)^+;$$

$$\tilde{V}_{T_{p-j},T_{p-j+1}} = \mathbb{E}_{T_{p-j}}\left[\frac{M_{T_{p-j+1}}}{M_{T_{p-j}}} V_{T_{p-j+1}}\right],$$

for j=1,...,p.

$$V_{T_{p-j}} = \left(\tilde{V}_{T_{p-j},T_{p-j+1}} + c_{T_{p-j}} - y_{T_{p-j}}\right)^+,$$

for j=1,...,p.

Then, the wealth-based goal is affordable if, and only if, the initial wealth satisfies $A_0 \geq V_0$. If this condition is satisfied, then the goal is secured by investing A_0 in a roll-over of compound exchange options expiring on dates $T_1,\dots,T_p$ with payoffs $V_{T_1},\dots, V_{T_p}$.

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Proof. See Appendix 6.1.6.

V_0 is the minimum capital that the investor must hold in liquid wealth at date 0 in order to secure the goal, which can also be interpreted as the price of the cheapest goal-hedging portfolio in the sense defined in Section 2.2.4.1. Its backward recursive definition may look complex, but the mechanics is simple. The investor wants to ensure that wealth after the last date (T_p) is nonnegative. If the income of this date covers consumption, this will be the case, whatever the wealth just before date T_p. If income is less than consumption, then the wealth before income and consumption, i.e. the quantity A_{T_p-}, must be greater than ($c_{T_p} - y_{T_p}$). It must also be nonnegative, so A_{T_p-} must be greater than $V_{T_p} = \left(c_{T_p} - y_{T_p}\right)^+$. Thus, the investor has an implicit wealth-based goal of horizon T_p. By absence of arbitrage opportunities, the wealth at date T_{p-1} must be greater than the present value of V_{T_p}, the minimum wealth to attain one step further. This condition means exactly that $A_{T_{p-1}}$ must be greater than $\tilde{V}_{T_{p-1},T_p}$. The reasoning is then the same as for date T_p : $A_{T_{p-1}}$ must be greater than $\left(\tilde{V}_{T_{p-j},T_{p-j+1}} + c_{T_{p-j}} - y_{T_{p-j}}\right)$ if this quantity is positive. Because it must also be nonnegative, it must in fact be greater than $V_{T_{p-1}}$. A backward induction which is formally written in Appendix 6.1.6, shows that A_0 must be greater than V_0.

Appendix 6.1.6 also shows that the following bounds hold for V_0:

$$\left(\tilde{G}_0 - \tilde{H}_0\right)^+ \leq V_0 \leq \tilde{G}_0,$$

where $\tilde{H}_0$ denotes the present value of all future income payments, i.e. the human capital, at date 0.

The upper bound $\tilde{G}_0$ has a very intuitive interpretation: due to the existence of future income, the minimum capital requirement is less than the price of the bond whose cash flows match the consumption expenses. V_0 will approach this upper bound as the income payments shrink to zero. In general, an investor endowed with income uses this income rather than liquid wealth to finance consumption expenses. Liquid wealth will be used to purchase compound exchange options that make up for the "funding gap", which at date T_j is formally defined as the payoff V_{T_j}, and can be loosely thought of as the excess of future consumption over future income. In general, both future consumption and future income are stochastic, so that the option, which has a stochastic strike price, can be regarded as an exchange option.

The lower bound is the minimum initial liquid wealth that would be required if no nonnegativity condition was imposed to liquid wealth. Indeed, the investor would be allowed to borrow against future income, and negative liquid wealth would be possible, as long as the sum of liquid wealth and the human capital stays nonnegative. Definition 8 imposes a tighter condition because it precludes negative wealth, so the initial capital requirement is more severe. V_0 will approach the lower bound as the nonnegativity condition on liquid wealth is progressively removed. This corresponds to the case where the investor cannot rely on future income to finance consumption, because income is too low. More specifically, V_0 will be equal to the lower bound if income is systematically lower than consumption. This statement can in fact be extended. As shown in Appendix 6.1.6, we have:

$$V_0 = \tilde{G}_0 - \tilde{H}_0$$

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provided that the present values of the goals and the income payments satisfy:

$$\tilde{G}_{T_j-} \geq \tilde{H}_{T_j-},$$

for all $j = 1,\ldots, p$,

(We recall that the left limit $\tilde{G}_{T_j-}$ is the present value of future consumption payments, with date T_j included, while $\tilde{G}_{T_j-}$ is the present value without date T_j.)

2.2.4.5 Examples of Strategies Securing the Goal in the General Case

As appears from the analysis of the examples, an investor endowed with non-portfolio income has (at least) three possibilities to protect a consumption-based goal.

1. LIQ: Use liquid wealth only

This strategy consists in purchasing at date 0 a bond that pays the excess, if any, of consumption over income. The minimum capital required is:

$$E_0 = \sum_{j=1}^{p} \mathbb{E}\left[M_{T_j}\left(-e_{T_j}\right)^+\right],$$

where $e_{T_j} = y_{T_j} - c_{T_j}$ is the net income;

2. INC-ZER: Use income assuming a zero re-investment rate for future excess income

In this strategy, at date T_1, the investor uses income to finance the largest possible fraction of consumption. The excess income, if any, is invested in a zero-coupon bond that matures on the next consumption date, T_2. If nominal rates are nonnegative, then in the worst case, the rate of return on this investment is zero. On date T_2, the capitalised excess income of date T_1 and the new income are used to finance consumption, and the excess, if any, is invested in a zero coupon maturing on date T_3. This operation is repeated at dates T_1, ..., T_{p-1}. Mathematically, if c_{T_j} is the consumption of date T_j, y_{T_j} is the income and $u_{T_{j-1}}$ is the excess of date T_{j-1} invested at a zero rate (with the convention $u_{T_0} = 0$, the deficit to finance on date T_j is:

$$\left(-e_{T_j} - u_{T_{j-1}}\right)^+,$$

so the minimum capital requirement is:

$$U_0 = \sum_{j=1}^{p} \mathbb{E}\left[M_{T_j}\left(-e_{T_j} - u_{T_{j-1}}\right)^+\right].$$

The recursion relationship between the quantities u_{T_j} is $(u_{T_j} = u_{T_{j-1}} + e_{T_j})^+$. Proposition 9 below formally shows that this strategy secures the goal.

3. INC-CMP: Use income and a compound option

This is the strategy corresponding to Proposition 8. At date 0, the investor purchases a compound option of price V_0 maturing at date T_1. At this date, he uses the option payoff, V_{T_1}, plus income, to finance consumption. By definition of V_{T_1}, we have:

$$V_{T_1} = \max\left(\tilde{V}_{T_1,T_2}, y_{T_1} - c_{T_1}\right),$$

so the investor can afford the option that pays V_{T_2} at date T_2, and moreover, this option can be purchased by using only the option payoff and the received income. This strategy is repeated at dates T_2, ..., T_{p-1}. The minimum capital requirement is V_0.

The following proposition formally states that strategy INC-ZER does secure the goal.

Proposition 9 (Sufficient Affordability Criterion for Consumption-Based Goal with Income).

If $A_0 \geq U_0$ and nominal rates are nonnegative, the strategy INC-ZER can be implemented and it secures the goal.

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Proof. See Appendix 6.1.7.

Because strategy INC-ZER secures the goal, Propositions 8 and 9 imply that U_0 is greater than or equal to V_0.[19] Moreover, it is clear (since the quantity $u_{T_{j-1}}$ defined above is nonnegative) that U_0 is less than or equal to E_0. Eventually, the three costs of protection are ordered as follows:

$$V_0 \leq U_0 \leq E_0.$$

This result provides two sufficient affordability conditions for the goal: it suffices to test verify that $A_0 \geq U_0$ or $A_0 \geq E_0$ These conditions have a practical interest because the price of the compound option, which is the cost of the cheapest protection, is difficult to compute. But it should be acknowledged that U_0 also involves compound options because $u_{T_{j-1}}$ is itself defined as a positive part.

19 - This property can also be checked directly from the definitions of U_0 and V_0, without using the conclusions of Proposition 8.

2.2.4.6 Other Examples in the Case of the Retirement Goal

A practically important situation is the case of retirement investment decisions. In this case, the investor is assumed to have a net positive saving in the first part of his life (accumulation phase), while consumption exceeds income in the second phase (decumulation phase). The retirement goal is formally defined as a goal for which there exists a date T_r such that consumption is less than income until T_r and greater afterwards:

$$c_{T_j} \leq y_{T_j} \text{ for } j = 1,\ldots, r$$

$$y_{T_j} \leq c_{T_j} \text{ for } j = r+1,\ldots, p$$

This goal is affordable if, and only if, there exists a strategy such that the wealth of date T_{r+1} (before consumption and income) satisfies:

$$A_{T_{r+1}-} \geq \tilde{G}_{T_{r+1}-} - \tilde{H}_{T_{r+1}-},$$

where $\tilde{G}_{T_{r+1}-}$ and $\tilde{H}_{T_{r+1}-}$ are the respective present values at date T_{r+1} of the consumption expenses and the income payments, with date T_{r+1} included. It should be noted that this necessary and sufficient affordability condition obtains because net consumption is nonnegative after retirement. Hence, there is a wealth-based goal of horizon T_{r+1}. To finance the purchase of the bond that pays net consumption during the decumulation phase, the investor can implement another strategy in addition to those listed in Section 2.2.4.5:

4. INC-ZER-RET: Use a modified version of strategy INC-ZER

In the strategy INC-ZER above, it is assumed that unspent income is carried forward from one income/consumption date to the next one by being invested at a zero rate for one period. In the strategy INC-ZER-RET, on the other hand, the savings of the accumulation phase are assumed to be invested at a zero rate for a period equal to the time to retirement. In order to finance his consumption objectives, the investor must reach a wealth level equal to $[\tilde{G}_{T_{r+1}-} - \tilde{H}_{T_{r+1}-}]$ at least at date T_{r+1}. Thus, the deficit to finance at date T_{r+1} as seen from date 0 is:

$$U_{ret,T_{r+1},0} = \left[\tilde{G}_{T_{r+1}-} - \tilde{H}_{T_{r+1}-} - \sum_{k=1}^{r} e_{T_k}\right]^+,$$

where $e_{T_k} = y_{T_k} - c_{T_k}$ is the net income of date T_K (it is nonnegative because the investor consumes less than what he earns). To finance this deficit, the investor must purchase an option that pays off $U_{ret,T_{r+1},0}$, which has a cost denoted as $U_{ret,0,0} = U_{ret,0}$, the present value of the payoff, at date 0. The strategy is repeated at dates T_2, ..., T_r. At date T_j, an income payment y_{T_j} is cashed in, and is aggregated to liquid wealth. The forecasted deficit is now:

$$U_{ret,T_{r+1},j} = \left[\tilde{G}_{T_{r+1}-} - \tilde{H}_{T_{r+1}-} - \sum_{k=j+1}^{r} e_{T_k}\right]^+.$$

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The investor must purchase an option paying $U_{ret,T_{r+1},j}$, which has a cost denoted as $U_{ret,T_j,j}$.

To formally prove that the strategy INC-ZER-RET is feasible, one has to verify that the wealth of date T_j is sufficient to afford the desired option. This is the content of the following proposition:

Proposition 10 (Sufficient Affordability Condition for Retirement Goal).
If $A_0 \geq U_{ret,0}$ and nominal rates are nonnegative, the previous strategy can be implemented and it secures the goal.

Proof. See Appendix 6.1.8.
The details of the proof in Appendix 6.1.8 show that we have the following inequality:

$$U_{ret,T_j,j} \leq U_{ret,T_j,j-1} + e_{T_j},$$
$$\text{for } j = 1,\ldots, r.$$

This means that the price of the option to purchase at date T_j (in the left-hand side of the inequality) is less than or equal to the sum of the price of the option that was purchased at the previous date, plus the net income. That is, the new option can be purchased simply by liquidating the position in the existing one and using the unspent fraction of income, if any. An interesting consequence is that if the fraction of liquid assets that is not used to protect the goal is invested in some performance portfolio, then there is no need to liquidate a fraction of this portfolio in order to finance the new option.

By Propositions 8 and 10, we have:

$$V_0 \leq U_{ret,0}$$

It should be noted that $U_{ret,0}$ is in general different from U_0. But it is potentially easier to compute in applications because it is a simple exchange option (between the minimum level of wealth to attain at the retirement age and the sum of net income), while U_0 involves compound options.

As noted in the discussion related to the introductory examples, forward contracts can also be employed to secure the goal, which leads to the definition of the following strategy:

5. INC-FWD: Use income and forward contracts if they are available
If forward contracts are available, another hedging strategy consists in locking up as of date 0 the re-investment rates for the income inflows. In other words, the income cash-flow of date T_j will be invested at the rate $f_{T_j,T_{r+1}-T_j}$, which is the forward rate at date T_j for maturity $T_{r+1}-T_j$. The deficit that remains to be financed at date T_r is the excess, if any, of the wealth-based goal value over the cumulative value of income payments invested at the fixed forward rates:

$$W_{T_{r+1}} = \left[\tilde{G}_{T_{r+1}-} - \tilde{H}_{T_{r+1}-} - \sum_{j=1}^{r} e_{T_j}\left(1 + f_{T_j,T_{r+1}-T_j}\right)^{T_{r+1}-T_j}\right]^+.$$

Let W_0 be the price of this option at date 0. The investor can afford the option if, and only if, the available liquid wealth is such that $A_0 \geq W_0$. It is obvious that the strategy which consists in purchasing the option at date 0 and invest the income cash-flows at the forward rates will secure the goal.

In terms of usage of initial liquid wealth, this strategy is cheaper than INC-ZER-RET: this is obvious, since it assumes that income is invested at forward rates, which are positive, while INC-ZER-RET assumes a zero re-investment rate.

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Perhaps more surprisingly, the strategy INC-FWD can also be shown to be cheaper than the strategy INC-CMP in some contexts, in the sense that:

$$W_0 \leq V_0, \quad (8)$$

a property that has already been mentioned in the context of the simple examples with two or three income/consumption payment dates, and can be extended to an arbitrary number of dates. As a consequence, the strategy INC-FWD is, at least when (8) holds, cheaper than any of the aforementioned policies which also secure the goal.

A situation where (8) is verified is when all cash flows before retirement are deterministic and the inequality

$$\tilde{G}_{T_{r+1}-} - \tilde{H}_{T_{r+1}-} \geq \sum_{j=1}^{r} e_{T_j} \left(1 + f_{T_j, T_{r+1}-T_j}\right)^{T_{r+1}-T_j}$$

holds almost surely. Then, we have:

$$W_0 = b_{0,T_{r+1}} \times \left[-e_{T_{r+1}} - \sum_{j=1}^{r} e_{T_j} \left(1 + f_{T_j, T_{r+1}-T_j}\right)^{T_{r+1}-T_j} \right]^+$$

$$= \left[-b_{0,T_{r+1}} \times e_{T_{r+1}} - \sum_{j=1}^{r} e_{T_j} b_{0,T_j} \right]^+ = \left[\tilde{G}_0 - \tilde{H}_0\right]^+,$$

which is the lower bound for V_0.

2.3 Taxes and Affordability

Taxes are a typical example of frictions in real-world financial markets. They usually apply to cash flows such as dividend and coupon payments, but also to the capital gains generated by selling operations of financial securities. In this section, we make a general presentation of the taxation principles that we will apply in the case studies of Section 4 and we revisit the definition of affordable goals.

2.3.1 Taxation Principles

As noted in the introduction to this section, taxes apply to dividend and coupon payments and to the capital gains achieved when selling a share of a security at a higher price than the purchase price. To compute the taxes on cash flows, we simply multiply the cash flow by a tax rate ζ, so the investor effectively receives a net payment equal to (1- ζ) times the pre-tax dividend or coupon. A tax rate must be specified in applications.

The taxation of capital gains involves more degrees of freedom:

- Which tax rate should be used? In practice depending on whether the gains are treated as short-term gains (less than a year) versus long-term gains (more than a year), and trading in taxable versus non-taxable versus tax differed accounts, the effective tax rate will be different;
- How are taxable gains computed? The principle is to tax gains on sales operations: if an asset share is sold at a higher price than the price at which it was purchased, taxes are applied to the gain. To specify which shares of an asset are sold in the event of a partial liquidation of the position, the standard options are LIFO (last in first out), FIFO (first in first out) and HIFO (highest in first out);
- Is there an option to write off losses within the year or to have a compensation of gains and losses within the portfolio?

In the case studies, we will use for simplicity a unique tax rate for all operations, and we will take 20% as the base case value. Taxes will be computed on an annual basis, which means that the investor will pay taxes once a year, for all the gains that occurred within the fiscal year. Regarding the computation of taxable gains, we will use the LIFO option: to decrease exposure to an asset class, the investor liquidates the shares by starting from the most recently purchased ones. Appendix 6.6.5 gives the

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mathematical expression for the annual tax payment.

Finally, we will allow for the write off of losses within a given year up to the amount of capital gains earned for this particular year, and for compensation between the constituents within the portfolio: if losses are recorded for an asset, they decrease the amount of taxes to be paid at the end of the year, the final tax payment being floored at zero. But we will exclude any possibility of carrying forward losses for another year.

Because they represent cash withdrawals from the portfolio, taxes can be regarded as a form of consumption. But the analogy with a consumption goal is only imperfect because unlike the consumption goals, taxes are endogenous: the amount to pay depends on the investor's exposure to each asset class, and the taxes on capital gains crucially depend on rebalancing decisions.

2.3.2 Affordable Goals

We now reconsider the notion of goal affordability in the presence of taxes. The general idea behind the definition is not substantially modified with respect to the situation without taxes: a wealth-based goal is affordable if the wealth is above the minimum levels and a consumption-based goal is affordable if the wealth after consumption is nonnegative. The only modification with respect to the definitions given in Section 1.1 is the budget constraint, which has to incorporate the tax payments. Formally, we let $\Theta_{t_1}, \dots, \Theta_{t_m}$ be the tax payments, which occur on dates $t_1, \dots, t_m$. We recall that the differential element $dJ_{t_j,t}$ is equal to 1 when $t = t_j$, and 0 the rest of the time. In the absence of consumption outflows, the budget constraint reads:

$$dA_t = A_t\left[r_t + \underline{w}_t'\underline{\mu}_t\right]dt + A_t\underline{w}_t'\underline{\sigma}_t'd\underline{z}_t - \sum_{k=1}^{m}\Theta_{t_k}dJ_{t_k,t}. \quad (9)$$

A wealth-based goal is said to be affordable if there exists at least a portfolio strategy $\underline{w}$ such that the minimum wealth levels are attained at all goal horizons, subject to the budget constraint (9). It should be noted that this definition is independent from a particular set of taxation rules, and in particular does not depend on the specification of the tax rate or the rules applied in the computation of taxable capital gains.

Similarly, a consumption-based goal is said to be affordable if there exists a strategy $\underline{w}$ such that the wealth generated by the following budget equation:

$$dA_t = A_t\left[r_t + \underline{w}_t'\underline{\mu}_t\right]dt + A_t\underline{w}_t'\underline{\sigma}_t'd\underline{z}_t - \sum_{j=1}^{p}c_{T_j}dJ_{jt} - \sum_{k=1}^{m}\Theta_{t_k}dJ_{t_k,t}. \quad (10)$$

remains nonnegative. These definitions can be extended to situations with non-portfolio income without difficulty.

2.4 Hierarchical Classification of Goals

The distinction between wealth-based and consumption-based goals is useful from a technical standpoint to characterise the notion affordability, but it abstracts away from any concept of priority ranking among goals, which is of relevance in most practical applications since investors typically have an explicit or sometimes implicit hierarchy ranking among the goals. In this context, Chhabra (2005) proposes another key distinction between *essential goals*, *important goals* and *aspirational goals*.

Intuitively, essential goals are goals that the investor wants to achieve with full probability at all costs. Among essential goals, one can for example identify the long-term objective of protection from

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anxiety or poverty, which is often implicit in investors' preferences, and justifies home ownership or cash holdings. In addition to such implicit safety goals, explicit minimum (nominal or inflation-linked) wealth and/or consumption levels are also often included at various dates until horizon. Lastly, short-term safety goals, such as protection against drawdown risk, are also included in this category.

Important goals essentially relate to goals that are slightly lower in terms of priority, but that are still of high relevance to investors. These important goals may include ensuring a high probability of maintaining one's standard of living, or high probability of paying for children's education, etc. Since these goals are important but not essential, an investor may decide not to invest the required amount of wealth to secure them. Finally, aspirational goals typically relate to generating a reasonable probability of a substantial wealth increase or even wealth mobility for consumption or bequest objectives. Among such aspirational goals, one can precisely distinguish between ambitious performance goals remaining within a given affluence class (e.g. capital growth objectives over long-term horizons) versus the more dramatic (and less likely) opportunity of affluence class mobility, that is, moving upward substantially in the wealth spectrum of society.

While these notions are intuitively clear, it is necessary to introduce a formal definition for them. In what follows, we precisely provide a formal definition for these goal types in relation to the concept of affordability.

2.4.1 Essential, Important and Aspirational Goals

As explained before, a key input expected from the individual investor in a goals-based wealth management framework is an ordered list of goals, by which we mean a list of goals and the associated priority ranks. A first task is to classify these goals as affordable and non-affordable. If there are several goals, it is necessary to use the definitions of affordability given in Section 2.2.3. It is clear that if a set of goals is jointly affordable, then any smaller set of goals is also affordable. This leads to the definition of the maximal set of affordable goals, which we denote with $\mathcal{AF}$.

Definition 9 (Maximal Set of Affordable Goals).

Let the investor's goals (wealth-based or consumption-based) be represented by the symbols $G^1,\dots,G^{N_G}$, where N_G is the number of goals and the goals are ranked by order of decreasing priority. The maximal set of affordable goals, $\mathcal{AF}$, is defined as:

- *If all goals are jointly affordable, then $\mathcal{AF} = \{G^1,\dots,G^{N_G}\}$;*
- *Otherwise, $\mathcal{AF} = \{G^1,\dots,G^i\}$ where i is such that the goals $G^1,\dots,G^i$ are jointly affordable and $G^1,\dots,G^{i+1}$ are not.*

It should be emphasised that $\mathcal{AF}$ is not the set of goals which are individually affordable. This is obvious for consumption-based goals, which are additive: two goals may be separately affordable in the sense that the investor can afford to secure the more expensive of the two hedging portfolios which finance the related consumption expenditure, but may not be able to secure both of them simultaneously. Hence, we avoid referring to $\mathcal{AF}$ as "the set of affordable goals", a terminology which would be misleading.

Definition 9 serves as the basis for the formal distinction between the three classes of goals of Chhabra (2005). First, we define essential goals as goals that must be reached with a virtually 100% probability, which

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implies two requirements. First, they must be part of the maximal set of affordable goals. Second, the strategy chosen by the investor must secure each goal, that is, it must reach all of them with a 100% probability.[20] In contrast, important goals are also defined as being part of $\mathcal{AF}$, but differ from essential goals in the fact that they are not included as part of the set of goals to be secured. Finally, the goals that are not part of $\mathcal{AF}$ are said to be aspirational goals. The following definition summarises the distinction between the three types of goals.

Definition 10 (Essential, Important and Aspirational Goals).
Consider the maximal set of affordable goals, $\mathcal{AF}$, defined in Definition 9.
- *Essential goals are the elements of $\mathcal{AF}$ that the investor decides to secure (affordable and secured goals);*
- *Important goals are the elements of $\mathcal{AF}$ that the investor decides not to secure (affordable but non-secured goals);*
- *Aspirational goals are the goals which are not contained in $\mathcal{AF}$ (non-affordable, and therefore non-secured, goals).*

The first step in the classification of goals is thus the identification of the maximal set of affordable goals. It is important to keep in mind that the affordability of a goal depends on the asset mix available to the investor. The introduction of a new asset which is not redundant with the existing ones may turn a non-replicable goal payoff into an attainable payoff. An example is given by inflation-linked bonds: the goal of respecting a certain inflation-adjusted minimum level of wealth is not affordable until an inflation-indexed bond is introduced in the asset mix because the floor value cannot be exactly replicated with other securities. Making this minimum wealth level attainable is the first step towards making the goal affordable: the next requirement is that investor's liquid wealth, and possibly future income, covers the price of the indexed bond.

A non-affordable goal can also become affordable if suitable contracts are offered to the investor. For instance, as explained in Section 2.2.4.6, one can secure a consumption objective upon retirement by entering forward contracts to lock up the re-investment rates of future income: when the consumption and income payments are known in advance, this turns out to be the cheapest hedging strategy, even cheaper than the strategy which consists in purchasing the compound option that makes up for the gap between income and consumption (see Proposition 8). As a consequence, the investor's liquid wealth may be less than the option price, while being sufficient to secure the goal with the forward contracts. In other words, the goal would not be affordable without the forward contracts, but becomes so if the contracts exist.

Another example is an insurance contract which pays the goal value. Such contracts may be available in the context of goals related to events whose occurrence is not certain, such as health contingencies. In the case study section of the paper, we will consider the goal of paying for nursing home fees (see Section 4.2). In this case, the expense is triggered by an exogenous event, known as a long-term care (LTC) event. An insurer may propose a contract whose benefits cover the expense. This contract completes the market if no option otherwise exists that would pay the goal value exactly when the LTC event occurs. Even if the option exists, the contract may be less expensive if the present value of the premiums is less than the option price. This present value depends on the pricing policy of the insurer among other factors, but it

20 - By a probability 1, we mean 100% of chances of success in the absence of gap risk or default risk on the safe assets. On the other hand, this assessment of the 100% probability of success should not be subject to model or parameter uncertainty.

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will hopefully always be less than the price of the bond that would super-replicate the goal value by paying the fees in all states of the world, whether the LTC event occurs or not. Thus, the goal may become affordable thanks to the insurance contract.

2.4.2 Interpretation and Implications of the Classification of Goals

We now discuss some key implications of Definition 10. One first implication of the presence of an explicit hierarchy within and across types of goals is that cash-flows related to lower priority goals that occur before cash-flows related to higher priority goals should be paid only if the payment of these cash-flows will not have too strong an impact on the subsequent goals. More precisely, the presence of a formal hierarchy of goals implies the following set of rules:

- An important or aspirational goal with consumption cash-flows occurring at dates before the consumption dates for some essential goals will be satisfied if and only if the satisfaction of this goal will not turn any one of the essential goals into aspirational goals;
- An important goal with consumption cash-flows occurring at dates before the consumption dates for some other more important goals will be satisfied if and only if the satisfaction of this goal will not turn any one of these more important goals into aspirational goals;
- An aspirational goal with consumption cash-flows occurring at dates before the consumption dates for some important goals will be satisfied if and only if the satisfaction of this goal will not turn any one of the important goals into aspirational (that is, non-affordable) goals;
- An aspirational goal with consumption cash-flows occurring at dates before the consumption dates for some other essential goals will be satisfied if and only if the satisfaction of this goal will not decrease the probability of achieving any one of these other goals.

We now turn to a more detailed discussion of the interpretation of each type of goal.

2.4.2.1 Essential Goals

An essential wealth-based goal should actually be regarded as a floor in the sense that it represents a minimum level of wealth that the strategy must satisfy with probability 1. In case the initial wealth of the investor makes it impossible to ensure the achievement of the essential goals with full certainty, the investor must bring additional contributions, either now (immediate increase in the dollar budget) or later (under the form of higher saving rates). In case the investor proves to be unable or unwilling to increase these dollar budgets, then he/she should be willing to accept lower essential goal levels.

All the examples of essential goals given above fit into this definition. Indeed, the goal of home ownership can be seen as a wealth-based goal, where the investor wants his/her wealth to remain greater than or equal to the house value. This goal is affordable and secured for those investors who own their residence. The goal of having a minimum level of wealth available at all times, in order to finance minimal levels of short-term consumption needs in scenarios such that income is dramatically decreasing, can be secured by holding a roll-over position in cash, since the value of cash never decreases.

It should be noted that in the previous two examples, goals are in general not explicitly formulated by the investor, and are only implicit. Other essential goals are explicit, and can be secured with an investment in a GHP such as those described in Section 1.1. In the end, the qualification of a

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goal as an essential one depends both on the affordability of the goal and on the investor's decision to secure it.

2.4.2.2 Important Goals

By Definition 10, important goals are part of $\mathcal{AF}$. Hence, the investor would be able to secure them together with the essential goals, but decides not to do so. A typical motive for not securing otherwise important goals is a concern over performance. The reason why an investor may decide not to secure otherwise affordable important goals is to avoid investing an exceedingly large share of his/her wealth in hedging portfolios so as to allow for a higher level of investment in performance-seeking assets, and as a result generate more upside potential and increase the probability of achieving aspirational goals. In this case, the goals, which are not formally secured, may not be achieved with probability 1.

Eventually, the only difference between an essential and an important goal is that the investor decides to secure the former but not the latter. But it is also possible to secure an affordable goal only partially. Then, the goal is split in two new goals, respectively an essential one (corresponding to the fraction of the goal to secure) and an important one (corresponding to the unsecured fraction). For instance, if an annual expense of $100,000 is affordable given the investor's current liquid wealth and future income perspectives, one may decide to secure only $75,000 by purchasing the appropriate GHP. The remaining annual expense of $25,000 is then treated as an important goal. An example of partial protection of an affordable goal will be presented in the case study section of the paper (see Section 4.2).

2.4.2.3 Aspirational Goals

Aspirational goals are defined as goals which are not in $\mathcal{AF}$. They consist of two categories of goals. First, there are non-affordable goals, i.e. goals whose value is not replicable with the available assets, or goals that would be replicable but are too expensive to be affordable. For instance, a consumption-based goal such that the present value of the expenditure exceeds investor's wealth is an aspirational goal. The second class of aspirational goals consists of the elements of $\mathcal{AF}$ which are not labelled as essential or important. Being elements of $\mathcal{AF}$, these goals are individually affordable, but they cannot be secured together with other goals with higher priority ranking (that is, goals of the essential or important types).

2.5 Building Blocks in Goals-Based Wealth Management

The fund separation theorem, which is a fundamental cornerstone of dynamic asset pricing theory, suggests that risk and performance are two conflicting objectives that are best managed when managed separately within dedicated building blocks. In practical terms, it implies that all investors should allocate (in addition to long or short positions in the risk-free asset) some fraction of their wealth to a common well-diversified performance-seeking risky portfolio (Tobin (1958)) as well as to some dedicated hedging portfolios designed to help the investor obtain protection against unfavourable changes in risk factors that impact their income streams as well as their wealth and consumption goals (Merton (1971, 1973)). In this section, we describe in more detail the various building blocks that will be involved in the design of goals-based strategies.

These building blocks can be classified as follows:

1. Goal-hedging portfolios (GHPs), specific

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to each (group of) individual investor(s);
2. A well-diversified performance-seeking portfolio (PSP, which should theoretically be the maximum Sharpe ratio portfolio, MSR), common to all investors;
3. Wealth mobility portfolios (WMPs), specific to each (group of) individual investor(s).

The distinction between GHPs, PSP and WMPs is isomorphic to Chhabra's classification of the three main categories of risks that an individual investor faces (Chhabra (2005)), which he named *personal risks*, *market risks* and *aspirational risks.*

The *personal risk* bucket is a broad category that includes events specific to an individual or family, which can have a material financial impact on their wealth. Protecting an investor against personal risk means protecting the investor against the anxiety of a dramatic decrease in the investor's lifestyle. As a result, an individual may be willing to accept a low real return on a portfolio designed to help hedge these risks, a portfolio which is not designed to generate upside potential but instead offer protection against downside risk relative to the particular goals identified by the investor. More formally, the personal risk bucket is defined as the risk bucket containing all essential GHPs. On the other hand, investors need to take on *market risks*, and collect the associated risk premia, in order to grow with their wealth segment and maintain their standard of living. The design of this performance portfolio should be entirely dedicated to the efficient extraction of market risks via diversification so as to eliminate, or at least reduce as much as possible, the presence of unrewarded specific risk, and therefore increase the risk-adjusted performance (Sharpe ratio in the mean-variance context). In other words, the market risk bucket contains all tradable risky assets that are held by an investor for performance purposes, in contrast to those held for hedging purposes. Note that this particular building block should in principle be identical for all investors, since it is meant to capture broad market risks as opposed to risks related to a particular investor's specific goals. Finally, the *aspirational risk* bucket contains, if any, all assets privately held by investors, which have a strong upside potential and are typically the driving force that allows investors to achieve wealth mobility objectives within or across affluence segments. Typical examples of assets held within the aspirational risk bucket are human capital, stock and stock option compensation packages, ownership stakes in privately held companies, etc.

In what follows, we discuss these portfolios in more detail.

2.5.1. Essential Goal-Hedging Portfolios and Personal Risk Bucket

We first focus on the building blocks dedicated to the protection of essential goals.

To each replicable goal is associated one suitably designed GHP. The general objective assigned to this portfolio is to secure the goal with certainty. Its nature depends on the goal. For a consumption-based goal, the GHP is a bond whose coupon payments match the consumption expenses, or equivalently, a portfolio of pure discount bonds (see Proposition 4). For a wealth-based goal with multiple horizons, the GHP is a roll-over of exchange options which expire on the goal dates (see Proposition 3). This result simplifies in some instances (see Corollary 1), e.g. if the goal actually has a single horizon: the GHP is a zero-coupon bond which pays the minimum wealth level at horizon.

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The following list provides selected examples of goal-hedging portfolios:

- If the goal is to avoid ending up homeless in case of a major financial downturn, then the GHP is a residential home;
- If the goal is to avoid starvation in case of a major financial downturn, then the GHP consists of cash holdings;
- If the goal is a (minimum/target) nominal or inflation-adjusted wealth level at horizon, then the GHP is a (nominal or real) pure discount bond with maturity date corresponding to the goal/investment horizon;
- If the goal is a (minimum/target) consumption level at all dates, then the GHP is a portfolio of pure discount bonds, or an annuity if the terminal date is the investor's uncertain date of death;
- If the goal is a (minimum/target) nominal or inflation-adjusted wealth level at the end of every year until horizon, then the GHP is a roll-over of one-year (nominal or real) bonds;
- If the goal is a max drawdown level, then the GHP is cash (which actually is a super-replicating portfolio in this case, in the sense that its performance will strictly dominate the performance of the flat floor).

From a theoretical perspective, cash is not necessarily the asset suitable for protecting a minimum level of wealth at all times. If the objective is to secure a minimum level of wealth at a given horizon, the safe asset is a zero-coupon bond with horizon date matching the investment horizon. If the objective is to protect a minimum level of wealth at various horizons, the safe asset is a roll-over of exchange options, as shown by Proposition 3. On the other hand, cash may appear to investors as safe in the sense that its value never decreases, while the value of a zero-coupon bond may fluctuate in response to interest rate changes. More importantly, a zero-coupon may be subject to the default risk of the issuer, a risk that can be regarded as credible even for sovereign issuers. Finally, funds held in the form of numeraire are perfectly liquid, while the sale or the purchase of a zero-coupon will incur transaction costs. Overall, the holding of a cash reserve can be justified to finance an essential goal that can be related to a minimum wealth level at all dates.

By definition, the initial value of the GHP is the minimum capital to invest in order to secure the goal, and a goal is affordable if, and only if, the investor's wealth covers this minimum amount. If the goal is not affordable, the GHP may still be included in the investor's strategy, since it may secure a lower level of consumption or wealth, which is then formally regarded as the affordable essential goal for the investor.

In closing, GHPs tend to be concentrated portfolios with potentially unattractive performance, and their raison d'être is to ensure the highest possible probability of achieving some essential goals.

By definition, the assets that are held to secure (implicit or explicit) essential goals form the "personal risk bucket" of the investor. In addition to home ownership and holdings of a reserve of cash, GHPs which correspond to explicit essential goals are typically financial assets such as bond portfolios. In the case studies, the initial personal wealth designates the aggregate value of the non-tradable assets held to secure the implicit goals (residence and cash account). Hence, this wealth is already assigned to the protection of implicit goals, and cannot therefore be regarded as available to secure other goals.

2. Efficient Allocation Within Risk Buckets

2.5.2. Performance-Seeking Portfolios and Market Risk Bucket

Unlike GHPs, performance-seeking portfolios (PSPs) are well-balanced portfolios enjoying a high reward per unit of risk, which provide access to a fundamental source of performance.

2.5.2.1. Performance Portfolios

If the initial wealth is exactly sufficient to fully finance all GHPs, then the investor will not be able to achieve relative upside potential in the absence of additional contributions. Given the need to generate performance so as to reach important and aspirational goals with a non-zero probability, it is in general desirable for investors to allocate some fraction of their assets to a well-diversified PSP, in an attempt to benefit from risk premia on risky assets across financial markets.

Diversification (as opposed to *hedging*) is the risk management technique that allows investors to efficiently extract long-term risk premia out of performance-seeking assets. Indeed, by holding well-diversified portfolios, investors may be able to eliminate or at least reduce (*diversify away*) unrewarded risk in their portfolios, which allows them to enjoy higher rewards per unit of risk, and therefore a higher average funding ratio at horizon for a given risk budget.

While the benefits of diversification are intuitively clear, there is no straightforward definition of what exactly a well-diversified portfolio is. The most common intuitive explanation of naive diversification is that it is the practice of not "putting all eggs in one basket". Having eggs (dollars) spread across many baskets is, however, a rather loose prescription.[21] It should be noted, fortunately, that a fully unambiguous definition of *scientific* diversification has been provided by Modern Portfolio Theory: more precisely, the prescription is that the PSP should be obtained as the result of a portfolio optimisation procedure aiming to generate the highest risk-reward ratio. Portfolio optimisation is a straightforward procedure, at least in principle. In a mean-variance setting, for example, if there are no restrictions on leverage or short sales, the prescription consists of generating an MSR portfolio based on expected return, volatility, and pairwise correlation parameters for all assets to be included in the portfolio. Formally, and with the notation introduced in Section 2.1, the MSR portfolio is defined as:

$$\underline{w}_{MSR,t} = \frac{\underline{\Sigma}_t^{-1}\underline{\mu}_t}{\underline{1}'\underline{\Sigma}_t^{-1}\underline{\mu}_t}.$$

The vector $\underline{1}$ in the denominator is the vector of size $N\times 1$ filled with 1. The denominator is adjusted to ensure that the sum of weights of the MSR equals 1.

Once a set of input parameters are given, the optimisation procedure can be handled analytically in the absence of portfolio constraints. More generally, it can be handled numerically in the presence of minimum and maximum weight constraints. Introducing weight constraints can actually be regarded as a way to reduce estimation risk (see for example Jagannathan and Ma (2003)), which is a key issue in practice, especially for expected return parameters (see Merton (1980)).

2.5.2.2 Portfolio Diversification Across and Within Asset Classes

The standard alternative approach widely adopted in investment practice consists instead of first grouping individual securities in various asset classes as well as sub-classes according to various dimensions, e.g. country, sector, and/or style within the equity universe, or country, maturity,

21 - See Deguest, Meucci and Santangelo (2013) for a detailed introduction to factor risk parity strategies based on a formal analysis of what is the true meaning of "many" and "baskets".

2. Efficient Allocation Within Risk Buckets

and credit rating within the bond universe, and subsequently generating the optimal portfolio through a two-stage process. On the one hand, investable proxies are generated for MSR portfolios within each asset class in the investment universe. We call this step the *portfolio construction step*. While market cap indices are natural default choices as asset class benchmarks, academic and industry research has offered convincing empirical evidence that these indices tend to exhibit a poor risk-adjusted performance, because of the presence of an excessive amount of unrewarded risk due to their extreme concentration in the largest cap securities in a given universe, as well as the absence of a well-managed set of exposures with respect to rewarded risk factors (for example, cap-weighted indices have a natural large cap and growth bias, while academic research – such as the seminal work by Fama and French (1992) – has found that small cap and value were instead the positively rewarded biases). The combination of these empirical and theoretical developments has significantly weakened the case for market cap-weighted indices (Goltz and Le Sourd (2011)), and a consensus is slowly but surely emerging regarding the inadequacy of market cap-weighted indices as efficient investment benchmarks. In this context, a new paradigm known as smart beta equity investing has been proposed, the emergence of which blurs the traditional clear-cut split between active versus passive equity portfolio management (see for example Amenc, Goltz, et al. (2012)).

After efficient benchmarks have been designed for various asset classes or sub-classes, these building blocks can be assembled in a second step, the asset allocation step, to build a well-designed multi-class PSP. It should be noted that an interesting new framework, known as *risk allocation* framework, is increasingly used at the asset (or factor) allocation stage.

2.5.2.3 From Asset Allocation to Risk Allocation

This trend is related to the recognition, supported by recent research (e.g. Ang, Goetzmann and Schaefer (2009)), that risk and allocation decisions could be best expressed in terms of rewarded risk factors, as opposed to standard asset class decompositions, which can be somewhat arbitrary. More generally, given that security and asset class returns can be explained by their exposure to pervasive systematic risk factors, looking through the asset class decomposition level to focus on the underlying factor decomposition level appears to be a perfectly legitimate approach, which is supported by standard asset pricing models such as the intertemporal CAPM (Merton (1973)) or the arbitrage pricing theory (Ross (1976)). If the whole focus of portfolio construction is ultimately to harvest risk premia that can be expected from holding an exposure to rewarded factors, it seems natural indeed to express the allocation decision in terms of such risk factors.

In this context, the term "risk allocation" is a new paradigm advocating that investment decisions should usefully be cast in terms of risk factor allocation decisions, as opposed to asset class allocation decisions. A second interpretation for what the risk allocation paradigm might mean is to precisely define it as a portfolio construction technique that can be used to estimate what an efficient allocation to underlying components (which could be asset classes or underlying risk factors) should be. The starting point for this novel approach to portfolio construction is the recognition that a heavily concentrated set of

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risk exposures can be hidden behind a seemingly well-diversified allocation. In this context, the risk allocation approach to portfolio construction, also known as the risk budgeting approach, consists in advocating a focus on risk, as opposed to dollar, allocation. In a nutshell, the goal of the risk allocation methodology is to ensure that the contribution of each constituent to the overall risk of the portfolio is equal to a target risk budget. In the specific case when the allocated risk budget is identical for all constituents of the portfolio, the strategy is known as risk parity, which stands in contrast to an equally-weighted strategy that would recommend an equal contribution in terms of dollar budgets (see Roncalli (2013) for further details).[22]

2.5.2.4. Definition of the Market Risk Bucket

The market bucket consists of assets that are held for performance purposes, as opposed to being held for the purpose of securing an essential goal. This corresponds to assets which can be traded in the market, regardless of their liquidity, and could be in the custody of a financial advisor representative, such as equity and bond indices, federal, municipal and corporate bonds, mutual funds, hedge funds, etc. Ideally, these assets should be held in the form of a well-diversified portfolio, that is, some proxy for the MSR portfolio constructed using one of the aforementioned approaches. Illiquid assets such as private equities or hedge funds are considered as part of the market risk bucket, since they tend to be relatively diversified attempts to harvest market risk premia, including alternative risk premia not easily accessible with traditional investment vehicles in a long-only format.

Given that it contains the most liquid assets, the market risk bucket is the place in which excess non-portfolio income is re-invested and from which funds are withdrawn to finance non-essential consumption plans (the essential goals being, by definition, financed with personal assets).

2.5.3. Wealth Mobility Portfolios and Aspirational Risk Bucket

The third risk bucket contains wealth mobility portfolios, which are typically strongly concentrated positions in illiquid privately held assets, which are held for wealth mobility purposes and are not intended as proxies for efficient portfolios in the sense of portfolio theory.

2.5.3.1. Wealth Mobility Portfolios

In the absence of leverage constraints, if an investor wants to achieve an exceedingly high expected return needed to allow for wealth mobility with a positive probability, the efficient approach would involve a leveraged allocation to the mean-variance efficient PSP. In the presence of leverage constraints, however, the use of a dedicated portfolio with a concentrated exposure to high performance assets will be needed to deliver returns materially higher than those of a diversified portfolio of asset classes, returns that are needed to achieve a given ambitious goal with a positive probability. In the limit case of a required target expected return equal to the highest expected return of all assets, then the performance portfolio will be 100% invested in that particular asset, and therefore will be poorly diversified and not particularly attractive in terms of Sharpe ratio.[23] These concentrated speculative portfolios are typically restricted to assets that are already held by investors through the human capital component of their wealth, as opposed to being regarded as portfolios to be optimally designed by financial advisors.

22 - Orthogonalising the factors is useful to avoid the arbitrary attribution of overlapping correlated components in the definition of risk budgets allocated to each of these factors. Principal component analysis (PCA) can be used to extract uncorrelated versions of the factors starting from correlated asset or factor returns. Alternatively, to avoid the difficulties related to the lack of stability and interpretability of principal components, and to generate uncorrelated factors that are as close as possible to the original assets or factors, one can use the minimal linear torsion (MLT) approach recently introduced in Deguest, Meucci and Santangelo (2013).

23 - If the target expected return is higher than the highest expected return of all assets, including privately held businesses, then no performance portfolio can be designed to allow the investor to achieve their exceedingly ambitious performance goals, that is unless their human capital portfolio, which itself tends to be a heavily concentrated portfolio in terms of risk exposure, allows them to do so.

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2.5.3.2. Definition of the Aspirational Risk Bucket

The aspirational risk bucket consists of assets without an immediate publicly available price and traditionally not managed by a financial advisor representative. Examples include the human capital, a stock option compensation, a privately held business, an art collection, a piece of land, etc. Some of these assets may be held for wealth mobility purposes. Their presence reflects the desire, or at least the potential, to achieve ambitious wealth levels that are not attainable with a mere efficient harvesting of risk premia in the presence of leverage constraints, and which can only be achieved through investments that involve a large amount of idiosyncratic risk. More often than not, the individual investor does not expect a financial advisor to manage this pool of assets, which represent a portfolio that the investor holds for reasons that extend beyond the standard desire to generate performance from financial markets.

Overall, the three risk buckets represent a collectively exhaustive and mutually exclusive partition of the investor's wealth. The investor's total wealth is split across three buckets. It should be noted that a correspondence exists between risk buckets and goals. Indeed, while all assets contribute to the achievement of all goals, assets in the personal risk bucket are by definition required to ensure the achievement of essential goals with probability 1. In the same vein, assets in the market risk bucket contribute significantly to the achievement of important goals, while speculative assets are required to ensure the achievement of aspirational goals such as wealth mobility goals. In other words, one might loosely think of assets in the market portfolio as focusing on the body of the distribution of the wealth (or wealth relative to goal values), while safe assets in the personal risk bucket focus on the left tail of the distribution and risky assets in the aspirational risk bucket focus on the right tail of the distribution.

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3. Efficient Allocation Across Risk Buckets

The analysis presented in the previous section has allowed us to define the efficient composition of the risk buckets that every investor should have. The outstanding question that remains to be analysed is the design and implementation of an efficient allocation across risk buckets at different points in time.

From a general perspective, the definition of an allocation strategy is guided by the following two principles:

- The strategy must secure all essential goals;
- It should lead to the highest possible success probabilities for non-essential goals, i.e. important or aspirational goals.

The first principle is a clear prescription, keeping in mind that we require that the 100% success probability for essential goals be robust with respect to assumptions on the dynamics of asset prices and to parameter choices. In other words, if implementation frictions lead to the existence of shortfalls with respect to the goals, such shortfalls should in principle be non-existent, and in practice should be limited in size and probability.

The second principle, on the other hand, is a somewhat vague recommendation, stating that while the protection of non-essential goals is (by definition) not required, the chosen investment strategy should generate a reasonably high chance to reach them.
It turns out that the problem of finding optimal strategies in the presence of a goal has been extensively studied in the academic literature. In Section 3.2, we precisely present a series of theoretical optimality results drawn from the literature, and in Section 3.3, we introduce implementable heuristic proxies for theoretically optimal strategies.

3.1 From Buy-and-Hold to Dynamic Allocation Strategies

The broad question that we face here is to define what to do with the excess of liquid wealth, if any, which is left available after all essential goals have been secured within the personal risk bucket.

In the context of goals-based wealth management, one natural benchmark strategy consists in securing all essential goals, and investing the available liquid wealth (that is wealth in the market risk bucket, or equivalently the investor's total wealth minus the wealth held in the personal and aspirational risk buckets) in a performance portfolio allowing for the most efficient harvesting of market risk premia, that is, a proxy for the MSR portfolio.

We now describe in more detail this *buy-and-hold* strategy. Consider a wealth-based or consumption-based goal with no income coming from sources outside the portfolio. Once the GHP has been identified, the simplest way to secure the goal is to purchase the GHP at date 0 and to invest the remainder of liquid wealth in some PSP. This is a buy-and-hold strategy, which generates the following wealth at date t whether the goal is wealth-based or consumption-based:

$$\boldsymbol{A_{BH,t} = (A_0 - GHP_0)A_{PSP,t} + GHP_t,} \quad (11)$$

where $A_{PSP,t}$ is the value of the PSP with an initial investment of \$1. Note that the value of the GHP at date 0 is equal to the present value of the goal on this date, and that the goal is clearly secured by this strategy.

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It is known that the Strategy (11) can be interpreted as a constant proportion portfolio insurance (CPPI) with a multiplier equal to 1. To see this, it suffices to note that the weights are given by:

$$\underline{w}_{BH,t} = \left(1 - \frac{GHP_t}{A_{BH,t}}\right)\underline{w}_{PSP,t} + \frac{GHP_t}{A_{BH,t}}\underline{w}_{GHP,t},$$

where $\underline{w}_{PSP,t}$ denotes the weight vector of the PSP and $\underline{w}_{GHP,t}$ that of the GHP.

Clearly the strategy defined in Equation (11) is not the only one that leads to securing essential goals with probability 1. Also, it is not necessarily the strategy that leads to the highest probability of achieving important and aspirational goals among all strategies that secure the essential goals with probability 1. The intuition actually suggests that the buy-and-hold strategy is a specific example of a wider class of *dynamic* GBI strategies, which advocate that the allocation to the market risk bucket versus the personal risk bucket should be taken as a multiple different from 1 of the current wealth in excess of the present value of the goal. In fact, one can replace the GHP value by the present value of the goal, which is defined in Section 1.1 as the minimum amount of liquid wealth required to secure the goal (it coincides with the GHP value in the case of a consumption-based goal or a wealth-based goal with a single horizon).

More generally, the weights of a strategy that secures the essential goal(s) with probability 1 can be as follows:

$$\underline{w}_t = f(A_t, \tilde{G}_t)\underline{w}_{PSP,t} + [1 - f(A_t, \tilde{G}_t)]\underline{w}_{GHP,t},$$

where $f(A_t, \tilde{G}_t)$ is some function of current wealth and goal present value. For the goal to be secured, the strategy must keep the ratio $A/\tilde{G}$ above one at all times. Indeed, if the ratio falls below one, the goal becomes non-affordable, and thus becomes an aspirational one.

For the ratio to stay above one, it makes intuitive sense that the volatility of the ratio $A/\tilde{G}$ must shrink to zero when the ratio approaches one from above. Otherwise, unexpected fluctuations may occur that would cause the ratio to fall below one. By Ito's lemma, the volatility vector of $A/\tilde{G}$ is $\underline{\sigma}_t[\underline{w}_t - \underline{w}_{GHP,t}]$. Because the volatility matrix, $\underline{\sigma}_t$, is by nature independent from wealth, the only way to have zero volatility when $A/\tilde{G}$ gets close to 1 is to cancel the difference $[\underline{w}_t - \underline{w}_{GHP,t}]$. Thus, $f(A_t, \tilde{G}_t)$ has to shrink to zero as the difference $[A_t - \tilde{G}_t]$ approaches zero.

The heuristic line of reasoning suggests that a necessary and sufficient condition required for ensuring the protection of essential goals is that the investor's wealth be fully invested in the essential GHP in case the amount of available wealth is exactly equal to the minimum wealth required to secure the goal with the corresponding strategy. The simplest specification that satisfies this property is a linear function: $f(A_t, \tilde{G}_t) = m(A_t - \tilde{G}_t)$, were we note that the case m=1 is the static benchmark strategy introduced above.

While more complex functional forms that satisfy the limit condition at zero can be considered, we argue in the next section that such strategies (defined such that the allocation to the market risk bucket is taken to be a multiple to the distance between current wealth and the present value of the essential goals) are of particular relevance, not only because they represent

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the simplest form of GBI strategies and require no unobservable parameters as inputs, but also because they actually coincide with the formal solution to an expected maximisation problem with (implicit) goals for a leverage-constrained myopic investor.

Before turning to the formal analysis of the GBI strategies in the academic literature, it should be noted that in the presence of income, the buy-and-hold strategy has to be modified. As discussed in Section 2.2.4, a strategy that secures the retirement goal of horizon T_r is the following:

- At date 0, purchase the option which pays $\left(\tilde{G}_{T_r} - \sum_{s \le T_r} y_s\right)^+$ on date T_r and invest the remainder of wealth in the PSP;
- At each income date (T_j), purchase the option which pays $\left(\tilde{G}_{T_r} - \sum_{T_j < s \le T_r} y_s\right)^+$ on date T_r and invest the remainder of wealth in the PSP.

Since a rebalancing takes place on each income date, this is not a buy-and-hold policy, but a roll-over of buy-and-hold strategies (in the sense that the portfolio is buy-and-hold between dates T_j and T_{j+1}).

3.2 Review of the Related Literature

In addition to the aforementioned seminal work by Chhabra (2005), as well as recent papers on asset-liability management in private wealth management (see for example Reichenstein (2006), Reichenstein and Jennings (2003), Wilcox, Horvitz and DiBartolomeo (2006) or Amenc et al. (2009)), our paper is related to two main strands of the literature. The first strand of papers, mostly published in mathematical finance or operations research journals, focus on investment solutions that are meant to optimise the value of some ad-hoc criterion such the minimisation of shortfall probability, or the minimisation of expected shortfall with respect to a given objective. It should be noted that these papers often solve these programs in an option hedging context, where the problem is to find an optimal hedging strategy in an incomplete market setting where the perfect hedging strategy does not exist. The second strand of the literature, mostly published in finance journals, focuses on mainstream expected utility maximisation while the presence of the goal is accounted for by the introduction of performance constraints. In Sections 3.2.1 and 3.2.2, we present these two strands of the literature separately, and we then comment on their similarities.

3.2.1 Goals-Based Allocation Strategies

In this section, we provide a broad overview of the papers that have considered wealth-based goals with a single horizon.[24] The mathematical formulation of this goal is $A_T \ge G_T$, G_T being the minimum wealth level. The first natural objective is to maximise the success probability, i.e. the probability of reaching the goal:

$$\max_{\underline{w}} \mathbb{P}(A_T \ge G_T) \quad \text{subject to (3).} \qquad (12)$$

This problem is only interesting for non-affordable goals (which are aspirational in the sense of Definition 10), that is, goals that cannot be reached with probability 1. If a goal is affordable, there exists at least one strategy that yields a success probability of 1, so a solution to Program (12) is the strategy that secures the goal.

24 - The reader is referred to Milhau (2014) for a more detailed presentation of the results from this literature, including the derivations of the solutions to the optimisation programs.

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This is the reason why the literature has focused on the case where the goal is not affordable. Föllmer and Leukert (1999) solve for the optimal payoff, A_T^*, both in complete and incomplete markets. For simplicity, we focus on the complete case. The optimal payoff is that of a digital option, which involves the value of the growth-optimal portfolio, which is defined as the portfolio strategy that maximises the expected logarithmic return at horizon T:

$$\max_{\underline{w}} \mathbb{E}\left[\ln \frac{A_T}{A_0}\right]$$
$$\text{subject to (3).} \qquad (13)$$

A well-known result (see e.g. Long (1990)) states that the solution is:

$$\underline{w}_{go,t} = \Sigma_t^{-1}\underline{\mu}_t.$$

It can also be written as a function of MSR weights:

$$\underline{w}_{go,t} = \frac{\lambda_{MSR,t}}{\sigma_{MSR,t}}\underline{w}_{MSR,t},$$

$\lambda_{MSR,t}$ and $\sigma_{MSR,t}$ being the Sharpe ratio and the volatility of the MSR (see Amenc et al. (2010)). Unlike the MSR portfolio, the growth-optimal portfolio policy is in general not fully invested in the risky assets, and involves cash. It is sometimes called "myopic" in the literature because the weights do not depend on the investment horizon.

The probability-maximising payoff can then be written as:

$$A_T^* = G_T \times \mathbb{I}_{\left\{\frac{A_{go,T}}{G_T} \geq \frac{A_0}{KG_0}\right\}}. \qquad (14)$$

The constant K in this equation is adjusted in such a way that the budget constraint $\mathbb{E}[M_T A_T^*] = A_0$ holds.

Finding the optimal strategy requires computing the Greeks of the digital option. This can be carried out analytically only under restrictive assumptions on parameter values. For instance, having stochastic risk premia will imply that A_{go} has stochastic volatility, which prevents from obtaining a closed-form expression. A stochastic interest rate permits an analytical computation to the extent that the ratio $A_{go,T}/G_T$ remains log-normal, which is satisfied for example if the short-term interest rate follows a Gaussian model (e.g. as in Vasicek (1977)), but not if it follows a square-root process (as in Cox, Ingersoll and Ross (1985)). Browne (1999) derives the hedging strategy under the assumption that the goal is constant, and that expected returns, volatilities and the short-term rate are deterministic functions of time, which ensures that $A_{go,T}/G_T$ is log-normal and allows for the use of the Black and Scholes (1973) option pricing formula. The optimal strategy involves the MSR portfolio and cash. Since the risk-free asset coincides with the GHP when the goal is constant and the short-term rate is deterministic, it is equivalent to say that the optimal strategy is a dynamic combination of the MSR and the GHP. A remarkable aspect of these strategies is that the allocation to the former building block is decreasing in the ratio of current wealth to goal value. In other words, one allocates more to the "performance" block if wealth is far from the goal.

The payoff (14) has a clear drawback, which is that it can take the value zero with positive probability. It should be emphasised that this property is not penalised by the objective function in (12), and zero outcomes are tolerated as long as they allow the investor to increase the success probability. Nevertheless, a

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zero terminal wealth means that investor loses all money invested, which is hardly acceptable for most individuals. To address the bankruptcy issue, Browne (1999) maximises the success probability subject to the constraint that wealth remains above a floor, which represents the wealth necessary to afford a minimum standard of living. The optimisation program reads:

$$\max_{\underline{w}} \mathbb{P}(A_T \geq G_T)$$

$$\text{subject to (3) and } A_T \geq F_T \qquad (15)$$

The payoff F_T can be thought of as an essential wealth-based goal. As for the goal G_T, we assume that F_T is replicable with a "floor-hedging portfolio" (FHP) $\underline{w}_{FHP}$, an assumption that is made because it ensures that there exists at least a strategy that satisfies the constraint $A_T \geq F_T$ almost surely.[25] We let $\tilde{F}_t$ denote the present value of the payoff F_T, which is the wealth obtained by investing $\tilde{F}_0$ in the FHP. Browne (1999) solves (15) when F_T is a fraction less than 1 of G_T. The following proposition slightly extends his result by providing the optimal payoff when F_T is simply assumed to be less than G_T. However, we follow Browne in assuming that the floor and the goal are proportional in order to write the optimal strategy.

25 - The assumption of replicability is not necessary to ensure the existence of such a strategy. The necessary and sufficient condition is the super-replicability of the floor.

Proposition 10 (Probability-Maximising Strategy with a Floor).
Assume that:
- *The market is complete;*
- *$0 \leq F_T < G_T$ almost surely;*
- *$\tilde{F}_0 \leq A_0 \leq \tilde{G}_0$;*
- *There exists a constant K such that $\mathbb{E}[M_T X^*] = A_0$, where*

$$X^* = F_T + (G_T - F_T) \times \mathbb{I}_{\left\{A_{go,T} \geq \frac{A_0}{K(\tilde{G}_0 - \tilde{F}_0)}(G_T - F_T)\right\}},$$

Then X^* is the optimal payoff in (15).

Assume in addition that $F_T = \alpha G_T$ for some $0 \leq \alpha < 1$, so that $\underline{w}_{FHP} = \underline{w}_{GHP}$, and that the vectors $\underline{\lambda}_t$ and $\underline{\sigma}_t \underline{w}_{Gt}$ are deterministic functions of time. Then, the optimal strategy is:

$$\underline{w}_t^* = \varphi_t \underline{w}_{go,t} + (1 - \varphi_t) \underline{w}_{GHP,t},$$

with:

$$\varphi_t = \frac{1}{\kappa_{t,T}} \frac{\tilde{G}_t - \tilde{F}_t}{A_t^*} n \left[\mathcal{N}^{-1} \left(\frac{A_t^* - \tilde{F}_t}{\tilde{G}_t - \tilde{F}_t} \right) \right],$$

$$\kappa_{t,T} = \sqrt{\int_t^T \left\| \underline{\lambda}_s - \underline{\sigma}_s \underline{w}_{GHP,s} \right\|^2 ds},$$

n and $\mathcal{N}$ being respectively the probability distribution and the cumulative distribution functions of the standard normal distribution.

Proof. See Appendix 6.2.1.
Hence, the optimal payoff is of the digital type, as in the absence of a floor constraint, but the outcome zero is replaced by the floor value. It is also shown in Appendix 6.2.1 that the optimal allocation to the growth-optimal performance portfolio shrinks to zero in two situations: if current wealth approaches the goal present value ($\tilde{G}_t$) or the floor present value ($\tilde{F}_t$). In other words, whenever wealth approaches one of the lower or upper bounds, the agent invests only in the "safe portfolio" to secure the prevailing wealth level and prevent it from exceeding the target value of dropping below the floor value.

Another way of penalising low wealth levels is to minimise some measure of shortfall size. Following this idea, Cvitanic (2000) minimises the expectation of the discounted shortfall:

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$$\min_{\underline{w}} \mathbb{E}\left[\frac{(A_T - G_T)^+}{S_{0T}}\right].$$

The optimal payoff he obtains corresponds to a digital option, which pays either the goal value, or some fraction (comprised between 0 and 1) of the goal value. It is clear that by avoiding bankruptcy, such a payoff leads to lower shortfalls than the probability-maximising one. Föllmer and Leukert (2000) solve a related problem, which is to minimise the expectation of some loss function of the shortfall, a formulation that nests the expected shortfall minimisation as a special case. But again, an explicit computation of the optimal strategy is only possible in a simple Black-Scholes setting. In addition to these objectives, the literature on optimal option hedging also suggests a number of alternative criteria to be minimised. For instance, Föllmer and Schweizer (1990) minimise the expected squared difference between final wealth and the goal defined as an option payoff that has been sold to a counterparty. From a technical and mathematical perspective, this problem is easier to solve than the minimisation of expected shortfall. On the other hand, one conceptual problem with the "mean-variance" hedging criterion is that it equally penalises upside and downside deviations from the goal.

A related strand of the literature has considered another somewhat related problem, which is to minimise the expected time to reach a goal. While the expected time to success is not a standard risk management indicator, using it as an optimisation criterion reflects the idea that an investor seeks to secure a goal that was not initially affordable as soon as possible. In this case, the goal is represented by a process $(G_t)_{t \geq 0}$, and the problem has an infinite horizon. Note that this goal is a wealth-based goal with multiple horizons. Mathematically, the time to success is defined as the first hitting time of the goal process by the wealth process, and the objective to minimise is the expectation of this time:

$$\min_{\underline{w}} \mathbb{E}[\tau] \quad \text{subject to (3)},$$
$$\tau = \inf\{t \geq 0\,;\, A_t \geq G_t\}. \qquad (16)$$

If $A_0 \geq G_0$, then the expected time to success is trivially zero, for any strategy. In other words, strategies are indistinguishable. The non-trivial case is when $A_0 < G_0$. It has been solved by Heath and Sudderth (1984) in the case of constant risk and return parameters (i.e. constant volatilities and expected returns for risky assets and constant interest rate) and a constant goal. The optimal policy is shown to be the growth optimal strategy that is a mixture of the MSR and cash.

The solution to this problem also confirms the intuitive property that the minimal expected time is decreasing in the Sharpe ratio of the MSR portfolio (which is a fixed-mix given the model's assumptions). However, the extension of such results to more general economies, with possibly time-varying parameters, is a formidable challenge. Kardaras and Platen (2010) propose to solve a modified version of the problem, in which the hitting time is measured according to a "market clock", as opposed to the usual calendar time. The speed at which market time flows is proportional to the squared Sharpe ratio of the MSR portfolio. Hence, market time flows faster when investment opportunities are "good". In this context, the growth-optimal portfolio is still shown to be optimal.

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Because the market clock is equivalent to the calendar clock when parameters are constant, this result encompasses that of Heath and Sudderth (1984). Another variant of the original problem which leads to a closed-form solution is studied in Aucamp (1977). Overall, Problem (16) is very hard to solve explicitly in its original form.

Another criticism that can be made to the solution to (16) is that there is no guarantee that the goal is secured once it has been reached. Indeed, let us consider the case of constant parameters. In this context, the growth-optimal portfolio guarantees the shortest time to success, but even after it has reached the goal, its value can fall below the goal value. This can be regarded as an undesirable property since one might expect from a GBI strategy not only that it reaches the goal as soon as possible, but also that it secures it after the first hitting time. The second property is not taken into account in (16), but adding it as a constraint in the optimisation would likely further hinder the mathematical tractability of the problem.

3.2.2 Expected Utility Maximisation

The second strand of literature takes as an objective the maximisation of expected utility in the presence of a goal. This can be done in various ways, depending on how the goal is incorporated in the optimisation program. Broadly speaking, the goal can be introduced either directly in the objective function or in additional constraints. In this section, we briefly review the properties of utility-maximising strategies in the absence of a goal, and we then present a set of optimality results that account for the presence of the goal, which are of a highest degree of relevance in the context of goals-based wealth management.

3.2.2.1 Utility Maximisation in the Absence of a Goal

A general formulation of the expected utility maximisation problem reads:

$$\max_{\underline{w}} \mathbb{E}[U(A_T)], \text{ subject to (3).} \qquad (17)$$

This version assumes that the portfolio is self-financing. Another form, which allows for consumption, reads:

$$\max_{\underline{w}} \mathbb{E}[U(A_T)], \text{ subject to (4).} \qquad (18)$$

It should be noted that this formulation differs from the one of Samuelson (1969), who seeks to maximise intertemporal utility derived from consumption. Indeed, we take in this paper consumption as an exogenous variable, rather than a control variable. In other words, the consumption payments are fixed and cannot be optimised over. It turns out that the solution to (18) can be expressed as a simple function of the solution to (17) (see Proposition 14 below). Hence, we focus in what follows on Program (17).

The seminal contributions on utility maximisation in an intertemporal setting are Samuelson (1969), Merton (1971, 1973). These papers solve the optimisation program via the dynamic programming approach, which produces a non-linear partial differential equation (the Hamilton-Jacobi-Bellman equation). This equation can only be solved under specific assumptions on model parameters. A first important finding is that if all risk and return parameters (i.e. the short-term interest rate, the expected returns and the volatilities in (1)) are constant, and the utility function is of the Constant Relative Risk Aversion (CRRA) type, that is:

$$U(x) = \begin{cases} \dfrac{x^{1-\gamma}}{1-\gamma} & \text{for } x > 0 \\ -\infty & \text{for } x \le 0 \end{cases},$$

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then the optimal policy is a fixed-mix strategy. In detail, the optimal weights are given by:

$$\underline{w}^* = \frac{1}{\gamma}\underline{w}_{go} = \frac{\lambda_{MSR}}{\gamma\sigma_{MSR}}\underline{w}_{MSR}. \qquad (19)$$

Since the weights do not sum up to 1 in general, cash is used to make the balance: the weight allocated to cash is $[1-(\underline{w}^*)'\underline{1}]$.

Merton (1971) also solves the problem (17) for a broader class of utility functions, known as the "Hyperbolic Absolute Risk Aversion" (HARA) functions. Note that the CRRA class of utility functions has an interesting property, namely that marginal utility of wealth grows to infinity as wealth approaches zero, as can be seen from the expression of the first derivative of the utility function:

$$U'(x) = x^{-\gamma},$$

This property, which means an investor starting with a low wealth will enjoy a large welfare improvement if given a small additional endowment, is important because it ensures that terminal wealth cannot be negative at the optimum. For some other utility functions, marginal utility stays bounded as wealth approaches zero. This is the case, for instance, with the Constant Relative Risk Aversion (CARA) function, also a member of the HARA class, defined as:

$$U(x) = \exp(-\alpha x).$$

With such utility functions, solving the utility maximisation Program (17) without a nonnegativity constraint on wealth leads to an optimal wealth which can take on negative values (see Merton (1992), Chap. 6). Given that terminal wealth at the final date is not admissible, since it would mean that the investor would exit with a non-repaid debt, the nonnegativity constraint has to be imposed explicitly.

An alternative to the dynamic programming technique which allows to easily incorporate the nonnegativity constraint on final wealth is the "convex duality", or "martingale" approach of Cox and Huang (1989). The first step in this approach consists in computing the optimal terminal wealth as a function of the state-price deflator. The second step is to find the replication strategy for this optimal payoff, which provides the optimal strategy. This approach is particularly simple in a complete market setting because this assumption guarantees the uniqueness of the state-price deflator and the attainability of any payoff (up to technical measurability and integrability requirements). On the other hand, solving for the optimal payoff in an incomplete market setting involves the computation of the "minimax state-price deflator", which is the optimal deflator chosen by the utility maximiser among the infinity of possible deflators (see He and Pearson (1991) for more details), and as such depends on investor's horizon and risk aversion.

In a complete market setting, Cox and Huang (1989) show that the optimal terminal wealth has the form:

$$A_T^* = [U'^{-1}(\nu M_T)]^+,$$

where U'^{-1} is the inverse of the marginal utility function, M is the unique state-price deflator and ν is a constant, which is implicitly given by the budget constraint $\mathbb{E}[M_T A_T^*] = A_0$. For utility functions with finite derivative at zero, the nonnegativity constraint is not binding, meaning that the optimal terminal wealth may be zero, which complicates the pricing of the

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optimal payoff, hence the finding of the optimal weights (Cox and Huang (1989)). For this reason, we follow a large body of the literature on optimal portfolio choice by assuming a CRRA function in the remainder of this paper.

When risk and return parameters are not constant, the fixed-mix policy (19) is no longer optimal, and it is not even optimal to update the weights with time-varying parameter values. Indeed, as shown by Merton (1973), the optimal portfolio in this context involves additionally involves a number of "intertemporal hedging demands". For instance, if the uncertainty in investment opportunities can be reduced to the variations in R "state variables" $X_1,\ldots, X_R$, then the optimal strategy has the form:

$$\underline{w}_t^* = \frac{\lambda_{MSR,t}}{\gamma\sigma_{MSR,t}}\underline{w}_{MSR,t} + \sum_{r=1}^{R} a_{rt}\underline{w}_{hed,r,t}, \quad (20)$$

where the portfolio $\underline{w}_{hed,r,t}$ is the hedging portfolio against variable X_r, defined as the portfolio invested in risky assets that maximises the correlation with unexpected changes in X_r.[26] The coefficients a_{rt} are functions of a number of variables, including subjective parameters (risk aversion and investment horizon) as well as objective parameters (current values of state variables and parameters that govern the dynamics of the variables). One important effect of the presence of stochastic investment opportunities is that the optimal strategy depends on the investment horizon, a dimension which is absent in the case of constant parameters.

The task of computing the coefficients a_{rt} is sometimes difficult, especially when there are multiple risk factors. It is not the purpose of this section to present explicit derivations for the hedging demands. This subject has been the focus of a large body of literature. Selected references, in which analytical expressions can be found, are included in the following (far from exhaustive) list:[27]

- Stochastic interest rate and constant risk premia (Brennan and Xia (2002)), Munk and Sorensen (2004), Martellini and Milhau (2012));
- Stochastic risk premia and constant interest rate (Kim and Omberg (1996)) and Wachter (2002));
- Multiple stochastic state variables, including notably stochastic interest rate and risk premia (Munk, Sorensen and Vinther (2004)), Sangvinatsos and Wachter (2005), Liu (2007), Detemple and Rindisbacher (2010) , Martellini and Milhau (2013) and Deguest, Martellini and Milhau (2014)).

For instance, in the presence of a stochastic interest rate, a demand arises for the zero-coupon bond that matures on the investor horizon date. This pure discount bond is the long-term safe asset for the investor, since it leads to a payoff with zero variance at horizon. In the presence of a stochastic equity risk premium, the optimal strategy (20) contains a hedging demand against unexpected changes in equity premium risk. Its role is to hedge against unexpected changes in investment opportunities (see Merton (1973)). The hedging demand against unexpected changes in the equity risk premium is the portfolio of risky assets that has the highest squared correlation with changes in the equity risk premium. The design of that portfolio is a matter of empirical calibration. It is widely accepted that an increase (respectively, decrease) in realised returns on equities implies a corresponding decrease

26 - Nielsen and Vassalou (2006) show that the state variables which induce a hedging demand are those that impact the position of the intertemporal capital market line (ICML), namely its intercept (the nominal short-term rate) or its slope (the Sharpe ratio of the MSR portfolio). In particular, a state variable that affect volatilities without impacting the ICML does not give rise to a hedging demand.

27 - This short list is restricted to papers that provide closed-form solutions. A number of papers also compute optimal portfolios numerically.

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(respectively, increase) in expected return of equities, given that, everything else equal, equity have become more (respectively, less) expensive following the market rise (respectively, fall). In this context, it is hardly surprising that empirical research has found a strong negative correlation between expected returns on stocks and realised returns on stocks (see e.g. Campbell and Viceira (1999), Barberis (2000) and Martellini and Milhau (2013)). As a result, the intertemporal hedging demand against changes in the equity risk premium mostly contains a long equity position that comes as an addition to the equity allocation already present in the MSR portfolio. The expression for this hedging demand is given in several papers (see e.g. Kim and Omberg (1996), Munk, Sorensen and Vinther (2004) and Martellini and Milhau (2013)).

We conclude this section by reviewing typical cases where the hedging demands are zero. First, this happens if investment opportunities are constant, since there is no need then to hedge. A second case is when the highest correlation that can be achieved with the state variable is zero, in which case there is a need to hedge but no ability to effectively hedge. A third case is when the risk aversion γ is equal to 1. Indeed, the CRRA function then coincides with the logarithmic function, so Program (17) is equivalent to maximising the expected logarithmic return (Program (13)). The solution to this problem is the growth-optimal portfolio scaled by $1/\gamma$.

3.2.2.2 Optimal Strategy in the Presence of a Wealth-Based Goal

We now turn to expected utility maximisation in the presence of a wealth-based goal. To the best of our knowledge, the financial literature has not provided analytical solutions to problems with multiple horizons, so we also focus on goals with a single horizon.

Choice of Objective. Having a wealth-based goal G_T means that the investor is concerned with the relative position of terminal wealth with respect to G_T. There are two main options to compare wealth to the goal value:

- The surplus, $A_T - G_T$;
- The funding ratio, A_T/G_T.

The idea is that an investor prefers large surpluses and high funding ratios, with as little uncertainty as possible. These preferences are captured by an increasing and concave utility function U. Thus, if Z denotes the quantity of interest (surplus or funding ratio), the portfolio choice problem consists in the determination of the strategy that maximises expected utility:

$$\max_{\underline{w}} \mathbb{E}[U(Z)], \text{ subject to (2).}$$

The choice of the variable of interest (surplus or funding ratio) is linked to the choice of the utility function. Indeed, if the selected function has marginal utility that grows to infinity near zero, then the random variable Z will be positive almost surely at the optimum. Thus, if Z has been chosen to be the surplus, the final surplus is positive with probability 1. By the absence of arbitrage opportunities, it follows that the initial wealth must be such that $A_0 > E[M_T G_T]$ for any state-price deflator. If this condition is not satisfied, then any strategy yields a positive probability that $A_T \leq G_T$, so that expected utility equals minus infinity. As a conclusion, the choice of a utility function with infinite derivative at 0 (such as the CRRA one) combined with

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the surplus as an objective does not allow for goals which are not affordable in the sense of Definition 1.

For these reasons, we take U to be the CRRA utility function, and Z to be the funding ratio. The fact that marginal utility grows to infinity at 0 ensures that the optimal terminal funding ratio is positive. In particular, the optimal terminal wealth is also positive, so the investor's problem reads:

$$\max_{\underline{w}} \mathbb{E}\left[U\left(\frac{A_T}{G_T}\right)\right], \text{ subject to (3).} \quad (21)$$

To see the impact of the presence of a goal on the optimal investment strategy, it will be useful to compare the solution to this program to the solution to an otherwise identical program without the goal. In other words, our objective here is to find relationships between the solutions to (17) and (21). For clarity, we denote by A^{*G} and $\underline{w}^{*G}$ the optimal wealth and optimal portfolio weights in the program with the goal (Equation (21)), and with A^{*0} and $\underline{w}^{*0}$ the analogous quantities for the program without the goal (Equation (17)).

Optimal Strategy. If markets are incomplete, there are infinitely many state-price deflators and each expected utility maximisation program carries a specific deflator, known as the minimax state-price deflator (He and Pearson (1991)). In particular, (17) and (21) may lead to different minimax deflators. Introducing market incompleteness would thus severely complicate the task of expressing the solution to (21) as a function of the solution to (17). We thus assume in what follows that markets are complete, which in turn ensures that the goal is replicable given a sufficient level of initial wealth.

Before we state the result comparing the solutions to both programs, we introduce an auxiliary stochastic process, which is the price $b_{t,T}$ of the zero-coupon bond that pays \$1 at date T. Its dynamics reads:

$$\frac{db_{t,T}}{b_{t,T}} = \left[r_t + \sigma_{b,t,T}\lambda_{b,t,T}\right]dt + \underline{\sigma}'_{b,t,T}d\underline{z}_t.$$

Proposition 11 (Optimal Payoff and Strategy with Wealth-Based Goal). *Assume that markets are complete. Then, the optimal payoff in (21) is:*

$$A_T^{*G} = \nu_1 A_T^{*0} G_T^{1-\frac{1}{\gamma}},$$

where $\nu_1 = \mathbb{E}\left[M_T^{1-\frac{1}{\gamma}}\right] \Big/ \mathbb{E}\left[(M_T G_T)^{1-\frac{1}{\gamma}}\right].$

Moreover, if $\underline{\lambda}_t$, $\underline{\sigma}_{b,t,T}$ *and* $\underline{\sigma}_{Gt} = \underline{\sigma}_t \underline{w}_{Gt}$ *are deterministic functions of time, the optimal strategy in (21) reads:*

$$\underline{w}_t^{*G} = \frac{\lambda_{MSR,t}}{\gamma\sigma_{MSR,t}}\underline{w}_{MSR,t} + \left(1 - \frac{1}{\gamma}\right)\underline{w}_{Gt},$$

$$\underline{w}_t^{*G} = \underline{w}_t^{*0} + \left(1 - \frac{1}{\gamma}\right)\left[\underline{w}_{Gt} - \underline{w}_{bt}\right],$$

where $\underline{w}_{bt}$ *is the portfolio that replicates the zero-coupon bond maturing at date T.*

Proof. See Appendix 6.3.1.

The assumption of a deterministic $\underline{\lambda}_t$ is satisfied if the Sharpe ratios and the pairwise correlations of risky assets are constant in time, as can be seen by rewriting this vector as:

$$\underline{\lambda}_t = \underline{U}_t \underline{\Omega}_t^{-1} \begin{pmatrix} \lambda_{1t} \\ \vdots \\ \lambda_{nt} \end{pmatrix},$$

where $\underline{\Omega}_t$ is the (instantaneous) correlation matrix of the assets and $\underline{U}_t$ is its Cholesky factor, i.e. the upper triangular matrix such that $\underline{\Omega}_t = \underline{U}_t'\underline{U}_t$. The condition on $\underline{\sigma}_{b,t,T}$

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depends on the dynamics of the short-term rate. If this rate follows the Vasicek model (Vasicek (1977)), then $\underline{\sigma}_{b,t,T}$ is deterministic. More generally, this condition is also satisfied if the short-term rate is a combination of two mean-reverting processes with constant volatilities (see Brennan and Xia (2002)). Finally, the assumption of a deterministic $\underline{\sigma}_{Gt}$ is verified if the final goal value is proportional to realised inflation, the price index follows a Geometric Brownian motion and the short-term rate follows the Vasicek model (see Martellini and Milhau (2012)).

Under these conditions, the optimal strategy is a combination of the MSR portfolio and the GHP (plus cash, which is the third fund). The allocation to the MSR is decreasing in the risk aversion, and for an infinite risk aversion, it is optimal to invest only in the GHP, as intuition suggests. This in turn yields a constant ratio A_T^{*G}/G_T, which was also expected: an infinitely risk averse investor seeks to minimise the dispersion of wealth relative to the goal, regardless of upside performance potential. This property is in fact very general, and extends beyond the present framework to general economies and utility functions (see Wachter (2003)). A similar fund separation between the MSR and the GHP result can be found in Martellini and Milhau (2012). The presence of a stochastic risk premium for one of the risky assets, e.g. a stock index, does not modify the optimal terminal wealth as long as the uncertainty in this risk premium is spanned, thus allowing for the markets to remain complete. On the other hand, it gives rise to a dedicated "hedging demand" in the optimal strategy for Program (17). In this context, the optimal strategy for Program (17) is a combination of the MSR, the zero-coupon bond maturing at date T, the equity premium-hedging portfolio and cash (see the references mentioned in Section 3.2.2). If a goal is introduced (as in Program (21)), the joint presence of the stochastic Sharpe ratio and the goal gives rise to interaction effects, so that the optimal strategy with the goal is not simply a combination of the optimal strategy without the goal, the GHP and the zero-coupon bond. On the other hand, it can still be shown that the optimal strategy is a combination of the same four funds as in Program (17), plus the GHP (see Martellini and Milhau (2013) for detailed expressions of optimal strategies with liabilities and a stochastic Sharpe ratio). However, the relationship between the solutions to Programs (17) and (21) is not as straightforward as in Proposition 11.

3.2.2.3 Optimal Strategy Securing an Affordable Wealth-Based Goal

Optimal Payoff and Strategy. Even if the goal is affordable, that is if $A_0 < \tilde{G}_0$, the optimal strategy of Proposition 11 does not secure it with full probability (that is, except for an infinitely risk-averse investor who would invest all available wealth in the GHP). Indeed, there is always a positive probability for the terminal wealth A_T^{*G} to be less than G_T. To secure the goal, one may solve (21) subject to the additional constraint that $A_T \geq G_T$ almost surely. More generally, one may impose a floor on terminal wealth, as was done for the program focusing on maximising the probability of success:

$$\max_{\underline{w}} \mathbb{E}\left[U\left(\frac{A_T}{G_T}\right)\right],$$
$$\text{subject to (3) and } A_T \geq F_T. \qquad (22)$$

As for the probability maximisation Program (15), we assume that the floor is replicable, the FHP being denoted by $\underline{w}_{FHP}$. For (22)

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to have a solution, it must be the case that $A_0 \geq \tilde{F}_0$, that is, the goal F_T must be affordable. Thus, in Program (22), F_T can be thought of as an essential goal, while G_T represents an important or aspirational goal.

Program (22) is a portfolio optimisation subject to a performance constraint. The solution is derived in various papers such as Tepla (2001), El Karoui, Jeanblanc and Lacoste (2005) and Deguest, Martellini and Milhau (2014), among others. An important result from this literature (a version of which is presented in Proposition 12 below) is that the optimal strategy involves a long position in the strategy that would be optimal without the performance constraint, plus a long position in an option that compensates for the possible gap between the value of this position and the goal value. In order to make a distinction between the solution to (22) and the solution to the otherwise identical program without the constraint of securing the goal (Equation (21)), we denote by A_T^{*F} and $\underline{w}^{*F}$ the optimal payoff and weight vector in Program (22). The following proposition provides relationships between the solutions to (21) and (22).

Proposition 12 (Optimal Payoff and Strategy Securing a Wealth-Based Goal).

Assume that markets are complete and that $A_0 \geq \tilde{F}_0$. Then, the optimal payoff in (22) is:

$$A_T^{*F} = \nu_2 A_T^{*G} + \left[F_T - \nu_2 A_T^{*G}\right]^+, \qquad (23)$$

where A_T^{*G} is the optimal terminal wealth without the performance constraint (Program (21)), and ν_2 is the unique solution to the equation $\mathbb{E}[M_T A_T^{*F}] = A_0$ in the range [0,1] if $A_0 > \tilde{F}_0$, and 0 if $A_0 = \tilde{F}_0$.

Moreover, if $\underline{\sigma_{Ft}} = \underline{\sigma_t w_{FHP,t}}$ and $\underline{\sigma_t^{*G}} = \underline{\sigma_t w_t^{*G}}$ are deterministic functions of time, then the optimal strategy in (22) reads:

$$\underline{w_t^{*F}} = \left[1 - \frac{p_{t,T}\tilde{F}_t}{A_t^{*F}}\right]\underline{w_t^{*G}} + \frac{p_{t,T}\tilde{F}_t}{A_t^{*F}}\underline{w_{FHP,t}},$$

where $\underline{w_t^{*G}}$ is the optimal weight vector for Program (21), and $p_{t,T}$ is the probability (under the probability measure that makes asset prices expressed in the numeraire $\tilde{F}$ follow martingales) that the put option in (23) ends up in the money. It is given by:

$$p_{t,T} = \mathcal{N}(-d_{2t}),$$

$$d_{2t} = \frac{1}{\zeta_{t,T}}\left[\ln\frac{\nu_2 A_t^{*G}}{\tilde{F}_t} - \frac{1}{2}\zeta_{t,T}^2\right],$$

$$\zeta_{t,T} = \sqrt{\int_t^T \left\|\underline{\sigma_s^{*G}} - \underline{\sigma_{Fs}}\right\|^2 ds}.$$

Proof. See Appendix 6.3.2.

Hence, the introduction of the constraint to secure the goal in Program (22) implies that the optimal payoff is that of an exchange option between the goal value and the optimal "unconstrained" wealth, which is the wealth that would be optimal without the performance constraint. This payoff can be statically replicated by investing $\nu_2 A_0$ in the optimal unconstrained strategy, and (1-ν_2) A_0 in a put option written on this strategy, with a stochastic strike price equal to G_T. Because the put option pays the exact difference between the value of the goal and that of the position in the unconstrained strategy when this difference is positive, it can be called an "insurance put".

In terms of portfolio weights, the optimal strategy is a combination of the optimal "unconstrained" portfolio, which would be optimal for the same objective but without the performance constraint, and the FHP.

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As noted by Deguest, Martellini and Milhau (2014), it exhibits analogies with a CPPI strategy:

- The "risky" asset is the unconstrained strategy;
- The "safe" asset is the FHP, i.e. the portfolio that secures the floor;
- The floor is $p_{t,T}\tilde{F}_t$;

The (probability-adjusted) cushion is $[A_t^{*Gs} - p_{t,T}\tilde{F}_t]$.

The amount invested in the unconstrained strategy is shown to be the product of a multiplier times the risk budget (known as the cushion in the context of CPPI strategies). It should be noted that while the objective is to have $A_T \geq F_T$ at the final date, the floor is not equal to the present value ($\tilde{F}_t$) of the goal, but instead the present value multiplied by a probability $p_{t,T}\tilde{F}_t$. Thus, the risk budget in the optimal strategy is the distance between current wealth and a probability-weighted floor which depends on the goal value and the likelihood of the risk budget being spent before the horizon. Since the probability is less than 1, the risk budget of the optimal strategy is larger than current wealth minus the goal. As shown in Appendix 6.3.2, the property of a vanishing risk budget when wealth approaches the goal value from above is preserved for these strategies despite the fact that the risk budget exceeds the distance to the floor. Intuitively, this is because the probability $p_{t,T}$ can be shown to converge to 1 when asset value converges to the floor.

Appendix 6.3.2 also shows that when wealth grows to infinity, the risk budget $[A_t^{*F} - \tilde{F}_t\mathcal{N}(-d_{2t})]$ approaches A_t^{*F}. Hence, the optimal strategy coincides with the strategy that would be optimal without the performance constraint. This makes intuitive sense; if the wealth level is substantially higher than the floor, then the investor acts as if there was no floor.

3.2.2.4 Cost of Insurance for the Optimal Strategy Securing a Wealth-Based Goal

As explained in Deguest, Martellini and Milhau (2014), the coefficient ν_2 in Proposition 12 has two equivalent interpretations. First, since it equals the ratio of optimal constrained wealth over optimal unconstrained wealth in those states of the world where the put option expires out-of-the-money, it represents the access to the upside of the unconstrained strategy. Since ν_2 is less than 1, this access is less than 100%. In other words, when the insurance proves ex-post to be unnecessary, the final wealth is less than what would have been achieved without purchasing the put. This leads to the second interpretation: the price of the put is $(1-\nu_2)\ A_0$, so the quantity $1-\nu_2$ is the (relative) cost of insurance. This cost is always nonnegative, but a natural question is whether it is strictly positive. The following corollary shows that it is the case, unless of course the optimal unconstrained payoff always outperforms the goal.

Corollary 2 (Cost of Insurance with a Floor).

The coefficient ν_2 of Proposition 12 satisfies:

$$1 - \frac{\tilde{F}_0}{A_0} \leq \nu_2 \leq 1.$$

Moreover, if the optimal unconstrained wealth has a positive probability of underperforming the floor, i.e. if

$$\mathbb{P}(A_T^{*G} < F_T) > 0,$$

then, ν_2 is strictly less than 1, which means that the cost of insurance is strictly positive.

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Proof. See Appendix 6.3.2.

The lower bound on ν_2 means that the cost of insurance satisfies $(1-\nu_2)A_0 \leq \tilde{F}_0$. This is the standard property saying that the put is less expensive than the zero-coupon bond with the same horizon. Hence, the optimal strategy makes insurance less expensive than a hedging strategy that would consist of purchasing the zero-coupon bond paying the floor to secure the floor, and investing the remaining amount, that is $[A_0 - \tilde{F}_0]$, in the unconstrained strategy. This observation is related to the fact that the optimal risk budget is in general larger than the distance of wealth to the present value of the goal. Deguest, Martellini and Milhau (2014) provide a detailed study of the cost of insurance by showing that it is increasing in the level of the goal and decreasing in the risk aversion parameter.

Corollary 2 has an important implication for the distinction between essential and important goals, because it explains why it is justified to decide not to secure an affordable goal in some contexts. The trade-off is between the protection of the goal and the loss of upside potential which arises from the purchase of the put. Thus, an investor may decide not to explicitly secure a goal, even if this goal is affordable, because the cost of insurance is deemed to be too large.

3.2.2.5 Decreasing the Cost of Insurance by Imposing a Cap

In addition to accounting for the presence of floors, the dynamic asset allocation strategies can also accommodate the presence of various forms of *caps* or *ceilings*. The idea of imposing a cap on wealth in order to reduce the cost of insurance against downside risk is discussed in detail in Martellini and Milhau (2012) and Deguest, Martellini and Milhau (2014). These strategies recognise that the investor has no utility over a *cap* target level of wealth, which represents the investor's goal (actually a cap), which can be a constant, deterministic or stochastic function of time. From a conceptual standpoint, it is not clear *a priori* why any investor should want to impose a strict limit on upside potential. The intuition is that by forgoing performance beyond a certain threshold, where they have relatively lower utility from higher wealth, investors benefit from a decrease in the cost of the downside protection. This is equivalent to adding a short position in a convex payoff in addition to the long position, so as to generate a collar-like payoff, with a truncation of the wealth level distribution on the left-hand side (below the floor level) as well as on the right hand side (above the cap level). Putting it differently, without the performance cap, investors have a greater chance of failing an almost attained-goal when their wealth level is very high.

Formally, we let C_T denote the terminal value of the cap, which is the maximum wealth that the investor is willing to accept. In addition to this maximum, the investor still imposes the floor F_T, and potentially includes a third goal, G_T, in the objective function. We recall that $\tilde{F}_t$ and $\tilde{G}_t$ denote the present values of the payoffs F_T and G_T, and we adopt the notation $\tilde{C}_t$ for the present value of C_T. The cap-hedging portfolio (CHP) is denoted as $\underline{w}_{CHP}$. The constraint to have $A_T \leq C_T$ with probability 1 implies that $A_0 \leq \tilde{C}_0$, so that C_T can be thought of as a non-affordable or a "just affordable" (if $A_0 = \tilde{C}_0$) goal. The optimisation program reads:

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$$\max_{\underline{w}} \mathbb{E}\left[U\left(\frac{A_T}{G_T}\right)\right],$$

subject to (3) and $F_T \leq A_T \leq C_T$. (24)

As shown in Martellini and Milhau (2012) and Deguest, Martellini and Milhau (2014), the optimal payoff consists of long positions in the optimal unconstrained strategy and the insurance put, plus a short position in a call option written on the unconstrained strategy. This combination gives a collar profile, which confines the terminal wealth between the bounds F_T and C_T. The following proposition gives a detailed expression.

Proposition 13 (Optimal Payoff and Strategy Securing a Wealth-Based Goal with a Cap on Wealth).
Assume that $F_T \leq C_T$ almost surely, that markets are complete and that $\tilde{F}_0 \leq A_0 < \tilde{C}_0$.[28] *Then, the optimal payoff in (24) is:*

$$A_T^{*C} = \nu_3 A_T^{*G} + \left[F_T - \nu_3 A_T^{*G}\right]^+ - \left[\nu_3 A_T^{*G} - C_T\right]^+, \quad (25)$$

where A_T^{*G} is the optimal terminal wealth without the performance constraint (Program (21)), and ν_3 is the unique solution to the equation $\mathbb{E}\left[M_T A_T^{*C}\right] = A_0$ in the range $[\nu_2, \infty[$ if $A_0 > \tilde{F}_0$, and 0 if $A_0 = \tilde{F}_0$.

Moreover, if $\underline{\sigma}_{Ft} = \underline{\sigma}_t \underline{w}_{FHP,t}$ $\underline{\sigma}_{Ct} = \underline{\sigma}_t \underline{w}_{CHP,t}$ and $\underline{\sigma}_{Ct} = \underline{\sigma}_t \underline{w}_{CHP,t}$ are deterministic functions of time, then the optimal strategy in (24) is:

$$\underline{w}_t^{*C} = \left[1 - \frac{p_{F,t,T}\tilde{F}_t + p_{C,t,T}\tilde{C}_t}{A_t^{*C}}\right]\underline{w}_t^{*G} + \frac{p_{F,t,T}\tilde{F}_t}{A_t^{*C}}\underline{w}_{FHP,t} + \frac{p_{C,t,T}\tilde{C}_t}{A_t^{*C}}\underline{w}_{CHP,t}.$$

where $\underline{w}_t^{*G}$ is the optimal weight vector for Program (21), $p_{F,t,T}$ is a risk-adjusted probability that the put option in (25) ends up in the money and $p_{C,t,T}$ is a risk-adjusted probability that the call option in (25) ends up in the money. These probabilities are given by:

$$p_{F,t,T} = \mathcal{N}(-d_{F,2t}), \quad p_{C,t,T} = \mathcal{N}(d_{C,2t}).$$

$$d_{F,2t} = \frac{1}{\zeta_{t,T}}\left[\ln\frac{\nu_3 A_t^{*G}}{\tilde{F}_t} - \frac{1}{2}\zeta_{t,T}^2\right],$$

$$d_{C,2t} = \frac{1}{\chi_{t,T}}\left[\ln\frac{\nu_3 A_t^{*G}}{\tilde{C}_t} - \frac{1}{2}\chi_{t,T}^2\right],$$

$$\chi_{t,T} = \sqrt{\int_t^T \left\|\underline{\sigma}_s^{*G} - \underline{\sigma}_{Cs}\right\|^2 ds}.$$

Proof. See Appendix 6.3.3.

The optimal portfolio is now a combination of the portfolio that would be optimal in the absence of the floor and the cap, plus the FHP and the CHP. In order to better understand how the strategy allocates to these building blocks, it is interesting to analyse limit cases when current wealth approaches floor or cap levels. In the former case, the put option is deep in the money and the call option is deep out of the money, so the coefficients $d_{F,2t}$ and $d_{C,2t}$ go to minus infinity. Hence, the probabilities $p_{F,t,T}$ and $p_{C,t,T}$ approach respectively 1 and 0, and the optimal weight vector, $\underline{w}_t^{*C}$, converges to the FHP. When wealth approaches the cap, the situation is symmetric: $p_{F,t,T}$ and $p_{C,t,T}$ approach respectively 0 and 1, so $\underline{w}_t^{*C}$ converges to the CHP. Hence, when a cap is imposed, there are two situations where the allocation to the unconstrained strategy vanishes, that is, either when wealth is close to the floor or when it is close to the cap.

The coefficient ν_3 represents the access to the upside of the unconstrained strategy when both options expire out of the money. By definition, ν_3 is greater than or equal to ν_2. The following corollary shows that

28 - Observe that the second of these inequalities is strict.

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the inequality is in fact strict provided the unconstrained wealth is not always lower than the cap.

Corollary 3 (Net Cost of Insurance with a Floor and a Cap).
Assume that the unconstrained wealth has a positive probability of outperforming the cap:

$$\mathbb{P}\left(\nu_2 A_T^{*G} > C_T\right) > 0$$

Then, the coefficient $\boldsymbol{\nu}_3$ *of Proposition 13 satisfies* $\boldsymbol{\nu}_3 > \boldsymbol{\nu}_2$.

Proof. See Appendix 6.3.3.
Hence, by imposing a cap in addition to a floor, one captures a greater fraction of the performance of the unconstrained strategy. Equivalently, the net cost of insurance, i.e. the put option price minus the call option price, equal to $(1-\boldsymbol{\nu}_3)\, A_0$, is lower than the cost of insurance with a floor only.

3.2.2.6 Optimal Strategy in the Presence of a Consumption-Based Goal
As explained above, we consider an exogenous stream of consumption, which is not optimised over. Thus, maximising the utility deriving from intertemporal consumption, which is the standard objective in the literature on optimal consumption and portfolio choice, is not an appropriate program here. We choose instead to maximise the expected utility from bequest after all consumption expenses have been made. Mathematically, this corresponds to Program (18).

As for the wealth-based goal, we solve this program under the assumption of complete markets, in order to ensure the uniqueness of the state-price deflator, and thereby facilitate the comparison between the solution to the program with a goal (Equation (18)) and the program without a goal (Equation (17)). We recall that the optimal wealth and weight vector for Program (17) are denoted with A^{*0} and $\underline{\boldsymbol{w}}^{*0}$, and we use the notations A^* and $\underline{\boldsymbol{w}}^*$ for Program (18). We also recall that $\tilde{\boldsymbol{G}}_t$ denotes the present value of the goal, which is the price of the consumption stream and is uniquely defined if markets are complete. We let $\underline{\boldsymbol{w}_{GHP}}$ denote the GHP, which is a portfolio fully invested in the bond whose coupons match the consumption payments.

Proposition 14 (Optimal Wealth and Strategy with Consumption-Based Goal).
Assume that markets are complete and that $A_0 \geq \tilde{\boldsymbol{G}}_0$, *where* $\tilde{\boldsymbol{G}}_0$ *is the initial price of the consumption stream. Then, the optimal wealth in (18) is:*

$$A_t^* = \left(1 - \frac{\tilde{G}_0}{A_0}\right) A_t^{*0} + \tilde{G}_t,$$

and the optimal strategy is:

$$\underline{w_t^*} = \left(1 - \frac{\tilde{G}_t}{A_t^{*c}}\right) \underline{w_t^{*0}} + \frac{\tilde{G}_t}{A_t^{*c}} \underline{w_{GHP,t}}.$$

Proof. See Appendix 6.3.4.
A first observation is that the optimal strategy secures the consumption-based goal, since the terminal wealth is $A_T^{*c} = \left(1 - \frac{\tilde{G}_0}{A_0}\right) A_T^{*0}$, a quantity which is nonnegative. This is a difference with respect to the wealth-based goal. Indeed, the optimal strategy for Program (17) does not secure the goal, unless the performance constraint is explicitly introduced (see Program (22)). Secondly, the optimal strategy with the consumption-based goal is reminiscent of an extended form of CPPI strategy, where the floor is the present value of future consumption streams, and the "safe asset" is the coupon-paying bond. When current wealth approaches the floor value, the

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investor invests a larger fraction of the portfolio in this bond.

3.2.3 Comparison of the Two Strands of Literature

At this stage, it is interesting to compare the various optimisation programs presented in the previous sections. We first comment on the choice of the objective function, and then highlight the similarities and the differences between the solutions to the programs.

3.2.3.1 Optimisation Criteria

A first important difference between the two frameworks lies in the choice of the objective function. Since the success probability is 1 for any strategy that secures the goal, a strategy that invests only in the GHP and the strategy that protects the goal by purchasing a put written on an unconstrained strategy (as in Proposition 12) are strictly equivalent in terms of this criterion. But it is intuitive that they have very different terminal payoffs. In other words, the success probability focuses on one very specific aspect of the distribution of wealth and is not rich enough to allow for a distinction between two distributions that are obviously not equivalent. Hence, the choice of this criterion appears to be too restrictive. The expected shortfall is subject to the exact same criticism: both aforementioned strategies have zero expected shortfall. More generally, the expected shortfall focuses only on what happens when the goal is missed, and leaves aside the right tail of the distribution of wealth.

On the other hand, the expected utility criterion incorporates the mean and the variance of the entire payoff distribution, as well as the higher-order moments. In the previous example, the "safe" strategy that invests only in the GHP and the strategy that involves an insurance put will lead to different levels of expected utility. A related observation is that as explained in Section 3.2.1, the success probability and the expected shortfall criteria do not lead to a uniquely defined optimal solution in the case of an affordable goal: the two strategies given as examples above are optimal. The non-uniqueness of the solution can be regarded as a drawback. The expected utility criterion leads to a unique optimal solution whether the goal is affordable or not. Hence, this criterion applies to both types of goals. Putting it differently, the certain achievement of essential goals should be taken as a constraint, and not as an objective, of the optimisation program.

3.2.3.2. Optimal Payoffs

The mathematical derivation of optimal payoffs is in general possible both for heuristic risk management indicators and for expected utility maximisation. It is convenient to ask whether, beyond their formal optimality for a given criterion, these payoffs would be acceptable for an individual investor. The digital option payoffs display a clear disadvantage from this perspective, because they can be zero with positive probabilities. This drawback can be circumvented by imposing a floor on wealth in the probability maximisation, or by minimising the expected shortfall in order to penalise bankruptcy. But the optimal payoff remains of the digital type, which implies in particular that it is discontinuous. It is well known the replication of such options raises difficulties in practice. In contrast, the utility-maximising payoffs written in the previous propositions are all

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continuous and positive, and the positivity property is obtained because the marginal utility for the CRRA function grows to infinity at zero. This remark is important, because if a utility function does not verify this property, the optimal terminal wealth can be zero with positive probability. Overall, the utility-maximising payoffs appear to be more acceptable in practice.

Another difference between the two frameworks is that for a non-affordable goal, the probability-maximising and the shortfall-minimising payoffs never exceed the goal (Section 3.2.1), while the utility-maximising payoff can do so (Proposition 11). The property of being always less than or equal to the goal is not inconsistent with the fact of optimising a criterion: maximising the success probability or minimising the expected shortfall does not require the strategy to outperform the goal and large surpluses are not more valued than the small ones. In practice, individual investors are rarely purely concerned with a single goal. Beyond the explicitly formulated goal, they may have in mind an implicit objective related to the upside of the strategy, with no well-defined threshold: an example is the bequest goal, which is to maximise the amount of money left to children upon death. Because expected utility depends positively on expected return, utility-maximising strategies leave room for goal outperformance, unlike the probability-maximising and the shortfall-minimising strategies. It should be noted, however, that the upside potential of the utility-maximising strategy is reduced by the imposition of a minimal performance constraint: indeed, part of the initial capital must be devoted to the purchase of an insurance put, which leaves less money available to invest in performance assets (see Section 3.2.2.3).

3.2.3.3. Optimal Strategies: Building Blocks

The optimal strategies are the strategies that replicate the optimal payoffs. A first observation is that the goals-based allocation strategies permit an analytical derivation only under assumptions of deterministic Sharpe ratios and volatility vectors for goal processes. For the probability maximisation and the expected shortfall minimisation, it is easy to see that these difficulties are due to the option-like nature of the payoffs (digital options): stochastic Sharpe ratios or volatility vectors would imply stochastic volatilities for the underlying assets. Expected utility maximisation typically leads to more tractable payoffs. In the absence of performance constraints, utility-maximising strategies can often be derived analytically even in the presence of stochastic investment opportunities; on the other hand, the introduction of an explicit performance constraint leads to a call option payoff, and the assumption of deterministic parameter is again needed to arrive at an explicit expression for the optimal strategy (see Proposition 12).

Beyond this technical aspect, there are striking similarities between the dynamic strategies obtained in the two frameworks. In particular, they all involve the MSR and the GHP as building blocks. Intuitively, the presence of these blocks can be explained as follows. Investing in the MSR increases the expected return of the strategy: hence, it improves the chances to reach the goal, which has a positive impact on risk management indicators such as the shortfall probability or the expected

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shortfall. Because expected utility is increasing in expected return, the MSR also improves expected utility. As far as the GHP is concerned, an investment in this building block narrows down the uncertainty over the value of wealth relative to the goal. The reduction in variance has a positive impact on the expected utility from the funding ratio. When the objective is to minimise a shortfall indicator, investing in the GHP ensures that the performance of the strategy does not deviate too much from that of the goal, which again leads to higher success indicators.

In order to make a more complete comparison between the sets of building blocks involved in the two families of strategies, one would need to have general fund separation theorems in both frameworks. For expected utility, there exist general decomposition results (see e.g. Detemple and Rindisbacher (2010)), but for the shortfall indicators, the expressions for optimal strategies barely extend beyond the case of constant or deterministic investment opportunities. In particular, it is known that a stochastic opportunity set leads to hedging demands in utility-maximising strategies, so a question is whether these hedging portfolios would also appear in the probability-maximising or shortfall-minimising strategies.

3.2.3.4 Optimal Strategies: Investment Rules

In addition to the similarity of building blocks, there are also common points in terms of allocations to these blocks. To see this in detail, let us consider a specific case with two proportional wealth-based goals at the horizon T: F_T is an essential goal and is thus treated as a floor, and G_T is a non-affordable goal, hence an aspirational goal. F_T is equal to αG_T for some scalar α between 0 and 1, and the FHP coincides with the GHP since the essential and aspirational goals are perfectly correlated. As before, we take the notations $\tilde{F}_t$ and $\tilde{G}_t$ for the present values of the two goals. We also assume that all risk premia and volatility vectors are deterministic, which enables to use the expressions of Sections 3.2.1 and 3.2.2 for the optimal weight vectors.

By Proposition 10, the strategy that maximises the probability of reaching the goal G_T subject to the constraint of securing the floor F_T is:

$$\underline{w}_t^{*prob} = \frac{\lambda_{MSR,t}}{\sigma_{MSR,t}} \varphi_t \underline{w}_{MSR,t} + (1-\varphi_t)\underline{w}_{GHP,t},$$

with:

$$\varphi_t = \frac{1}{\kappa_{t,T}} \frac{\tilde{G}_t - \tilde{F}_t}{A_t} n\left[\mathcal{N}^{-1}\left(\frac{A_t - \tilde{F}_t}{\tilde{G}_t - \tilde{F}_t}\right)\right],$$

$\kappa_{t,T}$ being a time-dependent coefficient, the expression of which does not matter here. By Propositions 11 and 12, the strategy that maximises the expected logarithmic utility of the ratio A_T/G_T while securing the floor is:

$$\underline{w}_t^{*EU} = \frac{\lambda_{MSR,t}}{\sigma_{MSR,t}} \psi_t \underline{w}_{MSR,t} + (1-\psi_t)\underline{w}_{GHP,t},$$

with:

$$\psi_t = \frac{1}{\gamma}\left[1 - \frac{p_{t,T}\tilde{F}_t}{A_t}\right].$$

In both cases, the allocation to each building block (MSR or GHP) depends on current wealth and floor present value, and, as argued in Sections 3.2.1 and 3.2.2.3, the allocation to the MSR shrinks to zero as wealth approaches the floor. As a result, the optimal portfolio becomes fully invested in the GHP (and cash if the weights of the GHP do not add up to 1). In other words, both strategies have the same behaviour as wealth gets closer to

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the floor. A difference between the two portfolios is that the probability-maximising weights depend on the present value of the non-affordable goal, which is not the case for the utility-maximising one. In fact, with the former strategy, wealth always remains below the non-affordable goal value, and the allocation to the MSR becomes zero as wealth approaches the goal from below. With the utility-maximising rule, wealth is not bounded from above, and for a very large wealth level, the allocation to the MSR is $1/\gamma$, which is non-zero.

At this stage, it appears that the expected utility criterion is appropriate to find optimal strategies in the presence of both affordable and non-affordable goals, and is thus more general than probability maximisation or expected shortfall minimisation. Moreover, it yields more realistic payoffs than these two criteria, and it allows for the explicit derivation of optimal strategies in a broader class of models. Finally, the previous discussion shows that it leads to allocation recommendations that are similar to those of the probability criterion. For these reasons, we take expected utility maximisation as the reference criterion to construct allocation strategies in the remainder of this paper, and we now explain what exact proxy we recommend should be used in practice.

3. 3 Implementation Challenges

Once the PSP and GHP have been carefully designed, the next step is to determine what percentage of investor's wealth should be allocated to each one of these building blocks. The theoretical optimality results given in Section 3.2 provide useful guidance with respect to the question of allocating to the PSP and the GHP. As explained in Section 3.2.3, we take expected utility maximisation as the paradigm to build investment strategies. The purpose of this section is thus to describe the adaptation of utility-maximising strategies to a realistic context.

3.3.1 Strategies without Performance Constraints

By an unconstrained strategy, we mean a strategy that does not target at securing any goal (although it can incidentally do so under some parametric assumptions). In this context, the strategy which maximises the expected utility from terminal wealth is a combination of the MSR, cash and a series of hedging demands dedicated to hedging changes in the opportunity set (see Section 3.2.2.1 and the references therein). If the objective is to maximise the expected utility from the terminal funding ratio (i.e. the ratio of wealth to the goal), then the optimal strategy is a combination of the MSR, the GHP, cash, and possibly additional hedging demands (see Section 3.2.2.2). The computation of these hedging demands requires the following inputs:

- The investor's risk aversion and horizon;
- The parameters of the models that describe of the risk factors affecting the opportunity set.

For the second class of inputs, one has to specify a dynamic model for the stochastic variables of interest, and to calibrate the parameters to market data. Summarising changes in the opportunity set through a reduced number of factors is a parsimonious approach, but it inevitably generates model risk. In Section 3.4.1.2, we explain how the parameters required for implementing and simulating a GBI strategy can be

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estimated in coherence with available market information.

Regarding the investor's risk aversion, which is a non-observable attribute, we argue that since investors' preferences have been thoroughly described in terms of the specific goals that needed to be achieved, there is no need to assume an additional artificial degree of risk-aversion. For these reasons, we assume a risk aversion parameter (denoted with γ in Section 3.2.2) equal to 1, which implies that all hedging demands are equal to zero. This has the advantage that the utility-maximising strategy is model-free, in the sense that it depends neither on the particular dynamics assumed for the state variable (since it avoids the computation of the hedging demands against changes in risk premia). Hence, the strategy that maximises the expected utility from terminal nominal wealth is the growth-optimal portfolio strategy:

$$\underline{w}_t^* = \frac{\lambda_{MSR,t}}{\sigma_{MSR,t}} \underline{w}_{MSR,t}. \quad (26)$$

It should be emphasised at this point that for most reasonable parameter values the ratio $\lambda_{MSR,t}/\sigma_{MSR,t}$ is greater than 1, which implies that the investor is meant to hold a leverage exposure to the MSR portfolio financed with borrowing at the risk-free rate. The introduction of a PSP distinct from the MSR can be rationalised in the presence of leverage constraints. Indeed, the fund separation theorem saying that all investors endowed with mean-variance preferences should hold a combination of the MSR and the cash account breaks down if such constraints are imposed. In this setting, the portfolio selected by an investor is a combination of two mean-variance efficient portfolios that depend on the risk aversion (see the critical line method introduced by Markowitz (1956) and extended by Sharpe). This justifies having a set of PSPs as opposed to a single portfolio intended as a proxy for the MSR. In this context, we therefore obtain that GBI strategies for a logarithmic leverage-constrained investor will involve a dynamic allocation to essential GHPs and the PSP, in addition to whatever wealth mobility portfolio the investor is already endowed with.

3.3.2 Strategies Securing Essential Goals

We now precisely turn to the analysis of GBI strategies that incorporate the need to secure one or more essential goal(s).

3.3.2.1 Wealth-Based Essential Goal with a Single Horizon

Consider first the case of a wealth-based goal with a single horizon, a goal which is represented by a (replicable) payoff F_T, which is a minimum level of wealth to attain on date *T*. In the expected utility framework, the goal can be secured by imposing a minimal performance constraint, as is done in Program (22). Note that when the risk aversion is 1, the solution to this program is independent from the deflator of the terminal wealth in the objective function. Indeed, the utility function is the logarithmic function, so for any goal G_T, the expected utility from the terminal funding ratio is:

$$\mathbb{E}\left[U\left(\frac{A_T}{G_T}\right)\right] = \mathbb{E}[\ln A_T] - \mathbb{E}[\ln G_T],$$

and the value of the second term in the right-hand side is independent from the strategy. Hence, the utility-maximising policy is independent from the choice of G_T. Thus, we can assume without loss of generality that G_T=1.

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We let $\tilde{F}_t$ denote the present value of the goal to secure, and since the goal is treated as a floor here, we refer to the GHP as the FHP. We also assume that the initial wealth is such that $A_0 \geq \tilde{F}_0$, an assumption which is required for the goal to be affordable. Proposition 12 implies that the optimal strategy is to take a long position in the optimal unconstrained strategy and an insurance put. But with $\gamma = 1$, the optimal strategy is (26). Hence, the optimal policy is given by:

$$\underline{w}_t^{*Gs} = \frac{\lambda_{MSR,t}}{\sigma_{MSR,t}} \psi_t \underline{w}_{MSR,t} + (1-\psi_t)\underline{w}_{FHP,t}, \quad (27)$$

where:

$$\psi_t = 1 - \frac{p_{t,T}\tilde{F}_t}{A_t},$$

and $p_{t,T}$ is the probability for the bucket invested in the MSR to underperform the floor (see Section 3.2.2.3). This probability is a function of model parameters, in particular of the risk premia and the goal volatility (see the detailed expression in Proposition 12). This creates dependency with respect to the model and to unobservable parameters. In order to avoid this additional source of complexity, we suggest to set the probability equal to 1, which amounts to having a conservative assessment of the risk budgets. A robust simplified version of the strategy is thus:

$$\underline{w}_t = \frac{\lambda_{MSR,t}}{\sigma_{MSR,t}} \left(1 - \frac{\tilde{F}_t}{A_t}\right) \underline{w}_{MSR,t} + \frac{\tilde{F}_t}{A_t} \underline{w}_{FHP,t}. \quad (28)$$

One can obtain a more general family of strategies by replacing the coefficient $\lambda_{MSR,t}/\sigma_{MSR,t}$ by a multiplier m, which controls the allocation to the MSR. Note that in implementation, the multiplier can/should be time-varying, as a function of changes in estimated levels for the Sharpe ratios and volatilities. For parsimony, at the simulation stage, comparable to an asset-liability management exercise for a pension fund, we will take a constant multiplier in the strategies that we test in Section 4.

We make two additional practical modifications to the optimal strategy (27). First, we consider a more general version, where the performance block is not necessarily the MSR, but some performance-seeking portfolio constructed after the principles given in Section 2.5.2. Second, we take the weight of the FHP to be 1 minus the weight of the PSP. This ensures that the resulting portfolio contains no short or long cash holdings, in addition to cash that can be held in the personal risk bucket for hedging purposes.

We thus finally obtain the following strategy:

$$\underline{w}_t^{GBI,MH} = m\left(1 - \frac{\tilde{F}_t}{A_t}\right)\underline{w}_{PSP,t} + \left[1 - m\left(1 - \frac{\tilde{F}_t}{A_t}\right)\right]\underline{w}_{FHP,t}. \quad (29)$$

We refer to this strategy as the goals-based investing (in short, GBI) strategy protecting the goal F_T. It can be regarded as an extension of the CPPI strategies studied by Black and Jones (1987) and Black and Perold (1992), the floor being the present value of the goal to secure and the safe asset being the FHP. In fact, the exact strategy that we will implement in the case studies of Section 4 slightly differs from (29) in that we will also impose a no short-sales constraint in the PSP and FHP blocks. Indeed, for high values of the multiplier and large risk budgets, the exposure to the PSP prescribed by (29) can be larger than 100%. The detailed expression of weights is given in Appendix 6.6.3.

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Strategy (29) can be regarded as the simplest form of strategy combining the PSP and the FHP in which the allocation to the FHP becomes 100% as wealth approaches the floor. The optimal strategy in Equation (24) has this property too, but the dependence of weights with respect to current wealth and the floor is more complicated and non-linear, due to the presence of the probabilities $p_{t,T}$. In contrast, the amount invested in the FHP with (29) is simply a linear function of the wealth and the floor. When the multiplier m is taken to be equal to 1, we recover as a special case the intuitive buy-and-hold strategy that recommends that any excess of liquid wealth (that is excluding wealth in the aspirational risk bucket) remaining available after all essential goals have been secured through investments in dedicated hedging portfolios should be invested in the well-diversified performance seeking portfolio.

At this stage, one can ask whether it would have been possible to further simplify the optimal strategy while securing the floor. Although it is not possible to give a general form for the weights of all portfolio strategies which secure a goal, there are many functional forms to achieve this property (see Section 3.2), and the utility-maximising rule allows us to identify one such strategy, but it involves a non-linear function of the wealth and the floor. Strategy (29) removes this non-linearity.

Another advantage of Strategy (29) over (27) is that it involves only observable quantities, at least to the extent that the present value of the goal is observable. Indeed, the evaluation of the probabilities $p_{t,T}$ requires the knowledge of the full distributions of PSP and floor values. This estimation is bypassed by Strategy (29).

3.3.2.2 Wealth-Based Goal with Multiple Horizons

Consider now the case where the goal to secure is a wealth-based one with multiple horizons. Section 3.2.2 does not provide a utility-maximising strategy for this goal, but we construct a strategy combining the PSP and a safe asset according the same principles as in the case of a single horizon. First, the safe asset must secure the goal. By Proposition 3, the FHP is a roll-over of exchange options, and the discussion in Section 2.2.1.2 shows that it actually super-replicates the floor: indeed the FHP value at each goal horizon is greater than the goal value. Second, the allocation to the PSP should vanish as wealth approaches a floor. A natural floor here is the present value of the goal for two reasons: first, it is the choice of floor already made in the case of the single horizon; second, the present value of a goal with multiple horizons is the minimum capital to invest in order to secure the goal. Having made this choice of floor, we define the risk budget as the distance between wealth and the floor, and the sum allocated to the PSP is a multiple of the risk budget. Thus, the GBI strategy for a wealth-based goal with multiple horizons has exactly the same form as (29):

$$\underline{w}_t^{GBI,MH} = m\left(1-\frac{\tilde{F}_t}{A_t}\right)\underline{w}_{PSP,t} + \left[1-m\left(1-\frac{\tilde{F}_t}{A_t}\right)\right]\underline{w}_{FHP,t}. \quad (30)$$

Again, short-sales constraints will be applied in this strategy (see Appendix 6.6.3 for details).

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Strategy (30) has the form of a CPPI: the floor is $\tilde{F}_t$, which coincides with the price of an exchange option between two horizons, and the safe asset is the FHP, which is a roll-over of exchange options. It should be noted that the risk budget, $RB_t = m(A_t - \tilde{F}_t)$, is discontinuous on the dates $T_1, \dots, T_p$: it is because wealth is continuous, but the goal present value is not (see Section 2.2.1.2).

Unlike Strategy (29), Strategy (30) is not entirely model-free because the risk budget involves the price of an exchange option. But under the monotony assumption on goal values made in Corollary 1, the option price coincides at each date with the present value of the next goal value, so an option pricing model is not necessary. It is intuitive that Strategy (30) secures the goal. Indeed, the allocation to the PSP shrinks to zero as wealth approaches the present value of the goal, so the portfolio becomes fully invested in the FHP, which super-replicates the goal. In the case of a goal with a single horizon, it is known that this intuition is valid: Strategy (29) secures the goal. The following proposition is a less standard result, which says that the property is also true in the case of multiple goals.

Proposition 15 (Protection of Essential Wealth-Based Goal by GBI Strategy). *Consider a wealth-based goal F_T with multiple horizons. Then, if the initial wealth satisfies $A_0 \geq \tilde{F}_0$, where $\tilde{F}_0$ is defined in Definition 3, the strategy (30) secures the goal, i.e.:*

$$A_{T_j} \geq F_{T_j}, \text{ for } j = 1, \dots, p.$$

Proof. See Appendix 6.4.1.

Proposition 15 gives a theoretical justification for the use of the GBI strategy (30). By varying the multiplier m, one obtains a class of risk-controlled strategies which secure the essential goal. But there is an important restriction to the result of Proposition 15: it holds under the assumption that the portfolio is rebalanced continuously. In real-world applications, trading is discrete, which may cause gap risk to arise : it is the risk for wealth to fall below the floor if the PSP displays a large negative return between two rebalancing dates, a risk which becomes more important if the allocation to the PSP is large, hence if the multiplier is large.[29]

29 - However, gap risk is limited to the case where $m > 1$. Indeed, one can show that Strategy (30) implemented in discrete time with short-sales constraint on the PSP and the GHP is such that $A_t \geq \tilde{F}_t$ for all t. In particular, it secures the goal.

3.3.2.3 Consumption-Based Essential Goal

In the case of a consumption-based goal, the utility-maximising strategy takes a simpler form than for a wealth-based goal because it involves no unobservable quantity such as the probability $p_{t,T}$. With γ=1, Proposition 14 implies that it is:

$$\underline{w}_t^{*c} = \frac{\lambda_{MSR,t}}{\sigma_{MSR,t}}\left(1 - \frac{\tilde{G}_t}{A_t}\right)\underline{w}_{MSR,t} + \frac{\tilde{G}_t}{A_t}\underline{w}_{GHP,t},$$

where $\underline{w}_{GHP,t}$ is the portfolio fully invested in the bond which pays coupons equal to the consumption expenses. Again, replacing the MSR by a more generic PSP, the ratio $\lambda_{MSR,t}/\sigma_{MSR,t}$ in front of the MSR weights by a generic multiplier m, and removing cash from the portfolio, we obtain the following proxy for the optimal strategy:

$$\underline{w}_t^{GBI,c} = m\left(1 - \frac{\tilde{G}_t}{A_t}\right)\underline{w}_{PSP,t} + \left[1 - m\left(1 - \frac{\tilde{G}_t}{A_t}\right)\right]\underline{w}_{GHP,t}. \quad (31)$$

As for the wealth-based goals, this strategy has the form of a CPPI, the floor being the present value of the forthcoming consumption expenses and the safe asset

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being the bond whose coupon payments match these expenses. This strategy does secure the goal, as stated in the following proposition.

Proposition 16 (Protection of Essential Consumption-Based Goal by GBI Strategy).
Consider a consumption-based goal represented by the payments $(C_{T_1},...,C_{T_p})$. *Then, if the initial wealth satisfies* $A_0 \geq \tilde{G}_0$, *where* $\tilde{G}_0$ *is the present value of the payments, the strategy (31) secures the goal, that is:*

$$A_t \geq 0, \text{ for all } t \text{ in } [0, T].$$

Appendix 6.6.3 describes the short-sales constraints applied to the portfolio at the implementation stage.

3.3.2.4 Multiple Essential Goals

The utility-maximising strategies presented in Section 3.2 are obtained under the assumption that there is a single goal. Thus, an adaptation is needed to handle several essential goals. Consider first the case of two wealth-based goals. As explained in Section 2.2.3.1, two wealth-based goals can be reduced to a single goal by taking the maximum of the two minimum wealth levels. Thus, one option would be to do this merging operation, and to compute the corresponding GBI strategy following the specification (30). While theoretically appealing, this approach raises practical difficulties, relating notably to the pricing of the exchange option between the two goals. In particular, the pricing exercise can only be carried out within the context of a particular model, which makes the strategy dependent on a number of unobservable parameters. We thus take a different approach, by taking as a floor the maximum of the two floor values, and as a safe asset the GHP which corresponds to the higher floor. This strategy relies on the simple intuition that the floor which is more likely to be breached is the one which is closer to wealth, i.e. the higher floor. Hence, it makes sense to favour the protection of this floor. The maximum is re-evaluated on each rebalancing date, so the strategy switches from one GHP to the other.

Mathematically, let $\tilde{F}_{1t}$ and $\tilde{F}_{2t}$ be the two floor values on date t, and denote the two GHPs with $\underline{w}_{FHP1}$ and $\underline{w}_{FHP2}$. The floor of the strategy is:

$$\tilde{F}_t = \max(\tilde{F}_{1t}, \tilde{F}_{2t}),$$

and the floor-hedging portfolio is:

$$\underline{w}_{FHP,t} = \mathbb{I}_{\{\tilde{F}_{1t} > \tilde{F}_{2t}\}} \underline{w}_{FHP1,t} + \mathbb{I}_{\{\tilde{F}_{1t} \leq \tilde{F}_{2t}\}} \underline{w}_{FHP2,t},$$

and the GBI strategy has the form:

$$\underline{w}_t^{GBI,MG} = m\left(1 - \frac{\tilde{F}_t}{A_t}\right)\underline{w}_{PSP,t} + \left[1 - m\left(1 - \frac{\tilde{F}_t}{A_t}\right)\right]\underline{w}_{FHP,t}. \quad (32)$$

This strategy may seem to be purely heuristic, but it can be justified to some extent by computing the portfolio that replicates the maximum of two floors at horizon T: Deguest, Martellini and Milhau (2014) perform this computation under the assumption that both floors follow Geometric Brownian motions, which leads to a closed-form expression for the price of the exchange option. The price of this option is of course not simply the maximum of the two floor values, but if one floor is much larger than the other, then the option price approaches the larger floor value. In this case, the portfolio which dynamically replicates the option also approaches the

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corresponding GHP. These two properties are also verified by the above floor value and floor-hedging portfolio. Hence, $\tilde{F}_t$ and $\underline{w}_{FHP,t}$ can be regarded as model-free proxies for the option price and dynamic replication strategy.

The specification (32) can clearly be extended beyond two wealth-based goals. It can be applied to consumption-based as well as wealth-based goals, and to an arbitrary number of goals, by taking the maximum over more than two floor values.

3.3.2.5 Imposing a Cap

As discussed in Section 3.2.2.5, imposing a cap decreases enables investors to capture a greater fraction of the performance of the unconstrained strategy. This suggests that the probability of reaching high wealth levels can be improved by imposing a maximum wealth level. We recall that when $\gamma=1$, the optimal strategy with a floor and a cap reads (see Proposition 13):

$$\underline{w}_t^{*C} = \frac{\lambda_{MSR,t}}{\sigma_{MSR,t}}\left[1 - \frac{p_{F,t,T}\tilde{F}_t + p_{C,t,T}\tilde{C}_t}{A_t}\right]\underline{w}_{MSR,t} + \frac{p_{F,t,T}\tilde{F}_t}{A_t}\underline{w}_{FHP,t} + \frac{p_{C,t,T}\tilde{C}_t}{A_t}\underline{w}_{CHP,t},$$

the coefficients $p_{FHP,t,T}$ and $p_{CHP,t,T}$ being probabilities, the vector $\underline{w}_{FHP,t}$ being the FHP and $\underline{w}_{CHP,t}$ being the CHP.

It would not be appropriate to simplify this expression by taking both probabilities to be equal to 1, because the strategy must respect the property that the allocation to the FHP (resp., the CHP) approaches 1 when wealth approaches the floor (resp., the cap), and that the allocation to the PSP vanishes in these two cases. In other words, one has to find coefficients $x_{PSP,t}$, x_{Ft} and x_{Ct} such that the strategy:

$$\underline{w}_t^{GBI,cap} = x_{PSP,t}\underline{w}_{PSP,t} + x_{Ft}\underline{w}_{FHP,t} + x_{Ct}\underline{w}_{CHP,t} \qquad (33)$$

has the aforementioned properties. A simple functional form for $x_{PSP,t}$ which guarantees that the allocation to the PSP becomes zero when wealth approaches the floor or the cap is:

$$x_{PSP,t} = m \times \frac{RB_t}{A_t},$$

where the risk budget is computed as $RB_t = A_t - \tilde{F}_t$ when wealth is below the threshold

$$\xi_t = \frac{\tilde{F}_t + \tilde{C}_t}{2},$$

and as $RB_t = \tilde{C}_t - A_t$ when wealth is above ξ_t. To ensure that the portfolio coincides with the FHP when wealth approaches the floor, we take $x_{Ft} = 1 - x_{PSP,t}$ when wealth is below ξ_t, and $x_{Ft} = 0$ otherwise. Symmetrically, we set $x_{Ct} = 1 - x_{PSP,t}$ when wealth is above ξ_t, and $x_{Ct} = 0$ otherwise.

When wealth is below the threshold, the allocation to the PSP and the FHP are the same as in the GBI strategy that protects a floor, regardless of the presence of the cap (Strategy (29)). When wealth is above the threshold, the portfolio rule is also similar to a CPPI, but the risk budget is computed as cap minus wealth, as opposed to being equal to wealth minus floor.

Having recognised that imposing a cap may be useful in some contexts, it remains to fix its level. This choice is more arbitrary than for floors: a floor is a minimum wealth level to protect, which is an input from the individual investor, but the cap does not correspond to an observable parameter. An option consistent with the expected utility paradigm is to set the cap equal to a wealth

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level that achieves satiation of investor's preferences: thus, the cap value on a date t (i.e. the quantity denoted $\boldsymbol{C}_t$ above) can be the present value of the highest goal expressed by the investor.

3.3.2.6 Impact of Income

As explained in Section 2.2.4, the presence of non-portfolio income decreases the minimum capital to invest in order to secure a goal. The general principle is that consumption expenses should be primarily financed with income, and that liquid wealth should be used only to finance the gap, if any, between consumption and income.

Let us consider the retirement problem already discussed in Section 2.2.4.6: the investor receives income during the first part of his life, and consumes more than what he earns during the second part (the retirement period). In order to secure the goal, he must be able to purchase on the retirement date a bond with cash flows equal to the consumption expenses. Thus, the consumption goal translates into a wealth-based goal with a horizon equal to the retirement date. If there were no income prior to retirement, the GHP to purchase at date 0 would be the bond itself. But in the presence of income, purchasing the bond at date 0 would consume an unnecessarily large fraction of liquid wealth and would leave less money available to invest in performance assets. The result would be a lower expected return on the liquid portfolio. The opportunity cost of this strategy can be measured as the difference between $\tilde{\boldsymbol{G}}_{\boldsymbol{0}}$, which is the amount effectively dedicated to the goal protection, and V_0, which is the minimum capital required to secure the goal.

Minimising the cost of the protection is a valuable effort because it leads to the largest access to the upside of performance assets, i.e. assets that are not used for hedging purposes. But the cheapest strategy, which is introduced in Proposition 8, involves a series of compound options. Since these options are unlikely to exist, one may consider replicating them through a dynamic strategy, but this requires the knowledge of their price and Greeks. The complex structure of the payoffs hinders these computations. In the case of the retirement goal (which will be considered in Section 4.3 below), a possible way out is the strategy INC-ZER-RET described in Section 2.2.4.6. It consists to secure at date 0 the positive part of the difference between the minimum level of wealth to attain at the retirement date and the sum of the income payments that will be received by then, while leaving the rest of the money available for investing in performance assets. At each income date, an inflow is cashed in, and the allocation decision made at date 0 is repeated: the payoff to secure is now the positive part of the difference between the minimum wealth at retirement and the sum of the forthcoming income payments. Using the same notations as in Section 3.2.2.6, we have that the wealth between dates T_j and T_{j+1} is:

$$A_t = \left(A_{T_j} - W_{T_j,j}\right) A_{PSP,t} + W_{t,j},$$
$$\text{for } T_j \leq t < T_{j+1}.$$

In this expression, $W_{t,j}$ is the price at date t of the option whose payoff at date T_r is:

$$W_{T_r,j} = \left[\tilde{G}_{T_r} - \sum_{k=j+1}^{r} y_{T_k}\right]^+ .$$

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This simple option is in principle easier to price than a compounded one, which facilitates dynamic replication.

A more extreme approach is to choose to secure the goal by using liquid wealth only, without relying on future income. Although it is extremely conservative, as argued previously, it may be preferred to a strategy that partially relies on income to secure the goal in contexts where future income is too uncertain. Indeed, if future income cannot be guaranteed (e.g. because of the possibility of a job loss), the investor may prefer to secure the goal with liquid wealth only, and the framework described in this paper can be used to provide an estimate for the implied opportunity cost in terms of the probability of achievements of important and aspirational goals.

3.3.3 Protecting Essential Goals in the Presence of Taxes

An overview of taxes was given in Section 2.3. In this section, we discuss the important question of protecting an essential goal in the presence of taxes. Indeed, because they represent constrained payments, taxes can cause deviations from the objectives if they are not anticipated. As explained in Section 2.3, a distinction must be made between taxes from cash flows such as dividends and coupon payments and taxes from capital gains. Indeed, the former are easily predictable while the latter depend on rebalancing decisions that will take place in the future and are more difficult to forecast accurately.

3.3.3.1 Taxes on Cash Flows

A typical situation that will be encountered in the case studies of Section 4 is that of an investor with a consumption goal represented by the payments $c_{T_1},\dots, c_{T_p}$ on dates $T_1,\dots,T_p$. In the absence of taxes, the GHP would be a bond with coupons equal to the target payments. If the coupons are taxed at the rate ζ, this GHP no longer secures the goal, as the investor will receive an after-tax payment of $(1-\zeta)\, c_T$ at date T_j. But it suffices to purchase $1/(1-\zeta)$ units of this bond to anticipate the taxes and fully secure the goal. This amounts to raising the price of the GHP, a simple adjustment that virtually cancels out the effect of taxes. Note that this technique applies both to constant and stochastic consumption expenses. But of course, it increases the initial capital requirement, so that depending on how high is the tax rate, a goal that was affordable in the absence of taxes might become non-affordable.

3.3.3.2 Taxes on Capital Gains

Taxes on capital gains cannot be anticipated in the same was as taxes on cash flows because the amount of taxes due at the fiscal year end depends on the transactions performed within the entire year: on a given date, the amount of trading that will be performed in the future is not known, which makes it impossible to compute an expected value of the final payment.

A simple way to avoid taxes on capital gains is to avoid any rebalancing, i.e. to take only buy-and-hold positions in the assets concerned by taxation. It should be noted that if the investor takes a buy-and-hold position in some building block which itself involves rebalancing in the taxable assets, he will pay taxes on these operations: indeed, the portfolio is buy-and-hold at the building block level, but not at the taxable asset class level. To take a concrete example, an investor may

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implement a GBI strategy of the form (31) in order to protect a consumption-based goal and take the multiplier *m* equal to 1. The portfolio is buy-and-hold in two building blocks, namely the PSP and the GHP, which is a bond paying coupons equal to the consumption expenses. While being buy-and-hold at the building block level, the portfolio may involve rebalancing if the PSP is itself a rebalanced portfolio, e.g. a fixed-mix portfolio of the constituents, with weights chosen to achieve a long-term MSR allocation. The fixed-mix nature of the portfolio implies a counter-cyclical rebalancing: constituents will be sold if they go up in order to maintain constant weights. This is likely to generate capital gains, hence taxes.

If a GBI strategy involves rebalancing between building blocks (the case where $m > 1$), there will be a second source of taxes. Due to the pro-cyclical nature of the strategy, the tax payments generated by the selling operations in the PSP should in principle be lower than those generated by the fixed-mix rebalancing within this block. Indeed, the idea of the strategy is to reduce the exposure to the PSP if the risk budget shrinks, which will in general coincide with a market downturn: in this context, the PSP will have negative returns, which decreases the amount of taxes to pay. Nevertheless, the tax payment is still positive.

It is unclear how to design a GBI strategy that protects an essential goal despite the presence of taxes on capital gains, but one can propose an ad-hoc adjustment to the weights designed to limit the frequency and the size of deviations from the goal. The motivation is as follows. A GBI strategy of the form (29) for a wealth-based goal or (31) for a consumption-based goal, aims at keeping wealth above a floor, which is the minimum capital required to secure the goal. If wealth becomes exactly equal to the floor, the portfolio becomes entirely invested in the GHP, which guarantees success in the goal. But in the presence of taxes, having wealth just equal to the goal present value is not enough to ensure a perfect protection: indeed, future taxes will decrease wealth and possibly take it below the present value. Thus, the idea is to raise the floor in order to acknowledge the presence of taxes. Ideally, one would want to increase the floor by an amount equal to the present value of the year-end tax payment, but this expected value depends on future rebalancing decisions, a complex dependency given the non-linearity of the tax payment with respect to portfolio weights (see Appendix 6.6.5). A more tractable option is to increase the floor by a tax provision, equal to the amount of taxes generated by the transactions that have taken place since the beginning of the year. The risk budget is thus computed as:

$$\boldsymbol{RB_t = A_t - \Theta_t - \widetilde{EG}_t^1},$$

where Θ_t is the tax provision.

In the implementation section of this paper (Section 4), we will test these various adjustments to GBI strategies to see their impact on the chances to reach essential or non-essential goals.

3.4 Inputs and Outputs of the Framework

The above framework proposes a classification of goals based on their funding status and the investor's preferences regarding their protection. It

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also leads a number of strategies designed to secure essential goals and to achieve the non-essential ones with high probabilities. In Section 3.4.1 below, we review what are the exact inputs required for a proper implementation of the framework, while we briefly review in Section 3.4.2 what the main expected outputs are.

3.4.1 Inputs

We first summarise the inputs of the framework, which can be classified as subjective inputs (to be obtained from the individual investor) versus objective inputs (to be specified by the portfolio advisor).

3.4.1.1 Subjective Inputs: Investor's Goals and Risk Allocation

As explained in Section 2.3, the categorisation of goals is a combination of investors' views and a formal analysis of whether these goals are affordable. The discussion leading to the analysis of the affordability of the goals a priori set by the investor, which leads to the definition of the goals to be formally treated as essential goals, is a key ingredient in the process of designing an investment solution in wealth management.

It should also be noted that the classification of goals is subject to periodic revisions. Indeed, the funding status of the goal (i.e. its affordability or non-affordability) depends on the present value of the goal, thus on market conditions and notably on interest rates, and the investor's current wealth. Moreover, the investor's priorities may vary over time. For instance, a birth may give rise to a new goal, which is saving for financing education. Another example is the following: if wealth has increased substantially since the initial date, an investor may wish to secure a higher wealth level, that is, introduce a new higher level of essential or important goal.

The decision to turn a formerly important goal into an essential one is the result of a comparison between the benefits drawn from the action of securing the target wealth or consumption level, and the associated opportunity cost that it implies given that a lower amount of risk taking eventually results in lower probabilities of reaching ambitious aspirational goal levels. Hence, every so often (say every year), the investor is expected to meet with the advisor an revise the *updated* list of goals, with an indication of which of the affordable goals, if any, should be treated as essential goals.

3.4.1.2 Objective Inputs: Parameter Values

Once goals have been identified, it is necessary to sort them as affordable versus non-affordable. The affordability criteria depend on the wealth-based or consumption-based nature of the goal (see the propositions given in Section 1.1). All of them require at some stage the computation of the present value of the goal (with a proper adjustment for income payments when the investor perceives non-portfolio income). It should be emphasised that these results are established in a very general context, with only minimal assumptions on uncertainty in the economy: no particular set of risk factors and no particular dynamics for interest rates, risk premia and volatilities have been assumed.

However, in order to use the affordability criteria in practice, it is sometimes necessary to specify a model to compute present

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values. For a wealth-based goal with a unique horizon, the present value is simply the discounted minimum wealth level, and for a consumption-based goal, it is the sum of discounted cash flows. These values can be directly obtained from the current zero-coupon yield curve (nominal yield curve for fixed cash-flows and real yield curve for inflation-linked cash-flows). For instance, the present value of a consumption-based goal on date t will be computed as:

$$\tilde{G}_t = \sum_{T_j>t} \frac{c_t}{\left(1 + y_{t,T_j-t}\right)^{T_j-t}},$$

where y_{t,T_j-t} is the zero-coupon rate of maturity T_{j-t} prevailing at the date where the present value is computed.

Zero-coupon curves are available at high frequencies (such as daily) for large sovereign issuers. They are constructed from the observed prices of sovereign bonds. Because there exists in general no zero-coupon for each cash-flow maturity, the zero-coupon rates needed to discount the cash flows are not readily observable but they can be recovered by bootstrapping or interpolation methods. For instance, the use of the Nelson-Siegel model for the yield curve represents a zero-coupon rate as the sum of three contributions from a level, a slope and a curvature factors (see Nelson and Siegel (1987)). In the end, discount rates are observable given knowledge of the current yield curve. Since the cash flows are specified by the investor, it follows that the present value of a consumption-based goal is observable too.

Restricting the discussion to interest rate risk for the moment, we therefore conclude that the knowledge of the current yield curve is in general sufficient to implement at each date a given GBI strategy. On the other hand, a dynamic model for the yield curve is required to simulate the subsequent performance of this strategy through Monte-Carlo generated scenarios. For instance, the equilibrium models such as those of Vasicek (1977) and Cox, Ingersoll and Ross (1985) lead to expressing each zero-coupon rate as a function of the maturity, the current value of a factor (here, the short-term interest rate), and a set of parameters that govern the dynamics of the factor. It is also possible to use a two- or three-factor model (see e.g. Duffee (2002) for a general presentation). The increased number of factors implies a higher flexibility of the model, but a higher degree of estimation risk. Such models require the estimation of the factor values and of the parameters that describe the evolution of the factors.

On any particular date, one natural approach is to calibrate the model, required to perform Monte-Carlo simulation needed to estimate probabilities of achievement of goals, by minimising the model pricing errors, that is, the distance between market prices and model-implied prices for a set of reference instruments. The calibration has to be performed on each date where the present value needs to be evaluated, which produces time-varying estimates for parameters, even though the model may involve constant parameters. This is similar to the extraction of the dynamic of implied volatilities from the Black-Scholes option pricing model, which itself assumes a constant volatility.

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An alternative approach consists of estimating, as opposed to calibrating, the parameters of the model. This can be done via various statistical techniques, which include likelihood maximisation, Kalman filtering or generalised methods of moments (see Duffee and Stanton (2012) for a survey of these methods). By combining cross-section and time-series information, these techniques improve the statistical efficiency of parameter estimates. In the case studies presented in Section 4, given that we do not refer to any particular date at which a calibration exercise can be performed, we have chosen to derive the yield curve from a term structure model with parameters estimated over 50 years of data, so as to be representative of the long-term behaviour of interest rates. As a result, the parameters we obtain are not consistent with the current yield curve. It is important to note that the use of a long sample, required in this process, generates estimates which are representative of the average behaviour of interest rates in the past, as opposed to current values. For this reason, the calibration procedure that is solely based current market information should in general be preferred when the exercise is to be performed at any particular point in time.

We emphasise again in closing that the simplified forms of GBI strategies analysed in this paper are based on observable quantities, and their implementation is therefore not subject to model or parameter risk. The specification of a model, and the associated parameter values, is only needed to compute probabilities to achieve non-essential goals. In other words, the benefits of the framework, including the ability to secure essential goals with probability 1 while generating a substantial access to the upside potential of performance-seeking assets, is extremely robust with respect to model and parametric assumptions. What is more subject to model and parametric assumptions is the quantitative assessment of probabilities to reach important and aspirational goals.

3.4.2. Ouputs

The framework is meant to be used both as an engineering tool for generating meaningful portfolio advice as well as a tool for facilitating the dialogue with the investors, and provides a set of subjective outputs (probability of reaching goals and associated expected shortfall) as well as objective outputs (allocation recommendations at all points in time).

3.4.2.1. Success Indicators for Goals

For a given allocation strategy (e.g. a fixed-mix rebalancing towards the investor's current allocation), it is possible to obtain a set of success indicators for the various goals. The following list gives examples of indicators that can be reported:

- The success probability for a goal is the probability of achieving this goal (at all horizons for a goal with multiple horizons);
- The expected maximum shortfall (abbreviated as expected shortfall) is computed as follows: first, we evaluate the shortfall with respect to the goal at each goal horizon; second, we take the maximum shortfall over all horizons; third, we compute the expectation of this maximum conditional on the event that at least one loss was recorded;
- The worst maximum shortfall is computed as the previous indicator, but the expectation in the final step is replaced by a maximum over all states of the world.

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When the investor is also concerned with drawdown risk, i.e. the risk of experiencing losses above a certain threshold, it is useful to add two other indicators:
- The expected maximum drawdown is obtained in two steps: first, the maximum drawdown is computed along each path; second, the expectation is taken;
- The worst maximum drawdown is computed by taking the maximum over all possible paths in the second step.

Formal definitions of these indicators are given in Appendix 6.6.4.

These indicators can be estimated by simulating future portfolio performance. Hence, their values depend on the assumptions made regarding the future performance of the various assets and the evolution of the risk factors which impact goal values (including notably interest rate and inflation). As indicated above, the various risk and return parameters can be re-estimated on each date, in order to generate updated success indicators.

Another type of useful output of the framework is an ex-ante measure of the opportunity cost associated with a given essential goal. Broadly speaking, the opportunity cost is a monetary measure of the opportunity cost implied by the requirement to secure a given goal. More precisely, it is given by the additional required amount of initial wealth needed to generate when the goal is secured the same probability of reaching an aspirational wealth level (or some other aspirational goal) as when the goal is not secured (and is therefore treated as an important rather than essential goal). One can in fact distinguish between two measures of opportunity cost, one related to the opportunity cost of the goal when it is optimally managed via a suitably-defined GBI strategy, and one related to the higher additional cost involved when the goal is managed via some inefficient strategy (one might use the investor's current allocation, or 100% in PSP, as base case benchmark for this inefficient non-GBI investment strategy). As a result, this analysis will not only allow the individual investor to assess the cost and benefit trade-off associated with setting various levels of essential goals; it will also allow the investor to measure the decrease in opportunity costs implied by the use of an efficient GBI strategy.

3.4.2.2. Allocation Recommendations

Based on their funding status and investor's priorities, the framework enables to categorise goals as essential, important and aspirational. In order to be admissible, a strategy has to secure all essential goals, that is, it must yield a 100% probability of reaching these goals.

It should be noted that this 100% probability must be robust to the choice of the model and the parameter values. For instance, one might find that under the assumption of a sufficiently high equity risk premium and with a sufficiently long horizon, a stock index has a 100% probability of reaching a certain level. But the strategy of investing in the stock only is not a reliable one to secure the goal because the realised return may significantly differ from the assumed expected return. Thus, the 100% probability is model- and parameter-dependent. We require instead strategies which secure essential goals for any choice of model and parameter values. For instance, GBI strategy (30) is suitable to protect a wealth-based goal because Proposition 15 shows

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that it secures the goal without referring to a particular model. Similarly, by Proposition 16, GBI strategy (31) secures a consumption-based goal, and this property holds under any model. Hence, these strategies are admissible to secure essential goals.

3.5. Mass Customisation Constraints

In closing, we would like to comment on the constraints on limited customisation. While providing each individual investor with a dedicated investment solution precisely tailored to meet their goals and constraints would be desirable, and while the proposed goals-based wealth management framework is precisely designed to achieve this objective, such a high degree of customisation would not be consistent with implementation constraints faced by financial advisors. In practice, it would be necessary to group individual investors in clusters with somewhat similar characteristics, and the outstanding question is whether the benefits of the framework are robust with respect with such mass-customisation implementation constraints. To answer this question, it is important to draw a key distinction between the building blocks and the allocation to the building blocks.

Turning first to the design of the building blocks, we note that there is a high degree of scalability involved in this process. For one thing, the composition of the market risk bucket, that is, the part of the investors' portfolio that is invested in a well-diversified performance-seeking portfolio, is in principle the same for all investors. From Modern Portfolio Theory, we know indeed that different investors with *different* expected return targets should invest in *different* proportions of the *same* two funds, namely the maximum Sharpe ratio portfolio and cash, with leverage used to achieve expected return targets that exceed the expected return on the maximum Sharpe ratio portfolio.

In other words, the best implementation proxy for the performance-seeking portfolio should in theory be offered to all investors, so mass customisation would involve no welfare loss at this level. In practice, however, the situation is somewhat different because of the presence of frictions such as the presence of short-sale constraints that justify more than one performance-seeking portfolio is needed to reach target expected return levels that extend beyond the expected return of the maximum Sharpe ratio portfolio. In the same vein, the presence of a home bias or any particular restriction on the menu of asset classes, could justify that different investors hold different performance-seeking portfolios, but these constraints can be accommodated in the context of a parsimonious approach involving a limited number of performance building blocks.

On the other hand, essential goals are specific to each investor, and the design of an essential GHP should therefore involve a high degree of customisation. Indeed, the most efficient approach to interest rate risk management, known as cash-flow matching, involves ensuring a perfect static match between the cash flows from the asset portfolio and the cash-flows required for consumption purposes. This technique, which provides the advantage of simplicity and allows, in theory, for perfect risk management, nevertheless

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presents at least two main limitations from a practical perspective. First of all, it will generally prove impossible to find fixed-income instruments whose maturity dates correspond exactly to the dates of the pension payments. Moreover, most of those securities pay out coupons, thereby leading to the problem of reinvesting the coupons. To the extent that perfect matching is not possible, financial advisors will have to resort instead to a technique called *immunisation*. Broadly speaking, the key difference is that immunisation strategies aim at ensuring a match between factor exposures in the goal-hedging portfolio and in the goal value process sides, which is a weaker requirement than ensuring a match between cash-flow payments; in other words cash-flow matching obviously implies interest rate exposure matching, while the converse is not true.

The most basic form of implementation of the immunisation approach can be performed in terms of duration matching, but the interest rate risk management technique extends to more general contexts, including for example hedging larger changes in interest rates (through the introduction of a convexity adjustment) or hedging against changes in the shape of the yield curve (see for example Fabozzi, Martellini and Priaulet (2005) for interest rate risk management in the presence of non-parallel yield curve shifts). It should be noted that these approaches can be implemented in principle either via cash instruments, typically sovereign bonds, or via derivatives such as interest rates swaps or futures contracts, even though the former approach is likely to be the preferred option in a wealth management context.
In conclusion regarding the design of hedging portfolios for essential goals, it appears from the previous discussion that financial advisors can achieve a robust implementation of GBI strategies provided they can have access to a series of bond portfolios (ideally with both a nominal and real versions available), with a limited number of target durations extending from the shortest to the longest durations, which can be used in most cases as reasonable proxies for goal-hedging portfolios. As a result, mass customisation can perfectly be applied with respect to the choice of the building blocks needed to implement GBI strategies. In an implementation stage, the appropriate granularity in terms of numbers and types of underlying building blocks can easily be assessed in terms of increases in probabilities to fail essential goals due to imperfect proxies for goal-hedging portfolios, with a key trade-off between increasing accuracy in implementing dedicated investment solutions and increasing costs of implementation.

On the other hand, turning from building blocks to allocation to building blocks, we note that it is in general impossible to offer a single strategy that would fit the need of several investors, even if these investors were sufficiently similar in terms of their goals to be offered the same menu of goal-hedging building blocks. Indeed, the allocation to the various building blocks typically depend upon ingredients that are specific to each investor, including current wealth levels and distance with respect to wealth- or consumption-based goals.

In this context, an outstanding question remains to determine whether a limited number of portfolios can serve as *reasonable proxies* for customised policy portfolios for a

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multitude of individual investors who would share a number of common characteristics. In particular, it is possible to re-interpret current practices from financial advisors, who use a number of model portfolios (say portfolios with an equity allocation of 20%, 40%, 60%, 70%, 80%, with the rest in bonds or cash) within the context of goals-based wealth management. According to this interpretation, such model portfolios can be regarded as arguably crude proxies for the aggregate wealth allocated to the personal and market risk buckets for various kinds of investors who have different levels of attention to essential GHPs, including protection against losses (justifying cash) and/or protection for long-term consumption needs (justifying bonds).

More generally, however, the proper mass-scale implementation of GBI strategies requires a dedicated allocation to a limited number of building blocks, which implies that a critical factor of success is the presence of an information technology system that can effectively process and update the key inputs of the framework at each point in time for each investor.

The critical importance of information systems, as well technological and transactional ability to implement in a cost-efficient way a large number of trades on behalf of individual investors, for the development of welfare-improving investment solution has been emphasised by Robert Merton in his Nobel lecture on December 9, 1997: "Deep and widespread disaggregation [of financial services] has left households with the responsibility for making important and technically complex decisions involving risk ... decisions that they had not had to make in the past, are not trained to make in the present, and are unlikely to efficiently execute even with attempts at education in the future. Financial engineering creativity, and the technological and transactional bases to implement that creativity, reliably and cost-effectively, are likely to become a central competitive element in the industry."

4. Case Studies

4. Case Studies

We want to apply the framework to three different problems that provide a fair representation for the variety of problems possibly encountered in wealth management.

The goal of these case studies is to show, through a dedicated Monte-Carlo simulation for all risk factors impacting the various risk buckets (see Appendix 6.5), that the opportunity costs implied by the need to respect the essential goals are significantly lower when these constraints are optimally addressed through dynamic goals-based investing strategies, as opposed to being inefficiently addressed through excessive hedging and an unconditional decrease of the allocation to risky market and speculative assets. Intuitively, the pre-commitment to reduce the allocation to risky assets in times and market conditions that require such a reduction so as to avoid over-spending risk budgets related to essential goals allows investors to invest on average more in such risky assets compared to a simple static strategy that is calibrated so as to respect the same risk budget constraints.

These insights will be developed on a number of case studies, according to the typical classification of individual investors that involves two main dimensions: life stage and affluence.

One may typically identify three clusters in terms of life stage:

- LS1: Accumulation (age less than 55 years)
- LS2: Transition (age between 55 and 65 years)
- LS3: Decumulation (age higher than 65 years)

One may also identify of three main clusters of affluence:

- A1: Mass affluent ($250,000 to $1m)
- A2: Affluent/high net worth ($1m to $5m)
- A3: Ultra high net worth (>$5m)

Among these nine finer clusters, 3 coarser clusters can be formed:

- C1: Accumulation/transition < $5m (LS1/A1, LS1/A2, LS2/A1, LS2/A2)
- C2: Decumulation < $5m (LS3/A1, LS3/A2)
- C3: UHNW, whatever the life stage (LS1/A3, LS2/A3, LS3/A3)

In what follows, we will provide a detailed analysis of three case studies, each one related to one of the three clusters C1, C2, C3, so that our case study selection is as follows:

Case 1 - HNW/UHNW Transition: A HNW/UHNW couple with substantial assets in the transition phase. This is a proxy for cluster C1.

Case 2 - HNW Retiree: A HNW couple at the beginning of the decumulation / retirement phase. This is a proxy for cluster C2.

Case 3 - Affluent Accumulator: An affluent young couple in the middle of the accumulation phase. This is a proxy for cluster C3.

4.1. Case Study 1 (HNW/UHNW Transition)

In this first case study, the investor is an executive with a net worth of $4.5m, holding a substantial concentrated stock position. His highest priority goal is to maintain a minimum net worth of $3m at all times.

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Table 1: Investor 1 - Current Risk and Asset Allocation and Goals.
(a) Risk and asset allocation

	Value ($)	% of Total		Value ($)	% of Total		Value ($)	% of Total
Personal Bucket	900,000	20.0	Market Bucket	2,150,000	47.8	Aspirational Bucket	1,450,000	32.2
Residence	1,500,000	65.3	Equity	1,500,000	69.8	Concentrated Stock	1,250,000	86.2
Cash	100,000	4.3	US Fixed Income	600,000	27.9	Executive Stock Option	100,000	6.9
Adjustable Rate Mortgage	(700,000)	30.4	Cash	50,000	2.3	Investment Real Estate	100,000	6.9

(b) Goals

Name	Goal	Time horizon (years)	Threshold
Goal 1 (wealth-based with multiple horizons)	Maintain minimum wealth (within liquid and aspirational)	1-35	$3m (inflation-adjusted)
Goal 2 (wealth-based with multiple horizons)	Avoid large drawdowns (within market)	1-35	15%
Goal 3 (wealth-based with single horizon)	Significantly increase wealth (within liquid and aspirational)	15	$7.2m (inflation-adjusted)

Panel (a) describes the current risk and asset allocation of Investor 1. Panel (b) describes his goals, which are ranked by order of decreasing priority.

4.1.1. Current Allocation and Goals

4.1.1.1. Description of Risk Buckets

The detailed composition of the current risk and asset allocation (through the lens of Wealth Allocation Framework[30]) is given in Table 1. The investor owns a house of value $1.5m, and is repaying a mortgage loan with a face value of $700,000. The personal risk bucket also contains a position of $100,000 in cash. As explained in Section 2.5.1, personal assets play the role of a guarantee for a minimum standard of living. In other words, they serve as a collateral against extreme adverse events: the investor does not want his family to end up homeless even in the event of huge losses within the other two buckets (market and aspirational) – hence the home ownership – and he wants to afford a minimum level of consumption – hence the cash reserve. Because of their special status, namely a guarantee for essential needs, the long positions in the house and the personal cash account will be left buy-and-hold throughout the case study.

There is also cash in the market bucket (only for a small proportion, of 2.3%), but unlike the previous one, this position is liquid and tradable. The market bucket is otherwise dominated by equities, which represent 69.8% of the allocation, versus 27.9% for US fixed income. In the remainder of this case study, we will model equity as a broad US stock index, and the fixed income class as a sovereign bond index (for brevity, we refer to the latter class simply as bonds). The last bucket consists of aspirational assets, that is, assets dedicated to wealth mobility. The dominant asset is a concentrated stock position, which represents 86.2% of the current bucket value. Executive stock options and investment real estate account for the remaining 13.8% of aspirational wealth,

30 - Based on the paper of Chhabra (2005).

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with equal contributions. One important difference between the market and the aspirational assets lies in their respective liquidity. Indeed, while the equity and bond indices are liquid assets, the investment real estate is subject to significant transaction costs, and it may be difficult to find counterparties for the concentrated stock and the executive stock options. Thus, dynamic trading is conceivable only within the market bucket. On the other hand, the aspirational bucket is either left buy-and-hold, or liquidated in one time at date 0 (we will study both situations below).

In the absence of information on the fixed rate of the mortgage rate and the amortisation scheme, we abstract away from the presence of this loan within the balance sheet and the budget equations. This means that the annuities (which encompass interest payment and principal repayment) are covered by an exogenous and non-modelled source of income. As a consequence, we take the initial personal wealth to be equal to the sum of the values of the residence and the cash position, that is $A_{per,0}$ =\$1.6m.

Liquid wealth is the sum of market wealth and the sum invested in the liquid personal assets: with the current risk allocation, it is equal to market wealth, since none of the assets held within the personal portfolio are liquid. But when we introduce in the personal bucket an asset dedicated to the hedging of an essential goal (i.e. a goal-hedging portfolio), liquid wealth will be the sum of market wealth and the sum invested in the GHP.

For parsimony, we model the values of the residence and the investment property with a single stochastic process Y. Similarly, we use a single process for both the concentrated stock value and the executive stock option, which we gather under the name of "illiquid stock value".[31] This process is denoted with X, and we model it as a stochastic process with the same expected return as the stock index (12%), but twice higher volatility (39.8% versus 19.9%).

Table 1 summarises our notations for the various stochastic processes (these notations will also apply to the other case studies). They are consistent with those of Section 2: wealth is still denoted A_t, and we use sub-indices to make a distinction between personal, market and aspirational wealth. However, for notational clarity, we use different letters for asset prices, as opposed to denoting them with S_{1t}, S_{2t},... The dynamics of the processes and the parameter values are given in Appendix 6.5.

Because it is useful to interpret some of the subsequent results, we report in Table 7 descriptive statistics on the simulated returns of the risky assets: expected return, volatility and maximum drawdown. These statistics are first computed in time series on each of the 10,000 simulated paths. The 10,000 values obtained are then averaged to produce the numbers contained in the table. The statistics are ordered as expected: the equity index and the illiquid stock have the highest expected returns (12%), but also the highest volatilities (19.9% and 39.8%) and the highest maximum drawdowns (46.3%, and a large 86.1% for the illiquid stock).

4.1.1.2 Description of Goals

Goals are also summarised in Table 1. The highest priority goal is Goal 1 (referred to as G1): it is a wealth-based goal that consists

31 - Alternatively, one could derive the value of the stock option from an option pricing model (e.g. the Black-Scholes model).

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of protecting a minimum level of $3m for the sum of liquid and aspirational wealth. This level is adjusted for inflation, which means that the goal value in year *t* is:

$$G_t^1 = G_0^1 \times \frac{\Phi_t}{\Phi_0},$$
$$t = 1, \ldots, 35.$$

(By convention, G_t^1 is zero if t is distinct from 1, ..., 35.) In this equation, Φ denotes the price index and G_0^1 is the face value of the goal, which is $3m. The ratio Φ_t/Φ_0 represents realised inflation between years 0 and *t*. In what follows, we shall normalise the current price index to 1, so we will omit the Φ_0 in the denominator. With the previous notation, G1 can be expressed as:

$$A_{liq,t} + A_{asp,t} \geq G_t^1,$$
$$\text{for } t = 1, \ldots, 35.$$

Goal 2 (G2) is a wealth-based goal that applies to liquid wealth only: liquid wealth is the sum of market wealth and the sum invested in the GHP, and the objective is to protect at least 85% of the maximum liquid wealth ever attained. Denoting the maximum-to-date of wealth with $\overline{A}_{liq,t}$, we can write this objective as:

$$A_{liq,t} \geq (1 - \delta)\overline{A}_{liq,t},$$
$$\text{for } t = 1, \ldots, 35,$$
$$\text{with } \delta = 15\%.$$

Finally, the third goal (G3) is to double the sum of current liquid and aspirational wealth in real terms at the horizon of 15 years. Mathematically, this objective reads:

$$A_{liq,t} + A_{asp,t} \geq G_t^3,$$
$$\text{for } t = 15,$$

$$G_t^3 = G_0^3 \times \frac{\Phi_t}{\Phi_0},$$
with G_0^3 = $7.2m.

4.1.1.3 Funding Status of Goals and Goal-Hedging Portfolios

For G1, a necessary and sufficient affordability criterion is given by Proposition 3. However, as explained in Section 2.2.1.2, the criterion takes a simple form if the 1-year real rate is nonnegative at all dates. We explicitly impose this condition of nonnegative 1-year real rates in our simulations.[32] Under these conditions, Corollary 1 shows that the present value of the goal, i.e. the minimum capital to invest in order to secure G1, is:

$$\tilde{G}_0^1 = G_0^1 \times \exp[-x_{0,1}^r].$$

(We recall that $x_{t,1}^r$ denotes the 1-year real rate at date t.) By Proposition 3, a strategy that secures G1 consists in investing $\tilde{G}_0^1$ in a roll-over of 1-year indexed bonds that pay G_0^1 plus realised inflation at the end of each year. The value of the GHP for G1 is the value of this roll-over. If $I_{s,t}$ is the price at date *s* of the indexed zero-coupon that pays Φ_t at date *t*, we have, by (6):

$$GHP_{G1,s} = \exp\left[\sum_{u=1}^{t-1} x_{u,1}^r\right] \times G_0^1 \times I_{s,t},$$
$$\text{for } t-1 < s \leq t \text{ and } t = 1, \ldots, 35,$$

$$GHP_{G1,0} = \tilde{G}_0^1. \quad (34)$$

Thus, G1 is affordable if, and only if, the investor is able to invest $\tilde{G}_0^1$ in the roll-over strategy. Given our parameter values, the constraint of a nonnegative real rate is binding at date 0 in the simulations, so $\tilde{G}_0^1$ is simply the face value of the goal, namely $3m. Thus, the minimum capital requirement to secure G1 is $3m.

For the third goal (G3), the minimum capital to invest is given by Proposition 1: it is the price of an inflation-indexed

32 - We do this by imposing a floor on the nominal short-term rate in the simulations (see Appendix 6.5). If a lower floor, or no floor at all, is imposed, then negative 1-year real rates can occur, and the roll-over strategy does not reach G1 with probability 1.

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zero-coupon bond that pays $7.2m plus inflation (that is, G^3_{15}) after 15 years. Our parameter assumptions imply a numerical value of $4,810,724.

Table 8 summarises the investor's balance sheet. The asset side consists of market and aspirational assets and the liability side contains the goals. It appears that G1 cannot be secured with current market wealth only, but would be affordable if aspirational assets could be liquidated: indeed, current market wealth is not greater than the face value of G1, but the sum of market and aspirational wealth is. Hence, G1 is part of the maximum set of affordable goals. In what follows, we treat it as an essential goal, i.e. as a goal that the investor would like to secure. We thus refer to this goal as Essential Goal 1 (in short, EG1).

In contrast, G3 cannot be funded with current assets: even if aspirational assets can be liquidated, the indexed zero-coupon bond that secures this goal is not affordable. Hence, G3 cannot be treated as an essential or important goal, and will therefore be categorised as an aspirational goal (in short, AG). The last line of Panel (ii) is the translation of Proposition 5 in the context of the case study: the minimum capital needed to secure two wealth-based goals is greater than or equal to the maximum of the two minimum capital requirements. Of course, since AG is not individually affordable, it is not jointly affordable with EG1 either.[33]

Finally, the drawdown-based goal (G2) is affordable regardless of the initial wealth, as a consequence of Proposition 3: it suffices to liquidate the current market wealth and to re-invest it in cash only, which guarantees that market wealth keeps growing. The categorisation of this goal as essential, important or aspirational, depends on whether it is jointly affordable with EG1. If it is, then it is part of the maximum set of affordable goals, and can thus be treated as essential or important, depending on whether or not the investor wants to secure it. Otherwise, it has to be considered an aspirational goal. The question is thus whether G2 can be secured along with EG1. The results below show that there do exist strategies that secure both goals: for instance, the GHP for EG1 or the cash account (see Section 4.1.2.4), and a dynamic GBI strategy that aims at protecting the maximum of two floors (see Section 4.1.3.3). These results confirm that G2 is jointly affordable with EG1. As a consequence, there are two possible statuses for this goal: essential or important. We will treat it as essential, that is, as a goal that the investor would like to secure: it will be referred to as Essential Goal 2 (in short, EG2).

As a conclusion:
- If aspirational assets are illiquid (i.e. cannot be liquidated at the initial date), G1 cannot be secured and must therefore be regarded as an aspirational goal;
- If aspirational assets are liquid, G1 is affordable and is treated as an essential goal;
- G3 cannot be secured, whether aspirational assets are liquid or not, and will be treated as an aspirational goal;
- G2 is affordable, and moreover it can be secured together with EG1. Thus, it will be treated as an essential goal.

In what follows, we look at the success indicators for various strategies.

4.1.2 Static Strategies

By static strategies, we mean strategies with weights that do not depend on current

33 - The exact computation of the minimum capital requirement for two wealth-based goals involves the pricing of an option written on the maximum of the two goals. In fact, in this case study, it could be shown that under reasonable assumptions on the bounds of real rates, the capital requirement for the two goals is equal to the capital requirement for AG.

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wealth, as opposed to GBI strategies, which will be tested below.

4.1.2.1 Current Strategy and Impact of Liquidity of Aspirational Assets

The "current strategy" is defined as a fixed-mix strategy that would maintain the same weights of equity, bond and cash as today, with an annual rebalancing frequency. Mathematically, the weights of this strategy are (see Table 6 for the definition of the notations):

$$\underline{\mathbf{w}}_{liq,CUR,t} = \underline{\mathbf{w}}_{liq,CUR,0}, \quad \text{for } t = 1, \ldots, 35,$$

where $\underline{\mathbf{w}}_{liq,CUR,0}$ is the current weight vector within the liquid bucket. For this strategy as well as for the subsequent ones, we compute a number of "success indicators" which quantify the degree of achievement of goals. The values of these indicators are shown in Figure 1. It appears that the current strategy has a probability of 40% of missing EG1. Since this goal is said to be "essential", this is a serious concern for the investor. Moreover, the expected shortfall for this goal is relatively large, around 20%, and in the worst case, the gap between wealth and the goal value can be as high as 80% of the goal value. One might argue that the investment strategy cannot be blamed in itself for this poor result, because current market wealth is too low to secure EG1 (see Table 8). This argument is admissible, because by absence of arbitrage, it is impossible to reach a goal with certainty if wealth is too low, whichever strategy is taken. To see whether an increase in market wealth would lead to better success chances, we recompute the success indicators by assuming that aspirational assets are liquidated at the initial date, and that the proceeds are re-invested in the market assets, with the same breakdown of weights as in the current market bucket. Figure 2 shows that although EG1 can now be secured, the shortfall probability for EG1 is 21.4%, which is lower than in the illiquid case, but still far from 0: hence, the investor has a significant probability of not having the desired minimum wealth level. That EG1 is not secured with the current asset allocation comes as no surprise, since the investor has no inflation-indexed bonds in his portfolio, but the numbers reported here show that shortfall risk is quantitatively important.

The success indicators for the drawdown-based goal (EG2) are by construction independent from the initial market wealth: indeed, the maximum drawdown of a fixed-mix strategy (and, more generally, for any strategy whose weights do not depend on wealth) is independent of the initial investment. Thus, the success indicators are the same whether aspirational assets are liquid or not. As appears from Figure 1, the current strategy is very unlikely to meet the objective of a 15% maximum drawdown, and high levels of maximum drawdown are to be expected: for instance, the expected maximum drawdown after 35 years is 25.2%, and in the worst case, the maximum drawdown takes the extremely high value of 73.9%. These high drawdown levels are due to the high proportion of stocks (69.8%): this asset class has the highest maximum drawdown among the market assets, and its weight is never revised, regardless of market conditions. Finally, Figure 1 and Figure 2 show that the current strategy has more than 50% of chances to reach AG. Although these probabilities may seem attractive as far as a secondary goal is concerned, they

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do not compensate for the low probabilities of reaching the essential goals.

We next investigate the existence of a relationship between the probability of reaching high wealth levels and the presence of aspirational assets. Indeed, the expected returns on the aspirational assets may make them attractive for wealth mobility, although the aspirational bucket is not meant to be a well-diversified portfolio in the sense of Modern Portfolio Theory. First, Figure 3 shows the distribution of market plus aspirational wealth, which is the wealth used to define the aspirational goal. In accordance with this definition, we focus on real wealth (i.e. wealth divided by the price index) after 15 years. It should be emphasised that the sum of wealth on date 0 is the same in both cases, so that the total wealth after 15 years can be compared with each other. The presence of aspirational assets clearly spreads the distribution: the minimum (0.76m of today's dollars) is slightly lower than when the aspirational bucket is liquidated ($1.20m), but the maximum of the distribution is much higher ($504.73m versus $93.10m). The larger span of the distribution in the illiquid case is of course due to the high volatility of the illiquid stock (39.8%). On the other hand, the median of the distribution slightly increases when aspirational assets are sold out (from $8.28m to $9.38m), which means that wealth levels around this median are more likely to be reached if aspirational wealth is re-invested in the market assets. As a conclusion, the presence of aspirational assets increases the uncertainty over future wealth levels, and does not increase the probability of reaching "medium" target levels (hence the decrease in the success probability for AG observed from Figure 1 to Figure 2). But these assets help to attain "very ambitious" wealth levels, which otherwise would be out of reach (here, the wealth levels comprised between $93.10m and $504.73m).[34] Figure 4 provides further evidence for the link between the performance of aspirational assets and the success chances for AG: it compares the success probability for this goal when the expected return of the illiquid stock equals that of the equity index (that is, μ_X=12%), and when it is twice as high (μ_X=24%). The second situation can model a private business with a very high expected return. The impact has the expected direction and it is substantial: the probability shifts from 57.6% to 81.9%.

34 - In unreported results, we have computed the success probabilities for a "very ambitious" aspirational goal, which would be to multiply the sum of market wealth and aspirational wealth by 10 in real terms at the 15-year horizon (the investor's AG is to multiply wealth by 2 only). This probability is 1.1% if one gives up aspirational assets, and 4.8% if they are kept in the portfolio.

In summary, if aspirational assets are not liquidated and the current market allocation is kept as it is today (in the form of a fixed-mix policy), the investor has substantial probabilities of being short each of the two essential goals. The probability of reaching EG1 increases if the aspirational bucket is liquidated, but remains low in view of the essential nature of this goal. These low success probabilities for essential goals are not compensated by the rather good success probabilities for AG.

Because EG1 cannot be secured with the current market wealth alone, we focus in what follows on the situation where aspirational assets are liquidated at the initial date, and we consider the illiquid case as a robustness check. Thus, unless otherwise stated, the investor's initial liquid wealth is:

$$A_{liq,0} = A_{mkt,0} + A_{asp,0} = \$3.6\text{m}.$$

4.1.2.2 Impact of Savings

The intuition suggests that the probability of success for the various goals can be improved by saving money. To give a

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quantitative assessment of this effect, we introduce a non-portfolio income stream in the dynamics of market wealth. This stream occurs at the end of each calendar year, and has a constant real value Sav_0. Thus, the savings in year t are:

$$Sav_t = Sav_0 \times \frac{\Phi_t}{\Phi_0}, \text{ for } t = 1,\dots,35,$$

$$Sav_0 = \$0,\ 50k, 100k,\ 200k,\ 250k,\ 500k, 1m.$$

Because of this payment does not come from the assets held in the portfolio, we must specify how it is invested in the portfolio: we assume that it is invested in such a way that the weights of stock, bonds and cash remain the same immediately after the payment as before (see Appendix 6.6 for formal expressions for the number of shares of each asset before and after the savings).

Figure 5 displays success indicators for the various goals. We recall that the success probability for a goal is the probability of reaching this goal; the expected shortfall is the expectation of the maximum shortfall recorded across goal horizons, conditional on the event of a shortfall; and the expected maximum drawdown is the expectation of the maximum drawdown recorded over the 35 years. Of course, the numbers obtained for savings equal to zero are identical to those reported in Figure 2. Unsurprisingly, higher savings imply higher chances to reach each goal, and lower average deviations from the targets. For instance, when savings grow from 0 to \$200k, the success probability for EG1 grows from 78.6% to 90%, and the expected shortfall decreases from 14.9% to 7.94%. Nevertheless, the shortfall risk does not completely disappear, even for a level of savings of \$1m per year, which is huge compared to the investor's current wealth (\$4.5m including the personal risk bucket).

For the drawdown-based goal (EG2), the situation is worse, because the probability of keeping the drawdown below 15% remains capped at 29.6%, a value attained only with the unrealistic level of \$1m of savings per year. The expected maximum drawdown is still 19.7% in this case. Hence, the drawdown of the portfolio is not under control. It is only for the aspirational goal that the success probability reaches 100% with \$1m of annual savings, but this does not make up for the high chances of missing the essential goals.

As a conclusion, the current strategy does not reach essential goals with a satisfactory confidence level, even when the investor infuses substantial amounts of money into his market portfolio every year. In other words, the investor cannot rely only on savings to secure the most important goals.

4.1.2.3 Using Diversification: MSR Strategy

In view of the impossibility of securing essential goals with the current strategy, one can think of using scientific diversification in order to improve the success chances. This approach can be justified to some extent by the literature on goals-based wealth management (see Section 3.2 above). Indeed, the MSR portfolio is a building block of the optimal strategies for many optimality criteria: maximisation of success probability, minimisation of expected time to reach the goal, maximisation of expected utility with or without performance constraint, etc. This collection of optimality results suggests that the MSR has merits in the context of goals-based wealth management, and motivates the introduction of a strategy that invests only in the MSR.

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Constructing an MSR portfolio requires the knowledge of risk and return parameters. The introduction of parameter uncertainty is beyond the scope of this paper, so we assume that these quantities are perfectly known to the investor. However, this assumption does not sound realistic for aspirational assets, which have low liquidity and for which it may be difficult or impossible to find enough historical data to perform a reliable estimation. Thus, we only consider an MSR portfolio of the equity and bond indices, and we leave the aspirational bucket outside the optimisation. The parameter values given in Appendix 6.5 imply that the MSR portfolio is:

$$\underline{w}_{MSR} = \begin{pmatrix} 0.8042 \\ 0.1958 \end{pmatrix},$$

where the first element is the stock weight and the second one is the bond weight. This allocation is different from the current-stock bond allocation. Indeed, after removing the leverage effect in the market bucket, the stock weight is 0.698/(0.698+0.279)=0.714. We thus consider a fixed-mix strategy that maintains constant weights within the market bucket. We do monthly rebalancing: this is a relatively high frequency, but this choice prevents the weights from drifting too far away from the target, which allows in principle to take the most of diversification benefits. A first observation from Figure 6 is that these benefits are not sufficient to reach EG1 with certainty. The success probability (74.5%) is even slightly lower than for the current strategy (78.6%, in Figure 2). This arises because the MSR portfolio contains a lower fraction of bonds than the current market bucket (19.6% versus 27.9%): indeed, bonds, even though they are fixed-income securities, are better proxies than stocks for the roll-over of 1-year indexed bonds that secures EG1. The success probability for EG2 is also lower than for the current strategy (5.9% versus 12.2% in Figure 1): again, this is an effect of the higher stock weight in the MSR portfolio (the equity index has higher maximum drawdown than the bond index). This higher stock allocation also accounts for the higher probability of reaching AG.

Overall, scientific diversification implemented through a maximum Sharpe ratio portfolio does not enable the investor to secure essential goals such as protecting a minimum level of real wealth and avoiding large drawdowns. It should be noted that this result has nothing to do with imperfect parameter estimation, which is one of the main concerns raised by the implementation of mean-variance efficient strategies. The reason for the lack of success in reaching essential goals is simply that the construction process of the well-diversified portfolio does not explicitly aim at avoiding losses.

4.1.2.4 Using Hedging: Safe Strategies

In order to make sure that EG1 is attained with probability 1, we consider a strategy that fully invests in the roll-over of 1-year indexed bonds. Because aspirational assets are liquidated at date 0, the initial wealth ($3.6m) exceeds the goal face value ($3m). Section 4.1.1.2 shows that in this context, the roll-over policy secures EG1. In other words, the portfolio is not well diversified in the sense of mean-variance theory (because it is invested in a single asset and does not target any risk-return trade-off), but it is safe with respect to EG1: hence, it can be called a hedging portfolio. The success indicators reported in Figure 7 show that not only EG1 is secured, as was expected, but EG2 is reached with certainty too. That is not to say that the roll-over has no drawdown at all, but it turns out that the worst maximum

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drawdown after 35 years is only 4.12%: this means that over the 35 years and across all simulated paths, the roll-over strategy never loses more than 4.12% of its maximum-to-date. This value lies comfortably below 15%, so EG2 is secured. This result is interesting because the roll-over has not been explicitly designed to ensure the achievement of EG2. At this stage, the hedging strategy represents an improvement over the current and the MSR strategies, in that it secures both essential goals. But as appears from Figure 7, the success probability for AG is severely decreased with respect to the other two strategies: it falls to 8.1% only, while the current and the MSR strategies displayed probabilities of 67.8% and 69.5% respectively. The reason for this sharp decrease is that the roll-over is a rather conservative strategy, which invests in assets with low expected returns compared to stocks: indeed, from Table 7, the equity index has an expected return of 12% per year, approximately twice as high as that of the roll-over (5.7%). Moreover, the volatility is low (only 2% for the roll-over), which implies a relatively narrow distribution for wealth, and consequently, leaves little chance to reach high wealth levels. This analysis exemplifies the limit of hedging as a risk management technique: it effectively eliminates downside risk, but does so at an exceedingly high opportunity cost, which compromises the ability to reach ambitious goals.

The previous strategy favours EG1 ex-ante, and turns out to secure EG2 too. One could take another perspective, by favouring EG2. This leads to a safe strategy invested in cash only. As can be seen on Figure 8, the results are similar. EG2 is secured by construction (the value of cash never decreases), and EG1 turns out to be secured too. The latter property can be explained by two factors. First, the condition of nonnegative 1-year real rates implies that the short-term rate cannot be lower than a positive floor, which has a positive impact on the performance of cash. Second, by liquidating the aspirational assets, one starts with an initial wealth of \$3.6m, which is well above the face value of EG1, which is \$3m: the difference of \$600,000 provides a safety margin to absorb large positive inflation shocks. As a result, investing in cash secures EG1. But the upside potential of this second safe strategy is not much better than that of the first one: the probability of reaching AG is only 9.7%, and the expected shortfall with respect to this goal is also close to 30%.

As a conclusion, the comparison between the MSR and the safe strategies highlights a trade-off between performance and hedging: the MSR strategy has an interesting probability of reaching AG, but does not attain the essential goals with sufficiently high probabilities, while the safe strategies secure these goals, but have low upside potential.

4.1.2.5 Combining Diversification and Hedging: Buy-and-Hold Strategy

Given the aforementioned trade-off, it is natural to seek to combine the respective advantages of the MSR and the GHP. Indeed, EG1 can be secured by purchasing one share of the GHP, which has a cost $\widehat{EG}_0^1$ (the present value of the goal). The remainder of wealth, $\left[A_0 - \widehat{EG}_0^1\right]$, is then invested in the MSR portfolio. Although the MSR is a fixed-mix portfolio of stocks and bonds which is rebalanced every month and the GHP is a roll-over of indexed bonds, the mixture strategy regarded as a portfolio

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of the GHP and the MSR is buy-and-hold. It should also be noted that there is no obligation to take the MSR as the second building block: for instance, this block could be fully invested in stocks, or it could be the result of an expected utility maximisation performed without the goal. The reason why we choose the MSR is that this portfolio has theoretical grounds, and unlike the utility-maximising policies, it does not depend on an unobservable risk aversion parameter.

Let $A_{MSR,t}$ denote the value of the MSR portfolio rebalanced on a monthly basis with an initial investment of \$1. The value of the buy-and-hold strategy is thus:

$$A_{liq,t} = GHP_{EG1,t} + \left(A_0 - \widetilde{EG}_0^1\right)A_{MSR,t}.$$

Of course, for this quantity to be greater than $GHP_{EG1,t}$, it is necessary to have $A_0 \geq \widetilde{EG}_0^1$. This condition is satisfied since we have assumed that aspirational assets are liquidated at date 0. However, the amount of money invested in the MSR strategy is low, since the price of one share of the GHP represents a significant proportion of initial wealth. We have:

$$A_{liq,0} = \$3.6\text{m}, \quad \widetilde{EG}_0^1 = \$3\text{m},$$

so that only \$600,000 are invested in the MSR. In this context, we expect the buy-and-hold strategy to be closer to the safe strategy than to the MSR one, and hence to be a conservative policy.

The initial weights of the buy-and-hold strategy are shown in Table 9, and can be compared with those of the current strategy, in Table 1. For both strategies, the personal risk bucket contains the residence and the cash reserve, which serve to fund the implicit goals. But while the current allocation does not involve any asset dedicated to the protection of EG1, the buy-and-hold strategy assigns a positive weight to the GHP: since the role of this asset is to secure an essential goal, it is included in the personal bucket. It even turns out to be the dominant asset in this bucket, since it represents 56.6% of personal wealth. The conservative aspect of the buy-and-hold strategy is reflected in the fact that the safety assets that constitute the personal bucket account for 86.7% of total wealth, while the market bucket represents only 13.3%.

Figure 9 confirms that the strategy resembles the safe strategy more than the MSR one. First, EG1 is secured, as it should be, due to the buy-and-hold position in the GHP. Second, drawdown risk is slightly higher than with the safe strategy (see Figure 7 for comparison): the expected maximum drawdown over the 35 years is 5.57%, versus 1.91% for the roll-over. Moreover, some of the drawdowns exceed the 15% threshold, so that the success probability for EG2 falls from 100% to 93.4%: this remains a large probability. Finally, the success probability for AG is 27.4%, which is between the values obtained for the GHP (8.1%) and the MSR (69.5%).

Overall, the buy-and-hold strategy secures EG1, and gives success probabilities for EG2 and AG that lie between those of the separate building blocks. But EG2 is not fully secured although it is essential (there remains a 6.6% failure probability), and the probability of reaching AG seems low compared to what can be achieved with the MSR. Hence the idea to test alternative strategies that still secure EG1 and improve the success indicators for the other two goals.

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4.1.3 Goals-Based Investing Strategies

In this section, we turn to the implementation of GBI strategies, following the description in Section 3.3.2.

4.1.3.1 Goals-Based Investing Strategy Securing EG1

The GBI strategy that we implement here is an adaptation of the one described in Equation (30). The difference between this equation and the strategy that we actually test is that we impose short-sales constraints and we take into account the possibility of floor violations caused by gap risk. The strategy has the same form as a CPPI, with the GHP playing the role of the safe asset and the MSR that of the performance asset. As follows from the discussion in Section 3.3.2.1, the floor is the present value of the goal, that is:

$$\widetilde{EG}_s^1 = EG_0^1 \times I_{s,t},$$

for t -1< $s \leq t$ and t =1,...,35,

$$\widetilde{EG}_0^1 = EG_0^1 \times I_{0,1},$$

$I_{s,t}$ being the price of the indexed bond that pays Φ_t at the end of year t. This floor is discontinuous at the end of each year. Indeed, we have:

$$\frac{\widetilde{EG}_{t+}^1}{\widetilde{EG}_t^1} = \frac{EG_0^1 \times I_{t,t+1}}{EG_0^1 \times \Phi_t} = \exp\left[-y_{t,1}^r\right],$$

where $y_{t,1}^r$ is the real rate of maturity 1 year prevailing at date t. Because real rates are nonnegative by assumption, the floor exhibits a negative jump on this date, unless the real rate is zero, in which case the jump vanishes. It should be noted that the floor is distinct from the GHP value, which is a difference with respect to a standard CPPI, where the safe asset replicates the goal value. The discrepancy between the two values comes from the fact that the goal has multiple horizons. One could envision an alternative version of (30) where the floor is taken to be the GHP value, but since the GHP super-replicates the floor, this would lead to lower risk budgets. These lower budgets would likely result in reduced access to the performance of the MSR.

The next step in the definition of the GBI strategy is the computation of the risk budget. From the definition of the goal, the reference wealth to take into account here is the sum of liquid and aspirational wealth. Because the aspirational bucket has been liquidated, this sum coincides with liquid wealth. In the continuous-time framework, the risk budget is computed as the difference between the reference wealth and the goal present value (see Equation (30)). With continuous rebalancing, this difference is always nonnegative, but with discrete rebalancing, it may become negative. Should this happen, we set the risk budget equal to zero. Hence, the risk budget is computed as:

$$RB_t = \max\left[0, A_{liq,t} - \widetilde{EG}_t^1\right].$$

A second modification is the imposition of a no-short sale constraint in the GHP. Indeed, by Equation (30), the amount invested in the MSR is $m \times RB_t$, which may exceed the liquid wealth if the risk budget and/or the multiplier is large. As a consequence, the investor would have a short position in the GHP. In order to avoid this, we cap the amount invested in the MSR to the liquid wealth, so that this amount is given by:

$$q_{MSR,t} = \max\left[A_{liq,t}, m \times RB_t\right].$$

Appendix 6.6.3.1 provides a detailed expression for the sums invested by the strategy in the various locally risky assets (i.e. all assets except cash).

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The fact that the GHP value differs from the floor has a noteworthy implication when it comes to the GBI strategy with $m = 1$: if the two values are equal, the GBI with m = 1 would collapse to the buy-and-hold strategy tested in Section 4.1.2.5. But because they are distinct, the GBI with $m = 1$ is still a dynamic strategy, which involves rebalancing.

We implement the GBI strategy with a monthly rebalancing period, and we take a base case value of 5 for the multiplier. There is no particular justification for this exact value: on the one hand, m must be sufficiently high to guarantee decent access to the upside potential of the MSR; on the other hand, a too high value will increase gap risk.

Table 10 shows the initial allocation according to the GBI strategy, with assets sorted in personal and market buckets. The composition of the market bucket is the MSR allocation to stocks and bonds, and is therefore the same as for the buy-and-hold strategy. But a striking difference with respect to the latter strategy is that the personal bucket, which consists of all assets held for safety motives, accounts for 33.3% of total wealth, which is much lower than the fraction of 86.7% obtained for the buy-and-hold strategy. In view of this number, we expect the GBI strategy to be less conservative.

To check whether this is the case, we look at the success indicators in Figure 10. First of all, the success probability for EG1 is 100%. This result is not surprising in view of Proposition 15, which shows that the goal is secured if the portfolio is rebalanced continuously. But floor violations could have been observed due to the discrete (monthly) rebalancing: the number here shows that this is not the case (but we will see below that gap risk arises for larger values of m). In contrast, the success probability for EG2 is very disappointing: there is only a 8.7% chance of maintaining the drawdown below 15%, and the expected maximum drawdown after 35 years is as high as 27.4%. This result stresses the need to add a specific control for drawdown risk in addition to the risk control for EG1. Finally, the strategy yields a 62.7% probability of reaching AG, which represents a substantial improvement over the buy-and-hold portfolio implemented in Section 4.1.2.5.

4.1.3.2 Impacts of Multiplier, Trading Frequency and Stock Performance

In this section, we study the impacts on the GBI strategy of the multiplier and the trading frequency.

Table 11 shows the allocation to personal assets as a function of the multiplier. It reports both the composition of the personal bucket, and the weight of this bucket within the investor's risk allocation. The aspirational bucket is not shown in the figure because it is always empty, and the market bucket is not shown either, because its composition is independent from the multiplier (it is the MSR allocation to stocks and bonds) and its weight is simply one minus the weight of the personal bucket. It should be noted that the sums invested in the residence and the cash reserve are constant (these are the values given in Table 1), but the weights of these two assets within the personal bucket vary with m because the allocation to the GHP depends on m.

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This table simply describes the mechanics of the GBI allocation formula. The allocation to the personal bucket is decreasing in *m*, shifting from 86.7% for *m* = 1 to 20.0% for *m* = 7. This evolution reflects the growing allocation to the performance assets (stock and bond indices) contained in the market bucket as m increases. The weight of the GHP also turns out to be decreasing in *m*. This is because the sum invested in the GHP is by definition decreasing in m. Indeed, we have (see Appendix 6.6.3):

$$q_{GHP,t} = \max[0, A_{liq,t} - m \times RB_t].$$

As a result, the residence and the cash account represent increasing fractions of the personal bucket. For a very large *m*, the allocation to the GHP shrinks to zero because of the lower bound set at zero in $q_{GHP,t}$. Hence, the personal bucket is entirely invested in the residence and the cash account, and its weight is the same as with the current allocation, that is 20.0% (see Table 9). It turns out that this limit is reached for *m* = 7.

We next perform a robustness check of the success indicators of the GBI strategy with respect to the multiplier and the trading frequency. The first purpose of this study is to check that EG1 is still secured. Indeed, a lower rebalancing frequency may increase gap risk, and conversely, it is expected that increasing m will lead to violations of the floor.

Figure 11 presents the results of the comparative static analysis with respect to these two parameters. In the static analysis with respect to frequency, *m* is kept equal to its base case value, namely 5, and throughout the analysis with respect to *m*, monthly rebalancing is assumed. We first let *m* vary from 0 to 10 (*m* = 0 is a limit case where the GBI strategy is fully invested in the GHP). It appears that EG1 is secured for any choice of *m* between 0 and 6 (included), but gap risk starts to materialise as of *m* = 7. Nevertheless, deviations from the goal remain extremely small, with an expected shortfall less than 1%, even for m as large as 10.

On the other hand, decreasing the trading frequency from monthly to quarterly, semi-annual or annual, has more impact on the shortfall: with one rebalancing per year, the shortfall probability is almost equal to what was obtained with monthly trading and *m* = 9, but the expected shortfall exceeds 4%, while it was less than 0.7% in the other case. Hence, as far as the achievement of EG1 is concerned, decreasing the trading frequency appears to be more detrimental than increasing *m*.

When it comes to EG2, it appears that the trading frequency has very little impact on the success indicators: the success probability remains close to 8%, which is very low, and the expected maximum drawdown is above 25%, much higher than the maximum tolerated level of 15%. The impact of *m* is weak as well, at least in the range [2,10]. Only the values *m* = 0 and 1 stand out, with lower shortfall probabilities and drawdown levels. Indeed, these strategies are rather conservative, with a low (and even zero for *m* = 0) allocation to MSR.

But these better scores for EG2 come at the cost of modest probabilities to reach AG, especially for *m* = 0 (as we already know from Figure 7). It is only for *m* = 2 that the success probability exceeds 50%, reaching a

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maximum of about 62% for $m = 5$. The fact that the probability reaches a ceiling can be explained as follows. With high values of m, and given the initial risk budget, the portfolio is fully invested in the PSP at date 0, but one day happens when the PSP value falls below the goal present value (see Figure 6). On that date – the same for all values of m greater than or equal to 5 –, the portfolio is invested in the GHP only, and remains so until wealth is back above the goal present value: this can happen, since, by Equation (34), the GHP super-replicates, rather than replicates, the goal. Thus, all portfolios have the same composition until the GHP value exceeds the goal present value, which explains the closeness of the shortfall probabilities. The graph of the expected shortfalls highlights the usual trade-off between return and risk: a higher m increases the probability of arriving at the goal but it also creates volatility, which increases the size of potential deviations. The other parameter tested, namely the rebalancing frequency, has no visible impact on the success indicators for this goal.

Apart from the choice of a higher m, another way of improving the likelihood of high wealth levels is to invest in a stock index with a higher expected return. This can be achieved, for instance, by switching from a cap-weighted index to a smart-weighted index, which exhibits better performance (see Section 2.5.2.2 for a brief presentation). We model this change by raising the expected return of the stock index by 25%, that is, from 12% (its base case value) to 15%. Figure 12 shows that gap risk is still negligible here, since the success probability for EG1 is 100%, and that the success probability for AG has increased appreciably, to 80.7%. But the issue of drawdown is left unaddressed: the probability for EG2 is only 15.3%, certainly better than 8.7%, but still low.

As a conclusion, a GBI strategy intended to secure EG1 always reaches this objective, unless the rebalancing takes place less frequently than every month. With monthly rebalancing, EG1 is secured, except for a very large m. The largest values of m (≥ 7) entail gap risk, but the deviations are very limited in size. Thus, the choice of a value for m has to be done on the basis of other criteria than the achievement of EG1. Broadly speaking, a larger m will improve the chances to reach ambitious wealth levels, but will increase the drawdown of the strategy. Hence, at this stage, the choice of m depends on how much upside potential the investor is ready to sacrifice in order to secure EG2. To solve this dilemma, it is of interest to consider strategies which secure both essential goals. This extension is all the more important because the current GBI strategy does not control the drawdown in a reliable way, except in the degenerate case where $m = 0$.

4.1.3.3 Goals-Based Investing Strategy Securing EG1 and EG2

As explained in Section 3.3.2.4, we protect the two essential goals by implementing a GBI strategy of the form (32). The floor is the maximum of the floors associated with EG1 and EG2. For EG1, the floor on a given rebalancing date is the present value of the goal, namely $\widehat{EG}_t^1$. The floor associated with EG2 is the drawdown floor, which is 85% of the maximum wealth ever attained. The two GHPs are respectively the roll-over of 1-year indexed bonds and the cash account. The strategy uses as a safe asset the GHP that corresponds to the higher floor. The detailed

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expression for the weights of the strategy can be found in Appendix 6.6.3.2.We still take a base case multiplier of 5 and assume monthly rebalancing. For clarity, we refer in what follows to this strategy as GBI2, and to the GBI strategy that protects only EG1 as GBI1.

First, Table 12 shows the risk allocation at date 0 implied by the GBI2 policy. Note that the personal bucket now contains two GHPs. The allocation to the GHP protecting EG1 is zero because the drawdown floor is higher. Indeed, it is equal to 85% of the initial liquid wealth, which is \$3.6m: it is thus \$3.06m, which is larger than \$3m, the value of the floor associated with EG1. Logically, since this strategy aims at protecting two goals, it is more conservative than the GBI1 one, so the personal bucket represents a larger fraction of investor's wealth: 40% versus 33.3%.

As appears from Figure 13, the first benefit of the GBI2 strategy over the GBI1 one is that it fully secures EG2: indeed, the maximum possible drawdown, across all dates and states of the world, is less than 15%. This improvement is all the more appreciable because the GBI1 rule respects the drawdown constraint in only 8.7% of paths, which is a very low score. Among all the strategies that have been tested so far, only the safe strategies, i.e. the one invested in the roll-over of indexed bonds and the one invested in cash, displayed probabilities of 100% for both essential goals (Figure 7 and Figure 8). But the key improvement here with respect to these conservative policies is the success probability for AG: while safe strategies achieve this goal with a less than 10% probability, the GBI2 strategy displays a much higher probability of 54.5%. This is a consequence of the non-zero position in the MSR: this portfolio has higher expected return than the safe assets, so its presence increases the potential for performance at the strategy level.

Nevertheless, AG is less likely to be achieved with the GBI2 strategy than with the GBI1 one. Indeed, the success probability with GBI1 was 62.7% (but it should be recalled that it came with a low probability of meeting the drawdown objective). This reduction reflects the opportunity cost of imposing the constraint of a 15% maximum drawdown. Indeed, the floor of the GBI2 strategy is by definition higher than that of the GBI1 one, so the risk budget is mechanically lower. This means a lower allocation to the MSR, hence a reduced access to upside.

As for the GBI1 strategy, we study the impacts of the multiplier and the rebalancing frequency. The objective is in particular to see whether some choices of these two parameters lead to violations of the floor constraints. The success probabilities for EG1 shown in Figure 14 display the same pattern as for the GBI1 strategy: EG1 is secured for any choice of m below 6, and violations arise as of $m = 7$, but they remain very limited in size, with an expected shortfall less than 1%. But the most striking difference between GBI1 and GBI2 is that while GBI1 has little chance to reach EG2, the success probability for this goal is 100% for any m between 0 and 8, and it is hardly less than 100% for $m = 9$ or 10. Even for these two values, the expected maximum drawdown is less than 15%, which shows that the violations of the 15% constraint are caused by a few extreme scenarios. But the success probabilities for AG are always

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less with GBI2 than with GBI1, which is a manifestation of the lower allocation to the MSR.

The fact that the GBI2 rule strategy allocates less to the MSR than the GBI1 one may also explain why it is less subject to gap risk when the rebalancing frequency is decreased: indeed, both essential goals are always respected for monthly and quarterly rebalancing, and the success probabilities obtained for semi-annual or annual rebalancing, while being less than 100%, are higher than those achieved with the GBI1 policy. The downside deviations from EG1 are also smaller than for GBI1, and the expected maximum drawdowns are greatly reduced. This clearly shows the usefulness of the drawdown control. But this reduction of drawdown comes at the cost of a decreased probability of reaching ambitious wealth levels, as can be seen from the success probabilities for AG, which are lower with GBI2 than with GBI1 for any choice of rebalancing frequency.

In order to have a direct measure of the opportunity cost, we compute the additional initial capital which must be invested in the GBI2 strategy in order to generate the same probability of reaching AG as with the GBI1 policy. This indicator is inspired by the monetary utility gain, which is often reported in the literature on optimal portfolio choice and is the additional initial investment needed to achieve the same expected utility as with a benchmark strategy (see e.g. Sangvinatsos and Wachter (2005) and Martellini and Milhau (2010)). Here, expected utility is replaced by the success probability for AG, and the benchmark is the GBI1 strategy. In Figure 15, we compute the success probability and the opportunity cost for various values of the maximum drawdown (DD) introduced in the GBI strategy: this parameter is the maximum DD that the investor is ready to accept, and it is used to compute the DD floor. By definition, a zero maximum DD leads to a portfolio fully invested in cash, and a 100% maximum drawdown to the GBI1 strategy, which does not attempt to control the drawdown. Of course, imposing a 100% maximum DD does not imply that the portfolio will effectively display such a large loss: the worst maximum DD of the GBI1 strategy is 75.3%, a value which represents the worst possible DD for a GBI2 strategy.

The figure confirms the previous explanation: indeed, the probability of reaching AG is increasing in the maximum DD, which means that a tighter drawdown constraint results in a lower probability of reaching ambitious goals. As a consequence, the opportunity cost measured in terms of additional initial wealth strongly increases when the tolerance for DD risk decreases: an investor who refuses any losses should invest in cash only, and he would have to multiply his initial investment by 1.57 in order to achieve the same success probability as if he was investing according to the GBI1 rule. Our base case maximum DD, which is 15%, yields a cost of 8.55%: this percentage is much lower than for the strategy fully invested in cash, but it is still non-negligible. This illustrates that the drawdown constraint has a significant cost in terms of performance.

As a conclusion, the GBI Strategy (32), which switches between the GHPs associated with the two essential goals, does secure these goals and proves to be rather robust to

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gap risk. This risk materialises only for high values of m (namely $m \geq 8$): it affects more the probability of reaching EG1 than that of reaching EG2, but as for the GBI strategy securing EG1 only, the deviations from the goal remain small on average. This strategy also displays improved robustness with respect to the choice of the rebalancing frequency. Indeed, it secures both goals not only when the portfolio is rebalanced every month, but also with quarterly rebalancing, while the GBI strategy dedicated to EG1 secures this goal only with monthly rebalancing. But the protection of two essential goals as opposed to one implies a more conservative investment policy, which results in lower probabilities of reaching high wealth levels. In view of these results, it is the values 6 and 7 for m which give the highest success probabilities for AG while ensuring that EG1 and EG2 are secured. The next step is to test another form of GBI strategy to try to improve the chances of reaching AG while keeping the protection of EG1 and EG2.

4.1.3.4 Goals-Based Investing Strategy Securing EG1 and EG2 with a Cap

We recall from Section 3.3.2.5 that the idea behind the imposition of a cap is to reduce the cost of insurance against downside risk. This approach is justified by the theoretical results of Proposition 13 and Corollary 3: when a cap is imposed, the optimal strategy involves a short position in a call option written on the performance assets, and the premium received by selling this option decreases the price of the put to purchase in order to secure the goal. Equivalently, this strategy captures a larger fraction of the performance of the MSR than the strategy which does not set any upper bound on wealth.

Because AG is not affordable, while EG1 and EG2 are, it represents the investor's highest goal. Hence, it is reasonable to assume that the investor has no utility in having a wealth level in excess of this goal. Thus, we first test a GBI strategy of the form (33) where the cap is the present value of AG. In order to protect both goals, we still take the floor to be the maximum of the floors respectively associated with EG1 and EG2, and the FHP of the strategy is the FHP that corresponds to the higher floor. In line with the definition of the cap, the CHP is the indexed zero-coupon bond that matures at the end of 15 years and pays \$7.2m in real terms. Of course, this bond is only available for the first 15 years, so we switch to the GBI2 strategy (i.e. the one protecting both essential goals) after the bond has expired. In the following comments, we refer to the GBI strategy with a cap as GBI3. The detailed expressions for the weights are given in Appendix 6.6.3.3. The initial risk allocation is in fact the same as with the GBI2 strategy (see Table 13) because the allocation at date 0 to the CHP is zero. Indeed, the numerical value of the threshold defined in Section 3.2.2.5 is (in millions of dollars):

$$\xi_0 = \frac{\tilde{F}_0 + \tilde{C}_0}{2} = \frac{3.06 + 4.81}{2} = 3.94,$$

which is greater than the liquid wealth of \$3.6m. Thus, at date 0, the investor acts as if there was no cap, and only takes care of floor protection.

It turns out from Panel (a) of Figure 16 that imposing a cap equal to the present value of AG leads to a lower probability of reaching this goal with respect to the GBI2 strategy. In particular, the success probability is 23.8%, versus 54.5% (see Figure 13) for the latter strategy, which was

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to be expected since risk taking is reduced before the cap/ aspirational wealth level is reached. Hence the reduction in success probability is not inconsistent with the fact that the strategy with a cap has in theory a higher access to the upside (see Corollary 3). Indeed, this property does not imply that imposing a cap increases the chances of reaching any wealth level: the levels in excess of the cap will never be attained, and it is only for "medium" levels, that the distribution of wealth is improved. As an attempt to increase the success probability for AG, we may decide to choose the cap level in such a way that the present value of AG lies between the floor and the cap. This is what we do in Panels (b) and (c), where we set the cap equal respectively to 2 and 3 times the present value of AG. The effect on the probability is positive since this indicator grows to 54.9% and 54.5%, but these values do not represent a significant improvement with respect to the GBI2 strategy, for which the probability was already 54.5%.

On the other hand, leaving aside the probability of reaching a goal, a key benefit of the introduction of a cap is that it involves a very significant positive impact on expected shortfall. For example, in the case of the cap taken to be at the AG level, we obtain that the expected shortfall has fallen by two-thirds, decreasing from 24.7% to 8.14%. As a result, we confirm that introducing a cap, which leads to securing the allocation strategy when the wealth process approaches target wealth levels, involves a reduction in the opportunity cost of downside risk protection, which in turn translates into lower expected shortfall excessive risk taking in situations when a goal is almost reached.

4.1.3.5 Impact of Illiquid Positions

The previous analysis has mainly focused on the case where aspirational assets can be liquidated, because this is a necessary condition for the affordability of EG1. If these assets cannot be liquidated, the goal cannot be secured. Thus, it has to be regarded as an aspirational goal. Despite this change of status, we still refer to it as Essential Goal 1 in what follows in order to have a terminology consistent with the one employed in the previous analysis. The purpose of this section is to see how the GBI strategy behaves in the presence of an illiquid bucket. Because aspirational assets are not liquidated, the initial liquid wealth is:

$$A_{liq,0} = A_{mkt,0} = \$2.15\text{m}.$$

First, the reference wealth to take into account to compute the risk budget is the sum of market and aspirational wealth, since the goal expressed by the investor is to keep the sum of these two quantities above $3m plus inflation. This leads to the following risk budget:

$$RB_t = \max\left[0, A_{liq,t} + A_{asp,t} - \widehat{EG}_t^1\right].$$

The total allocation to performance assets (i.e. assets contained in the market and the aspirational buckets) is the sum of market and aspirational wealth. Assuming that the market bucket is fully invested in the MSR of stocks and bonds, the total exposure to performance assets can be written as:

$$q_{perf,t} = q_{MSR,t} + A_{asp,t}.$$

According to the definition of the GBI strategy, this allocation should be equal to the cushion $m \times RB_t$. But when the cushion is less than the aspirational wealth, there is an overexposure to risky assets. If this happens, the allocation to the MSR is set to zero, in order to reduce as much as possible

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the size of the exposure. In all other cases, we take $q_{perf,t}$ equal to the cushion, which amounts to investing $[m \times RB_t - A_{asp,t}]$ in the MSR. As usual, this sum is capped to the value of liquid wealth. The sum invested in the GHP is then equal to liquid wealth minus the investment in the MSR.

Table 13 shows the weights of the GBI strategy at date 0. The inner composition of the market bucket is the MSR allocation to stocks and bonds, exactly as in the case where aspirational assets are liquid. Moreover, the sum invested in the GHP turns out to be the same as in the liquid case. Indeed, the initial cushion is (in millions of dollars):

$$m \times RB_0 = 5 \times (2.15 + 1.45 - 3) = 3,$$

which is greater than $A_{asp,0}$ (equal to \$1.45m). Thus, the sum invested in the MSR is:

$$q_{MSR,0} = m \times RB_0 - A_{asp,0},$$

and that invested in the GHP is:

$$q_{GHP,0} = A_{liq,0} - q_{MSR,0}$$
$$= A_{mkt,0} + A_{asp,0} - m \times RB_0,$$

which is exactly the same value as in the liquid case. This explains why the composition of the personal bucket is strictly identical in the liquid and the illiquid cases. In order to observe a difference between the two, one would have to choose a sufficiently low value of m, for the cushion to be less than $A_{asp,0}$. The allocation to the MSR would then be zero, while the investment in the GHP would be:

$$q_{GHP,0} = A_{mkt,0},$$

while it was equal to $[A_{mkt,0} + A_{asp,0} - m \times RB_0]$ when aspirational assets were liquidated.

Thus, the personal and market bucket compositions are the same in the liquid and the illiquid cases. But the relative weights of the various compartments are different. In the illiquid case, the investment in performance assets is split across the market and the aspirational buckets, so the sum of the weights of these two buckets is 57.7%, which is the weight of the market bucket in the liquid case (see Table 10).

Plain GBI Strategy. We first implement the GBI strategy in its "plain" form, that is, as it is described in Section 4.1.3.1. Figure 17 shows that the success probability for EG1 falls to 53.0%. There are two reasons why this probability is no longer 100%. The first is independent of the investment rule. The initial liquid wealth is \$2.15m (the sum of the current positions in stock and bond indices), which is less than the present value of EG1, which is \$3m. Hence, by absence of arbitrage opportunities, no strategy can reach EG1 with a 100% probability. The second reason is specific to the GBI strategy. As explained above, it implies an overexposure to performance assets whenever the cushion is lower than the aspirational wealth. As a consequence, the volatility of the ratio

$$\left(A_{liq,t} + A_{asp,t}\right) / \widehat{EG}_t^1$$

does not shrink to zero when the ratio approaches 1, that is, when the reference wealth approaches the floor. Because of this, we would expect violations of the floor, even if the initial liquid wealth was greater than \$2.15m (see the discussion in Section 3.3.2.1).

GBI Strategy with Ratchet Effect. The standard GBI strategy may, by chance, reach the present value at some date, even if it

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starts from a lower level at date 0. The goal would then change status by becoming affordable, but it is not secured by the strategy, because of the inability to cancel the exposure to performance assets when the risk budget shrinks to zero. In order to avoid breaching the floor after it has been reached, we implement a modified version of the GBI strategy which incorporates a "ratchet" effect: as soon as liquid wealth hits the present value of the goal, the goal is secured by investing liquid wealth in the GHP only. Hence, the access to the upside potential of performance assets is lost after the first hitting time, but this performance is used in the first phase to reach the non-affordable goal.[35]

The effect on the success probability is positive, but it is mild: the probability grows from 53.0% to 58.0%. This suggests that the lack of success in reaching EG1 with the plain GBI strategy is primarily due to the insufficient level of liquid wealth compared to the goal value. The other factor which contributed to the floor violations was the fact that the GBI strategy does not secure the goal after its present value has been hit, but it turns out that avoiding subsequent violations through ratcheting has only a small positive effect. This makes a case for the liquidation of aspirational assets in order to make the first goal affordable.

Partial Liquidation of Aspirational Assets. In order to address the issue of insufficient liquid wealth at date 0, one can attempt to liquidate a fraction of the aspirational assets in order to have liquid wealth just equal to the goal present value. After the partial liquidation, liquid and aspirational wealth are:

$$A_{liq,0} = \$3\text{m}, \quad A_{asp,0} = \$0.6.$$

Liquid wealth is exactly the minimum capital required to secure the goal, so the goal becomes affordable again. Figure 18 shows the impact of the multiplier on the success probabilities and expected shortfalls. It can be compared with Figure 11, where the aspirational bucket was entirely liquidated at date 0. The various indicators display a similar pattern: the probabilities of reaching essential goals decrease as m grows, while the expected shortfall tends to increase.

The main difference is that gap risk for EG1 arises for $m = 3$ in the partially liquid case, while it does not arise until $m = 7$ in the liquid case. For instance, with $m = 5$, the success probability is only 79.2%. This is due to the lower level of liquid wealth in the former case. Indeed, a partial liquidation of the aspirational bucket leaves an initial wealth of \$3m, instead of \$3.6m with a complete liquidation. Thus, the safety margin available to absorb adverse shocks on the value of the performance portfolio is lower, and the portfolio is more sensitive to gap risk.

GBI Strategy with Overlay. As explained above, a drawback of the plain GBI strategy is that it implies an overexposure to risky assets (that is, assets not dedicated to the hedging of essential goals) when the cushion is less than the value of the aspirational portfolio. This overexposure could be reduced, or even completely eliminated, if it was possible to sell short the aspirational assets within the liquid portfolio. This possibility is not completely unrealistic as far as the illiquid stock (which aggregates the concentrated stock and the executive stock options) is concerned, but it is certainly ruled out for investment

35 - Another choice of investment policy for the first phase would be the growth-optimal policy because under some assumptions, this strategy minimises the expected time to reach the goal (see Section 3.2.1).

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real estate. Thus, we consider the following cases: first, the illiquid stock can be sold short (which is the ideal situation); second, this short sale is not possible and the investor has to resort to an imperfect substitute.

In all cases, we set an upper bound to the size of the short sale: it is the value of the illiquid stock position. Thus, when the cushion exceeds the aspirational wealth, a short position of size [m × RB_t - $A_{asp,t}$] is taken in the shortable asset, up to a cap. The detailed expression of the weights is given in Appendix 6.6.3.4. It should be noted that with this specification, the strategy is overexposed to risky assets only when the cushion is less than the value of the investment real estate position. Thus, overexposure is less frequent, and of smaller size, than when short sales are prohibited. But it cannot be completely eliminated because of the non-tradable position in real estate.

Figure 19 shows the success probability for EG1 as a function of *m*, for various choices of the shortable asset. When the illiquid stock itself can be sold short, the situation is better than when short sales are ruled out for all values of *m* between 0 and 3: indeed, no violation of the floor is observed. The explanation depends on the value of *m*. For *m* = 0, the cushion is zero, hence less than the value of the investment real estate position. Thus, the investor has a systematic overexposure to risky assets. As a consequence, the amount allocated to the MSR is zero and it follows from the formulas in Appendix 6.6.3.4 that the amount invested in the GHP is:

$$q_{Ft} = A_{liq,t} + A_{X,t},$$

where $A_{X,t}$ is the value of the illiquid stock position. At the initial date, this amount is, in millions of dollars:

$$q_{F0} = 2.15 + 1.25 + 0.1 = 3.5,$$

which is greater than the goal present value ($3m). Thus, the possibility of selling short the illiquid stock makes the goal of maintaining a minimum level of wealth of $3m affordable. It turns out in our simulations that the condition $q_{Ft} \geq \widehat{EG}_t^1$ is satisfied at all rebalancing dates. Hence, at each of these dates, the investor can afford the roll-over of bonds which secure the essential goal, and the success probability is 100%.

For *m* = 1, 2 or 3, the probability of being overexposed is small, because the position in real estate is only $100,000, which represents a relatively small amount. In our simulations, the probability of being overexposed at one rebalancing date at least is less than 1%. For larger values of *m*, gap risk arises. It should be noted that this risk is made more important by the short position: indeed, violations can be caused not only by a bad return on the MSR but also by a good return on the shorted asset. That is why, for *m* above 7, the success probability is less than 50%.

If short sales of the illiquid stock are not possible, one can envision the use of a substitute. From Figure 19, the stock index appears to be a poor substitute: the success probability for EG1 hardly reaches 50%. It should be noted that for any choice of the shortable asset, the liquid wealth may become negative if the asset in question displays good returns. But when the asset is the illiquid stock itself, the returns in the liquid portfolio and in the aspirational

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buckets exactly offset each other, and the sum of liquid and aspirational wealth remains nonnegative. This is no longer the case when the returns on the shortable asset do not perfectly replicate those of the illiquid stock. As a result, not only the liquid wealth, but also the sum of liquid and aspirational wealth may become negative. Of course, in these scenarios, EG1 is missed. This has a negative impact on the probability of reaching this goal, as can be seen from the figure.

In order to better replicate the returns on the illiquid stock, an idea is to sell short a stock index that is better correlated than the broad stock index. Indeed, the correlation between the broad index and the illiquid stock is 50%, but a sector index representative of the activity sector of the concentrated stock is likely to have a higher correlation. We model this increase in the correlation by taking as a shortable asset an index with a correlation of 75% or 90% with the illiquid stock. Figure 19 shows that the success probability is increasing in the correlation, and higher than what is achieved with the broad index, but the change is apparent only for the lowest values of *m*, i.e. the values between 0 and 3. In any case, however, the probability is less than 100%, which indicates that the goal is never secured.

As a conclusion, the presence of an illiquid aspirational bucket makes the "Essential Goal 1" unaffordable, which turns this goal into an aspirational one. As a result, no strategy can secure the goal, and the decrease in the success probability is severe: for the tested GBI strategies, the probability falls below 60% (it was exactly 60% with the current strategy; see Figure 1). Affordability can be recovered if the investor can liquidate a fraction of his aspirational assets, but this leaves him with substantial gap risk, unless the multiplier is set to a low level (less than 3). The existence of the illiquid bucket raises also an issue specific to the GBI strategy: the strategy is overexposed to risky assets when the risk budget is close to zero. This overexposure can be reduced by taking a short position in a substitute for an aspirational asset, but the substitute must have a high correlation with the asset, and as for the partial liquidation, gap risk creates frequent deviations from the goal for large values of *m*.

4.1.3.6. Impact of Taxes

The last robustness test that we perform relates to the impact of taxes (see Section 2.3 for a general introduction to the question of taxes). We are interested in checking whether the protection of EG1 is still effective after taking taxes into account.

The assets involved in the GBI strategy aiming to secure EG1 are the stock and the bond indices and the GHP, which is an annual roll-over of 1-year indexed bonds. We assume that the dividends and the coupons paid by the constituents of the two indices are re-invested in the indices themselves, so that none of the three assets pays dividends. Hence, taxes only arise from capital gains. In detail, the sources of taxes are:

- the rebalancing of the MSR towards constant weights;
- the roll-over operations within the GHP: at the end of each year, the position in the GHP is virtually liquidated, which possibly generates profits;
- the rebalancing between the building blocks (MSR and GHP);

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As explained in Section 2.3, we apply a 20% tax rate on all capital gains, and we use the LIFO algorithm to compute the taxable gains (see Appendix 6.6.5 for details).

To account for the presence of taxes in the design of the GBI strategy, we raise the floor by an amount equal to a tax provision, as described in Section 3.3.3.2: the provision is computed as the amount of taxes accrued since the beginning of the year. It should be emphasised that it is not equal to the present value of the annual tax payment, so that the GBI strategy might not reach the goal with certainty. This can be verified in Figure 20, where we look at the impact of the multiplier on the success probabilities for EG1. While no violation of the minimum wealth constraint is observed for $m = 1$, a few deviations from the goal appear for $m = 3$ and 5. For $m = 3$, the probability of missing EG1 is only 0.2%, which is close to negligible. For $m = 5$, the shortfall probability is more significant, reaching 3.1%, but the deviations are of limited size, with an expected shortfall less than 1%.

In Figure 21, we let the tax rate vary across the values 0, 10% and 20%: the case of a zero tax rate is of course a reminder of the base case without taxes. With $m = 1$, there is no deviation from the goal, and with $m = 3$, deviations are so rare that the success probability is indistinguishable from 100%. Only the graph of expected shortfalls reveals that some deviations occur when the tax rate is 20%. It is only with the highest tax rate (20%) and the most aggressive strategy ($m = 5$) that the shortfall probability becomes non-negligible.

To complete this study, Figure 22 shows the impact of the tax rate on the success indicators for the Aspirational Goal. For the three values of m, the effect is the same: a higher tax rate lowers the probability of reaching the goal and increases the expected shortfall. The impact is material, but not substantial. For instance, for m = 5, the success probability decreases from 62.7% with no taxes to 61.5% with a 20% rate. A potential explanation for this low sensitivity with respect to the level of taxes is that the GBI strategy leads by definition to selling the performance assets (stock and bond indices) on the downside (that is, when the risk budget shrinks), so that it is unlikely that profits will be made from these operations, and the contribution to taxes is small.

As a conclusion, the GBI strategy is relatively robust to the impact of taxes in the sense that it still secures the essential goal, except for high values of the tax rate and the multiplier. Hence, a lower value of the multiplier should be chosen in order to limit gap risk.

4.2. Case Study 2 (HNW Retiree)

The second case study concerns a married and just retired couple. The husband and wife both are 67 years old and have no long-term care or life insurance coverage to start with. They have enough savings and retirement income to fund some of their goals but some goals remain aspirational (see Section 4.2.1.3 for more details) because they cannot be fully funded.

It should also be noted that Goal 4, because it does not involve a pre-specified target level of bequest, will always be achieved with probability 1. Indeed, after the death of the last surviving spouse, the remaining

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Table 2: Investor 2 - Current Risk and Asset Allocation and Goals.
(a) Risk and asset allocation

	Value ($)	% of Total		Value ($)	% of Total		Value ($)	% of Total
Personal Bucket	1,350,000	47.3	Market Bucket	1,400,000	52.7	Aspirational Bucket	0	0
Residence	900,000	66.7	USEquity	770,000	55.0			
Cash	450,000	33.3	US Fixed Income	420,000	30.0			
			Hedge Fund	140.000	10.0			
			Cash	70,000	5.0			

(b) Goals

Name	Goal	Time horizon (years)	Threshold
Goal 1	Retirement Lifestyle	1-26	$80,000 (inflation-adjusted 2.5%)
Goal 2	Long-term Care Contingencies	24-29	$100,000 (inflation-adjusted 4.5%)
Goal 3	Retirement Lifestyle	1-26	$40,000 (inflation-adjusted 2.5%)
Goal 4	Bequest to Children	29	Surplus Assets

Panel (a) describes the current risk and asset allocation of Investor 2. Panel (b) describes his goals, which are ranked by order of decreasing priority from the top to the bottom.

amount of wealth, however small or large, will be passed on to the children. In this context, we will not report probabilities to achieve this goal, but will show instead the distribution of final wealth.

4.2.1 Current Allocation and Goals

4.2.1.1 Description of Risk Buckets

The initial net worth of the household is equal to $2,750,000 and consists of $1,350,000 in personal assets and $1,400,000 in market assets that can be sold to design a bespoke GBI strategy. The personal risk bucket is divided into a residence whose value is equal to $900,000 and $450,000 in cash. The personal risk bucket will be considered illiquid, and therefore modelled as a buy-and-hold strategy. The market risk bucket contains US equities (55%), US fixed-income (30%), hedge-funds (10%) and cash (5%). We will proxy the equity asset class as a broad US equity index and the fixed-income asset class as a sovereign US bond index, and assume that both classes are liquid. The hedge fund will also be assumed liquid so that the household can liquidate their entire market wealth to form a goals-based portfolio (mixture of MSR and Goal-Hedging portfolios). Note in this case that the investor owns no aspirational assets.

The household receives an income of $65,000 (pre-tax) per year which consists of $30,000 from the social security and $35,000 from a personal pension (no Cost Of Living Adjustment). This income is taxed with a 20% rate. The same tax rate will be used for the capital gains obtained from rebalancing and receiving bond coupons.

4.2.1.2 Goals and Goal-Hedging Portfolios

The household has three explicitly formulated consumption goals. Goal 1 (G1) is a consumption-based goal that consists of protecting a minimum lifestyle on retirement: the investor wants to afford an annual expense of $80,000 growing at the annual rate of 2.5% between ages 67 and 93 (that is, at horizons comprised between 1 and 26 years). The goal value is

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thus given by:

$$G_t^1 = G_0^1 \times 1.025^t$$

for t =1,..., 26),

where G_0^1 =\$80,000. The GHP for this goal is a coupon-paying bond that pays fixed annual coupons equal to $G_1^1,\ldots,G_{26}^1$. Its price on date t is equal to the present value of the goal:

$$GHP_{G1,t} = \sum_{\substack{s=1 \\ s>t}}^{26} G_s^1 \times b_{t,s},$$

where we recall that $b_{t,s}$ is the price on date s of the nominal zero-coupon which pays \$1 at date t. By Proposition 4, G1 is affordable if, and only if, the wealth available to finance the consumption stream is greater than or equal to $GHP_{G1,0}$.

The second goal by decreasing order of priority is Goal 2 (G2). Similarly to G1, it can be seen as a consumption-based goal, but it has a non-zero probability p of occurring. This expenditure is meant to finance long-term care contingencies (LTCC): the horizon ranges from year 24 to year 29 with a face value of \$100,000 at the initial time. The working assumptions are as follows. When the couple reaches age 90, the husband will need a nursing home for 3 years and then pass away. Then, his wife will need 3 years of home care, starting at age 93 until she reaches 96 and dies. In our study we consider the following value p =65.39% for Goal 2 to occur, which has been tabulated to reflect the need for LTCC of a couple that is 90 years old. The cost of this care is \$100,000 per year today, and rises 4.5% per year in subsequent years:

$$G_t^2 = \begin{cases} G_0^2 \times 1.045^t & \text{for } t = 24,\ldots,29 \text{ with probability } p \\ 0 & \text{with probability } (1-p) \end{cases}$$

where G_0^2 =\$100,000. The price of this consumption stream on date t is the price of a bond with random coupons equal to $G_{24}^1,\ldots,G_{29}^1$. It is given on date t as:

$$P_{G2,t} = G_0^2 \times p \sum_{\substack{s=24 \\ s>t}}^{29} 1.045^s \times b_{t,s},$$

where for simplicity we assume that the actuarial risk is independent from the financial risk under the risk-neutral probability and that the risk-neutral probability for the LTCC coincides with the historical probability. The price $P_{G2,t}$ is the theoretical present value of Goal 2. However, because of its optionality this bond is not tradable in the market and cannot be directly replicated.

One approach to deal with random consumption goals is to design a GHP with deterministic coupon that can cover the random consumption streams in all states of the world, in which case we talk about *super-replication* as opposed to *replication*. The price of the bond that covers each stream at date t is given by:

$$GHP_{G2,t}^{surrep} = G_0^2 \sum_{\substack{s=24 \\ s>t}}^{29} 1.045^s \times b_{t,s}.$$

If one invests in such a GHP, then Goal 2 would be super-replicated since $P_{G2,t} < GHP_{G2,t}^{surrep}$ as long as the probability of LTCC is strictly below one: p<1.

A second approach to secure the LTCC is to buy insurance. As explained in Section 2.4.1, using insurance can be well suited for goals with uncertain cash-flows since it might lead to cheaper strategies than a full super-replication of the random cash-flows. In this study, we will consider an insurance

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policy where the conditions include an annual premium of $11,476 for a contract that covers half of the LTCC expenses. This annual premium is constant for life but premiums are no longer due once a claim is made at year 24. The insurance policy yearly benefit is $50,000 today, and rises 4.5% per year in subsequent years. In summary, the couple pays $11,476 per year in premiums for 23 years. Over the subsequent six years, either they remain healthy (this event occurs with probability *1-p*) and keep paying the premiums, or they file an insurance claim, which gives them the right to stop paying the premiums and to receive half of their LTCC expenses from the insurer (this event occurs with probability *p*). The premiums have been computed so that the following identity is satisfied with a load equal to 25%:

$$PV(Benefits) = (1 - load) \times PV(premiums).$$

Thus, for every dollar of premium, the insured can (in present term) expect to receive 75 cents of benefits. This 25% estimate for the load is consistent with standard practice, as cited by Brown and Finkelstein (2007). In the case of insurance, the GHP must be a bond that pays fixed coupons equal to the annual premiums $11,476 during the first 23 years, and pays fixed coupons over the last 6 years that cover the worst case scenario between paying the premiums without claiming insurance, or claiming insurance and paying the LTCC expenses that are not covered by the insurance. Since the insurance covers 50 percent of the LTCC expenses, the worst case scenario is to claim insurance. The GHP is therefore given by:

$$GHP_{G2,t}^{assur} = Q_0^2 \sum_{\substack{s=1 \\ s>t}}^{23} b_{t,s} + \frac{G_0^2}{2} \sum_{\substack{s=24 \\ s>t}}^{29} 1.045^s \times b_{t,s},$$

where Q_0^2=$11,476 represents the annual premiums paid to the insurer. With the parameters of the case study, Table 14 shows that $P_{G2,t} < GHP_{G2,t}^{assur} < GHP_{G2,t}^{super-rep}$ which means that the strategy involving the insurance is cheaper than the strategy with full super-replication of Goal 2.

The goal with the lowest priority rank is Goal 3 (G3). It is also a consumption-based goal very similar to G1, needed to improve the retirement lifestyle. Goals 1 and 3 originally form a unique important goal but in order to make it affordable, it has been split into two goals, one which is attainable (Goal 1) and one which is not (Goal 3) - see Section 2.4.2.2. The face value of Goal 3 is $40,000 per year, growing at the annual rate of 2.5% between ages 67 and 93 (that is, at horizons comprised between 1 and 26 years). The goal value is thus:

$$G_t^3 = G_0^3 \times 1.025^t$$
for t =1,..., 26

with G_0^3=$40,000. The present value of this goal is the price of a bond that pays fixed annual coupons equal to G_1^3,..., G_{26}^3. Its price on date t is equal to the present value of the goal:

$$GHP_{G3,t} = \sum_{\substack{s=1 \\ s>t}}^{26} G_s^3 \times b_{t,s}.$$

4.2.1.3 Funding Status of Goals

The first column of Table 14 looks at the goal affordability without using the income as a way to secure the goals. We notice that Goals 1 and 2 require $1,824,302 to be secured without any income, which is lower than the total liquid wealth (including the present value of guaranteed lifetime income). If we add the minimum requirement for Goal 3, then this sum

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exceeds the initial liquid wealth. Therefore using Proposition 6 we can conclude that that Goals 1 and 2 are affordable and that Goal 3 will remain aspirational.

- Goal 1 is affordable with liquid wealth and can thus be treated as an Essential Goal (referred to as EG1);
- Goal 2 is affordable with liquid wealth jointly with EG1 and can thus be treated as an Essential Goal (referred to as EG2);
- Goal 3 is not affordable jointly with EG1 and EG2, and thus represents an aspirational goal (referred to AG1, while AG2 will refer to the goal concerning the bequest to the children).

If Goals 1 and 2 are secured with the income, then the super-replication or the insurance approach for Goal 2 both lead to a minimum capital requirement that is lower than the initial liquid market wealth $1.4 million. The use of income to secure Goals 1 and 2 that are essential goal is justified in Sections 2.2.4.5 and 2.2.4.6. In a nutshell, by doing so the investor leaves a higher fraction of liquid wealth available to invest in performance assets and increase his chances of reaching non-essential goals.

4.2.1.4 Decision Rules for Goal Payment

In order to increase the probability of reaching all their essential goals, at a given time, the couple should pay their non-essential goals only if they are left with enough wealth to secure all the essential goals until the end of their lives. Hence, AG1 is paid in full at date t if the wealth on date t after paying the essential goals, but before paying the non-essential goals (and net of income and mortgage repayment) satisfies $A_{t-} \geq MCR_{EG,t} + AG1_t$, where $MCR_{EG,t}$ is the minimum capital requirement for securing the essential goals and $AG1_t$ represents the non-essential goal expenses of the aspirational goal at date t. The minimum capital requirement at a given date t is obtained by summing up the GHP values at the same date of all the essential goals.

In case the available wealth is not sufficient to pay a goal in full, the investor only pays the fraction of the goal such that the wealth after the payment is greater than the minimum capital level.

4.2.2 Strategies Securing Essential Goal(s)

4.2.2.1 Current Strategy

The current strategy is a fixed-mix that keeps the asset weights within the liquid bucket (US equity, US fixed-income, hedge-fund and cash) equal to their initial values at the beginning of each year. We assume that at each goal payment date, the household pays the goal if the liquid wealth is sufficient, otherwise it pays the largest possible fraction of the goal that can be covered by the liquid wealth. We report the results for the current strategy in Figure 23.

We notice that the goal with the highest probability of achievement with 76.9% is the aspirational goal. It shares the same cash-flow dates as EG1 but costs only half of EG1 therefore it can be met more often than EG1. Indeed the probability of attaining EG1 is equal to 76.5%, which is slightly lower than that of AG1. When looking at the expected shortfall figures for AG1 and EG1, we notice that the default occur in the same year 13 which is half way through the payment period of both goals. From year 13 on, the expected loss increases until the last payment on year 26.

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On the other hand, EG2 payment dates occur towards the end of the period, from year 24 to year 29. Therefore, despite the importance of this goal, it will be harder to achieve given that the liquid wealth will have been already spent on the two other goals before year 24. We observe a probability of achieving their goal equal to 61.2% and we observe a strictly positive expected shortfall on the first date of payment, which shows that the average investor who follows the current strategy cannot even fully afford the LTCC expenses on the first year they have to be paid.

Both essential goals cannot be attained with a very high level of confidence, and the goal that is satisfied with the highest probability is the least essential to the investor. These two drawbacks of the current strategy will be addressed with a GBI solution in the following sections.

4.2.2.2 Protecting Essential Goals 1 and 2 Without Insurance

In this section, we propose to secure the two essential goals 1 and 2 with a goals-based investment solution. We use the entire income to secure either EG1 or EG2 in order to reduce the future cash-flows that the investors will have to pay, hence reducing the initial position in the goal-hedging portfolios. In this section, we do not use insurance policy to protect the household against the risk of needing long-term care. Therefore, we have to super-replicate EG2 using $GHP_{G2,t}^{super-rep}$ as explained in Section 4.2.1.2. The results for this GBI strategy are provided in Figure 24.

The success indicators for the strategy securing EG1 and EG2 are equal to 100%, which shows that in each of our 1,000 Monte Carlo scenarios, the household can afford their retirement lifestyle (EG1) and their long-term care contingencies (EG2). However, the probability of reaching AG1 is equal to 16.4%, which is lower than that obtained with the current strategy (76.9%). This can be explained by the fact that paying EG1 and EG2 at each payment date is costly and therefore the investors no longer have enough wealth available to pay for AG1.

Falling to pay AG1 occurs earlier in the period compared to the current strategy because in a GBI strategy a significant percentage of the initial wealth is used to invest in GHP, and the non-essential goals can no longer be paid. Table 15 shows the percentage of wealth invested in the personal and market buckets at the initial date. We notice that in order to secure EG1 and EG2, the household needs to lock up \$1,068,898. Therefore the investment left in the market bucket is equal to \$331,102 which represents 12% of the total wealth of the household that is free to cover the non-essential goals. The initial percentage of wealth invested in the market bucket was 52.7% for the current strategy, which illustrates that the GBI strategy is very different from the current allocation.

4.2.2.3 Protecting Essential Goals 1 and 2 with LTC Insurance

Instead of fully super-replicating the long-term contingencies care expenses as in Section 4.2.2.2, the household now buys an insurance policy that covers half of their expenses in case of LTCC needs. Again, we use the income to partially secure EG1 and what is left to cover in EG2. As explained in Section 4.2.1.2, the goal-hedging portfolio is now given by

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$GHP^{assur}_{G2,t}$ assur and is computed as the sum of two components: the present value of the fixed policy premiums paid over the first 23 years, and the present value of the expenses that are not covered by the insurance between years 24 and 29. If the insurance policy is not too expensive, this strategy should be cheaper and therefore enables the investor to get more upside. In our simulation, we have considered a load of 25% meaning that for every dollar of premium, the insured can (in present value terms) expect to receive 75 cents of benefits. The results for this strategy are given in Figure 25.

Similar to the GBI strategy implemented without insurance, this strategy can secure both essential goals 1 and 2 with a probability of 100% as illustrated by the success indicators of Figure 25. The probability of securing the aspirational goal is equal to 21.7% which is 5.3% higher than that obtained without insurance. The explanation for improving the performance with respect to AG1 is that the insurance reduces the cost of EG2, freeing up more wealth to pay for the cash-flows of AG1. Indeed, the sum of both GHP for the two essential goals is equal to $1,041,125 with insurance, which is slightly lower than the amount of $1,068,898 obtained without the insurance. The difference is not very high because the coverage of the insurance is equal to 50% hence the remaining half of the expenses still has to be covered by a super-replication strategy.

In Figure 26, we show the terminal wealth distribution for both strategies. We notice that they share the same minimum and median wealth equal to 0. Indeed, since AG1 is not affordable, both strategies will pay cash-flows until the household runs out of wealth. This should happen more than 50% of the time since the probability that the investors will face LTCC expenses is equal to 65.39%. In that case the household will be spending a substantial amount of wealth in their last six years of life. In the other 34.61% of the time, they will not spend anything for LTCC and end their life with a significant surplus (a maximum of $30 million or $38 million depending on the strategies according to Figure 26) which represents the bequest to the children. In order to have a control on the minimum bequest level to the children, we can add a wealth-based goal to the strategy which is the purpose of the following section.

4.2.2.4 Introduction of a Minimum Wealth Constraint

We now consider the introduction of an additional goal, which is to secure at the 29-year horizon an amount of wealth equal to the initial liquid capital, $1,400,000. This wealth-based goal models the second aspirational goal of the household, i.e. a bequest objective (the investor wants to leave a certain amount of money to his children). We refer to it as Goal 4 (in short, G4).

From Figure 26, we notice that the median wealth is equal to zero in both approaches (with and without insurance) which means that at least one half of the scenarios leads to final wealth levels that remain below the target of $1.4 million, so the strategy fails at securing this wealth-based goal. Before designing a new strategy that take Goal 4 into account, the first question that we should address is to know whether or not this new goal is jointly affordable with the more essential goals EG1 and EG2. So we

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must first qualify the affordability of G4. By absence of arbitrage opportunities, if this goal, EG1 and EG2 are reached with certainty, then the liquid wealth at the initial date must satisfy:

$$A_0 \geq GHP_{G1,0} + GHP_{G2,0} + A_0 b_{0,29},$$

where $GHP_{G1,0}$ is the value of the goal-hedging portfolio for EG1, $GHP_{G2,0}$ the value of the goal-hedging portfolio for EG2 (either equal to $GHP_{G2,0}^{assur}$ if we use the insurance or equal to $GHP_{G2,0}^{super-rep}$ in the absence of insurance), A_0 is the initial wealth ($1,400,000) and $b_{0,29}$ is the price of a zero-coupon bond maturing at date 29. Conversely, if the inequality holds, then the investor can afford the bonds paying EG1 and EG2 cash flows and a zero-coupon that will deliver A_0 on date 29. Hence, the above inequality is a necessary and sufficient condition for the goals EG1, EG2 and G4 to be jointly affordable. In Table 17 we observe that the present value of securing $1,400,000 at year 29 is equal to $214,792 which shows that EG1, EG2 and G4 are jointly affordable. Indeed, in the net value version (where the income is used to secure EG1 and EG2) we find that the right-hand side of the above inequality is equal to $1,283,690 in the strategy without insurance and to $1,255,250 in the strategy with insurance. Both minimum capital requirements fall below the initial liquid wealth A_0. Therefore, the status of Goal 4 will be as follows:

- Goal 4 is affordable with liquid wealth jointly with EG1 and EG2, and can thus be treated as an Essential Goal (referred to as EG3);

In Figure 27, we notice that the GBI strategy securing the three essential goals EG1, EG2 and EG3 achieves its target whether the household buys insurance or not. The success indicators exhibit a 100% chance to attain the three essential goals. However, the aspirational goal becomes impossible at all to attain in both strategies. In order to see how the two approaches for handling EG2 differ, we look in Figure 28 at the distribution of terminal wealth and at the shortfall indicator for AG1. We observe that the strategy without insurance fails on average at paying AG1 on the second date whereas the strategy that uses insurance can afford the first and second payments. This comes from the higher cost of super-replication compared to the cost of insurance. When looking at the terminal wealth, we see that the minimum wealth coincides with the median wealth and is equal to $1.4 million. The maximum wealth is higher for the super-replication strategy because in the case where the household does not have to pay for LTCC, then the super-replication of the strategy leaves them with a lot of wealth towards the end of their life. When the household uses the insurance, they only super-replicate one half of the LTCC expenses, which will give them 50% less extra wealth in case they do not have to pay for LTCC. Moreover, if they remain healthy, they will have to keep paying the insurance premiums. This explains why the maximum terminal wealth level is higher for the strategy without the insurance, and differs from the strategy with insurance by a factor close to 2.

4.2.3 Impact of Taxes

To see if the results obtained in Section 4.2.2.2 remain robust with respect to the introduction of taxes, we now consider a uniform tax rate ζ equal to 20%, and adopt the LIFO convention to compute the taxable gains (see Appendix 6.6.5 for mathematical

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details). We run the robustness check on the strategy that secures both EG1 and EG2. This will adjust the previous GHP computations to take into account taxes in the GBI strategy.

4.2.3.1 Adjustment to GHP for Goal 1

In this case study, taxes arise from the selling operations in the stock and the bond indices held within the PSP building block and from the coupons paid by the bonds that secure EG1 and EG2, coupons which are equal to the consumption expenses.

As explained in Section 3.3.3.1, the effect of taxes on coupons can be virtually removed by purchasing $^1/_{(1-\zeta)}$ units of the bond. In other words, for the goal to be secured with certainty, the investor's wealth at the initial date must satisfy:

$$A_0 \geq \frac{1}{1-\zeta}\left(GHP_{G1,0} + GHP_{G2,0}\right),$$

where we recall that $GHP_{G1,0}$ is the present value of retirement lifestyle expenses and $GHP_{G2,0}$ the value of the goal-hedging portfolio securing the long-term contingencies care (either equal to $GHP_{G2,0}^{assur}$ if we use the insurance or equal to $GHP_{G2,0}^{surrep}$ the absence of insurance). Thus, in the presence of taxes, the GHPs for Goals 1 and 2 are simply the following portfolios:

$$GHP_{Gj,tax,0} = \frac{1}{1-\zeta} GHP_{Gj,0}$$

for j =1 and 2;

The tax rate being positive, ζ>0, it is clear that these GHPs are more expensive than the ones that secure the goals in the absence of taxes, leading to more costly GBI policies to attain the same level of goal achievement. The performance-seeking allocation of our portfolio which is represented by the MSR can also lead to additional taxes from periodical rebalancing. These taxes are not hedged by the adjustment to the GHP value so the taxes generated by selling operations can in theory cause violations of the essential goal.

To verify that EG1 and EG2 remain jointly affordable in the presence of taxes, we go back to Table 14 and multiply the minimum capital required by 1.25 (which corresponds to the multiplicative factor $^1/_{(1-\zeta)}$ with a 20% tax rate). We obtain the following

No Insurance:
1.25×$1,068,898=$1,336,122.5

Insurance: 1.25×$1,041,125= $1,301,406.5,

These minimum capital levels can be secured at the initial date since the liquid wealth is equal to $1,400,000 at date 0. Therefore the two goals remain jointly affordable, but the risk budget which corresponds to the difference between the liquid wealth and the minimum capital required to secure the essential goals has become very small. Therefore Goal 4 can no longer be affordable together with EG1 and EG2, and the aspirational goal AG1 will have to remain aspirational.

4.2.3.2 Test of Strategies Securing Essential Goals 1 and 2

As expected, the risk budget resulting from the introduction of taxes is too small to enable the household to afford AG1. In Figure 29, we have tested the strategy that secures both EG1 and EG2 without insurance for EG2. We observe that the probability of attaining AG1 is equal to 0. However, the probabilities of achieving both essential goals 1 and 2 remain equal to 100% which shows that GBI strategies are

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robust to the introduction of taxes as long as the minimum capital required is updated accordingly. We can notice that the impact of taxes coming from selling operations in the MSR portfolio did not cause violations. We do not report the results of the strategy with insurance in the presence of taxes since they are very similar to those of the strategy without insurance.

4.3. Case Study 3 (Affluent Accumulator)

The third case study relates to a younger investor (45 years old), with potential to move up the wealth level. Again, not all goals can be funded using current assets.

4.3.1. Current Allocation And Goals

4.3.1.1. Description of Risk Buckets

Table 3 shows the current risk and asset allocation of the investor. The personal risk bucket consists of the principal residence, whose current value is $300,000, and a cash account of value $10,000. As in the other two case studies, these assets are used to finance essential needs, i.e. not to be homeless and to afford a minimum standard of living. Thus, we will assume a buy-and-hold allocation to these assets. The investor must also repay a fixed-rate mortgage loan. In the absence of any other precision on the amortisation schedule, we will assume a constant-annuity scheme, which is the most widespread scheme. We also assume a borrowing rate of 4%. The expression for the constant annuity ℓ is derived in Appendix 6.5.2:

$$\ell = \frac{r}{1-(1+r)^{-T}} \times L,$$

with r =4%, L =$250,000 (the principal of the loan) and T=20 years. Numerically, the constant annuity is ℓ=$18,395.

The market risk bucket is dominated by equities, which represent 63.8% of market wealth. The remainder of this bucket is invested in US fixed income instruments (31.9%) and cash (4.3%). These proportions are similar to those of Investor 1 (see Section 4.1). We will proxy the equity asset class as a broad US equity index and the fixed-income asset class as a sovereign US bond index, and assume that both classes

Table 3: Investor 3 - Current Risk and Asset Allocation and Goals.
(a) Risk and asset allocation

	Value ($)	% of Total		Value ($)	% of Total		Value ($)	% of Total
Personal Bucket	60,000	6.0	Market Bucket	940,000	94.0	Aspirational Bucket	0	0
Residence	300,000	53.6	USEquity	600,000	63.8			
Cash	10,000	1.8	US Fixed Income	300,000	31.9			
Adjustable Rate Mortgage	(250,000)	44.6	Cash	40,000	4.3			

(b) Goals

Priority	Goal	Time horizon (years)	Threshold
Goal 0	Mortgage amortisation	1-20	$18,395 / year (constant)
Goal 1	Retirement lifestyle	21-50	$90,000 / year (inflation-adjusted)
Goal 2	Children's education	11-14	$50,000 / year(inflation-adjusted)
Goal 3	House purchase	5	$300,000(inflation-adjusted)

Panel (a) describes the current risk and asset allocation of Investor 3. Panel (b) describes his goals, which are ranked by order of decreasing priority from the top to the bottom.

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are liquid. Finally, the investor owns no aspirational assets.

In addition to these assets and liabilities, the investor is also endowed with a positive net income stream: every year, he receives $25,000 (net of taxes), an amount that grows at the annual rate of 2.5%:

$$y_t = \begin{cases} y_0 \times 1.025^t & \text{for } t=1,\ldots,20 \\ 0 & \text{otherwise} \end{cases}$$

4.3.1.2 Goals and Goal-Hedging Portfolios

First of all, it should be noted that the mortgage represents an implicit essential goal (referred to as Goal 0 in Table 3): indeed, the annual mortgage down payment is a constrained payment that the investor has no cannot cancel or postpone. By definition, the present value of this goal equals the face value of the loan of $250,000.

Besides, the investor has three explicitly formulated goals. Goal 1 (G1) is a consumption-based goal that consists of protecting a minimum lifestyle on retirement: the investor wants to afford an annual expense of $90,000 growing at the annual rate of 2.5% between ages 66 and 95 (that is, at horizons comprised between 21 and 50 years). The annual expense is fixed in real terms, which means that it is adjusted for inflation. The goal value is thus:

$$G_t^1 = \begin{cases} G_0^1 \times 1.025^t & \text{for } t=21,\ldots,50 \\ 0 & \text{otherwise} \end{cases}$$

where G_0^1 = $90,000. The GHP for this goal is a coupon-paying bond that pays fixed annual coupons equal to $G_1^1,\ldots, G_{50}^1$. Its price on date t is equal to the present value of the goal:

$$GHP_{G1,t} = \tilde{G}_t^1 = \sum_{\substack{s=21 \\ s>t}}^{50} G_s^1 \times b_{t,s},$$

where we recall that $b_{t,s}$ is the price on date s of the nominal zero-coupon which pays $1 at date t. By Proposition 4, G1 is affordable if, and only if, the wealth available to finance the consumption stream is greater than or equal to $GHP_{G1,0}$.

The next goal by order of decreasing priority is Goal2 (G2). As G1, it is a consumption-based goal. This expenditure is meant to finance children's education: the horizon ranges from 11 and 14 years, and the face value is $50,000. As G1, the annual expenditure grows at the annual rate of 2.5%:

$$G_t^2 = \begin{cases} G_0^2 \times 1.025^t & \text{for } t=11,\ldots,14 \\ 0 & \text{otherwise} \end{cases}$$

where G_0^2 =$50,000. The price of this consumption stream on date t is:

$$GHP_{G2,t} = \tilde{G}_t^2 = \sum_{\substack{s=11 \\ s>t}}^{14} G_s^2 \times b_{t,s},$$

The goal with the lowest priority rank is Goal 3 (G3). It is also a consumption-based goal, which is to purchase a house at the horizon of 5 years. The real value of the goal is $300,000, and so that its nominal value is:

$$G_t^3 = \begin{cases} G_0^3 \times 1.025^t & \text{for } t=5, \\ 0 & \text{otherwise} \end{cases}$$

with G_0^3 =$300,000. The present value of this goal is the price of a nominal zero-coupon that pays off $300,000 at the 5-year horizon:

$$GHP_{AG,t} = \tilde{G}_t^3 = G_5^3 \times b_{t,5} \times \mathbb{I}_{\{t<5\}}.$$

4.3.1.3 Funding Status of Goals

At the first level of analysis, it is possible

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to look at the affordability of goals by abstracting away from income. If the investor relies only on liquid wealth to achieve his consumption goals, a necessary and sufficient affordability criterion is given by Proposition 6: the initial capital must be larger than the sum of goal present values. Panels (i) and (ii) in Table 19 respectively show the asset side and the liability side of the balance sheet, and the second column of Panel (ii) contains the cumulated sum of goal present values (the other columns will be commented on later). Liquid wealth is sufficient to afford the most priority goal (Goal 1), but not to afford jointly Goals 1 and 2. However, if future savings could be turned into liquid wealth, Goal 2 would become affordable. In what follows, we will rule out this possibility because selling a claim on future savings would mean that the investor is allowed to borrow against future income. As a conclusion, if the investor does not use income to finance the goals:

- Goal 1 is affordable with liquid wealth alone, and can thus be treated as an Essential Goal (referred to as EG1);
- Goal 2 and 3 are not affordable jointly with EG1, and thus represent aspirational goals.

But as explained in Sections 2.2.4.5 and 2.2.4.6, the investor should use income to secure the goal, in order to leave a fraction of liquid wealth available to invest in performance assets and increase his chances to reach non-essential goals. The liquid wealth is then used to purchase an option whose payoff covers the fraction of the goal that is not covered by income. Among the various strategies described in Sections 2.2.4.5 and 2.2.4.6, we only test those that do not involve compound options, because the pricing of such options would raise technical challenges that are beyond the scope of this study. Specifically, we consider the strategies referred to as INC-ZER-RET and INC-FWD in Section 2.2.4.6.

The strategy INC-ZER-RET assumes that future income is invested at a zero rate for a period equal to the time to retirement. It works as follows:

- At date 0, the investor purchases an option that pays on the retirement date the excess of the goal present value over the cumulated value of income capitalised at a zero rate. With the notations of Section 2.2.4.6, this payoff is given by:

$$U_{ret,21,0} = \left(\tilde{G}^1_{21-} - \sum_{k=1}^{20} y_k\right)^+.$$

The remainder of wealth is invested in a performance-seeking portfolio, which we take to be the MSR portfolio of stock and bond indices;

- At the income date $j=1,\ldots,20$, the investor purchases an option that pays on date 21:

$$U_{ret,21,j} = \left(\tilde{G}^1_{21-} - \sum_{k=j+1}^{20} y_k\right)^+.$$

As explained in Section 2.2.4.6, this option is fully financed by the option purchased at the previous date and the received income. The remaining amount is invested in the MSR portfolio;

- As of date 20, the investor holds a long position in the bond that delivers the cash flows of the goal, which fully secures this goal.

In the context of our simulations, the pricing of the options is greatly simplified because it happens that $\tilde{G}^1_{21-}$ is greater than the cumulated income with probability 1. As a result, each option consists simply

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in a long position in the bond that pays the goal cash flows and a short position in a zero-coupon bond that pays, on date 21, the cumulated income. As a result, at each income date j, the investor can increase the dollar allocation to the MSR portfolio by the quantity:

$$U_{ret,j,j-1} + y_j - U_{ret,j,j} = (1 - b_{j,21})y_j,$$

where $b_{j,21}$ is the price of the pure discount bond which pays \$1 on the retirement date. This price is less than \$1, since nominal rates are assumed to stay nonnegative throughout our simulations.

The minimum initial capital required to implement this strategy is the price of the first option. With the assumed parameter values, it is \$709,181, which of course is less expensive than the bond that pays the goal cash flows. Interestingly, Goal 2 becomes jointly affordable with Goal 1 if this strategy is adopted, because the total capital requirement to fund both goals is \$879,074, which is less than the available \$940,000. As a consequence, Goal 2 can be treated as an essential or important goal, depending on the decision to secure it or not. However, Goal 3 is still not affordable jointly with the two more priority goals.

The strategy INC-FWD proceeds as follows:

- At date 0, the investor enters forward contracts to fix the investment rate for a loan starting at date $j = 1,\ldots,20$ and finishing at date 21. This rate is the forward rate $f_{j,21-j}$, which can be expressed as a function of zero-coupon prices:

$$f_{j,21-j} = \left(\frac{b_{0,j}}{b_{0,21}}\right)^{\frac{1}{21-j}} - 1.$$

The deficit to finance at date 21 will thus be

$$W_{21} = \left(\tilde{G}^1_{21-} - \sum_{k=1}^{20} y_k \left(1 + f_{k,21-k}\right)^{21-k}\right)^+.$$

It is covered by purchasing an option of price W_0. The remaining wealth is invested in the same performance portfolio as before, namely the MSR portfolio of stock and bond indices;

- At each income date $j = 1,\ldots, 20$, the received income is invested at the forward rate;
- At date 21, the investor is able to purchase the bond paying the goal cash flows by combining the option payoff and the cumulated income. Thus, the goal is secured.

It should be noted that with this strategy, unlike the INC-ZER-RET one, there is no additional investment in the MSR portfolio on an income date: income is entirely invested in a separate account and will grow at the forward rate. The initial cost of the protection is the option price, W_0, which is numerically equal to \$628,750. This strategy is cheaper than the strategy INC-ZER-RET, but the reduction in cost is not sufficient to make Goal 3 affordable. As a conclusion, if the investor uses income to protect EG1:

- Goal 2 is jointly affordable with EG1 and will be subsequently regarded as an important goal (IG);
- Goal 3 is not jointly affordable with EG1 and IG and thus corresponds to an aspirational goal (AG).

4.3.1.4 Decision Rules for Goal Payment

It is important to recognise that the willingness to protect EG1 has implications for the payment of the other goals. Indeed, in the context of this case study, the chronological order of goals is exactly

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the reverse of the priority order. As a consequence, the payment of AG, which is the goal with the lowest priority, could lead in some trajectories to downgrade the funding status of EG1 and turn it into a non-affordable goal. For all strategies that aim at protecting EG1, we thus adopt a payment rule that preserves the funding status of this goal. Moreover, if IG is funded at the 5-year horizon, we require that the payment of AG should not turn IG into a non-affordable goal. But since IG is not explicitly secured, it may be the case that it is no longer funded after 5 years. In this case, only the condition on EG1 is taken into account.

The minimum capital requirement for EG1 depends on the strategy and the decision to use or not income to protect the goal. With the strategy INC-ZER-RET, the minimum wealth requirement is the price of the option required to secure the fraction of the goal which is not covered by income:

$$U_{ret,t} = \mathbb{E}_t\left[\frac{M_{21}}{M_t}\left(\widetilde{EG}^1_{21-} - \sum_{\substack{k=1 \\ k>t}}^{20} y_k\right)^+\right].$$

Hence, IG is paid in full at date t if the wealth of date t before any goal payment (but net of income and mortgage repayment) satisfies $A_{t-} \geq U_{ret,t} + IG_t$. For AG, the condition is more severe as it states that IG must stay funded too. Hence, the condition is to have $A_{t-} \geq U_{ret,t} + IG_t + AG_t$. With the strategy INC-FWD, the conditions have a similar form, but the minimum capital requirement for EG1, $U_{ret,t}$, is now replaced by:

$$W_t = \mathbb{E}_t\left[\frac{M_{21}}{M_t}\left(\widetilde{EG}^1_{21-} - \sum_{k=1}^{20} y_k\left(1 + f_{k,21-k}\right)^{21-k}\right)^+\right].$$

In case the available wealth is not sufficient to pay a goal in full, the investor only pays the fraction of the goal such that the wealth after the payment is greater than the minimum capital level.

4.3.1.5 Current Strategy

As in the previous case studies, we start by computing the success indicators achieved with the current strategy, which is a fixed-mix policy that keeps the asset weights within the liquid bucket equal to their initial values. Since this strategy does not aim at protecting any goal, we do not apply the decision rules given in Section 4.3.1.4, and we simply assume that at each goal horizon, the investor pays the largest fraction of the goal that is covered by the liquid wealth. That is, if t is a goal horizon and $A_{liq,t-}$ is the liquid wealth before any goal payment, the effective consumption expense is min $(G_t, A_{liq,t-})$.

It turns out that the aspirational goal, which is the goal with the shortest horizon (5 years) is reached with certainty. This can be explained by the fact that the average annual arithmetic return of the simulated fixed-mix portfolio over the first five years is 10.40%, which is much larger than the inflation rate of 2.5%.[36] As a result, the expected wealth after five years (before savings and consumption) is approximately $1.54m, which is comfortably larger than the aspirational goal value ($339,420 including inflation adjustment). These numbers give a sense of why the wealth after five years is always larger than the aspirational goal value, hence why the goal is always attained.

Similarly, the important goal is always attained. Again, this is because the face

36 - The average annual arithmetic return at horizon T is defined as $\left(\frac{\mathbb{E}[A_{liq,T}]}{A_0}\right)^{1/T}-1$.

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value of this goal is relatively low compared to the available liquid wealth. Indeed, the average wealth after eleven years (still before savings and consumption) is $2.21m, while the goal threshold is only $50,000. Even though this amount is to be paid every year for four consecutive years, it is not difficult to guess that the consumption objective will always be attained.

The situation is different with the essential goal, for which there is a significant shortfall probability of 19.7%. The examination of maximum shortfalls shows that the deviations from the goal start to occur at the end of the first three years of retirement. In other words, the investor can afford the goal for the first two years with certainty, but the fixed-mix policy does not guarantee the respect of the goal thereafter. It is also worth noting that the expected shortfall grows very rapidly after two years, to reach and exceed 70%. These numbers reflect a very substantial shortfall risk, which motivates the design of strategies that guarantee the achievement of consumption objectives. This is what we turn to in the next section.

4.3.2 Strategies Securing Essential Goal(s)

4.3.2.1 Protecting Essential Goal 1 with Liquid Wealth

To secure EG1, the first option is to purchase a bond that pays the goal cash flows. This strategy (referred to as LIQ) relies only on liquid wealth to secure the goal, and disregards the presence of income: it could be adopted by an investor receiving no income and otherwise equivalent to Investor 3. The bond is the GHP, and as shown previously (see Table 19), it is affordable with liquid wealth alone. The remainder of liquid wealth is invested in a performance block, which we take to be the MSR portfolio of stocks and bonds. As appears from Table 20, the great majority of assets are held in the form of safe assets within the personal bucket: this bucket represents 73.7% of the total assets. The remaining part is held in the form of a claim on future savings and market assets. Following the definition of the aspirational bucket given in Section 2.5.3.2, the present value of future savings is assigned to this bucket because it is not a traded asset, and it does not have a publicly available price. Because the mortgage annuities are a constrained payment, they are subtracted from the annual savings in the computation of the present value of savings: it is as if the investor was receiving a diminished income. The present value of savings accounts for 15.3% of the total, and market assets (equities and bonds) for only 11.0%. The small percentage of market assets shows that the strategy is highly conservative.

The corresponding success probabilities and shortfall indicators are reported in Figure 31. The most striking observation is that AG is reached in only 1.5% states of the world. Indeed, in these trajectories, the investor gives up a part of this goal because paying the goal in full would compromise the ability to finance EG1 and IG. The associated deviations are also severe: on average, the investor finances less than 25% of the goal. These poor performances with respect to AG are not surprising in view of the risk allocation of Table 20: the protection of EG1 with a bond exhausts most of the initial wealth, which leaves little cash available to invest in performance-seeking assets.

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The decision rule has also a negative impact on IG: this goal is reached with a probability of 77.7% only, while it was systematically attained with the current strategy.

4.3.2.2 Protecting Essential Goal 1 with Income

In order to have a broader access to these assets, the investor must rely at least in part on the future income. The strategy INC-ZER-RET assumes a zero re-investment rate for future income, and, in every year of his working life, the investor secures the fraction of the goal that is not covered by income by purchasing a suitable option. The risk allocation is shown in Table 21. As implied by the theoretical analysis (see Section 2.2.4.6), the new GHP is less expensive than the bond paying $90,000 per year: we have $U_{ret,0}$ =$709,181, versus $\tilde{G}_0$ =$810,256. This decrease in the price of the GHP implies a decrease in the personal bucket size by the same amount and a related increase in the market bucket size. The difference between the two values is $101,075, which is less than the present value of future savings ($181,506). This is a numerical illustration of the general property

$$\tilde{G}_0 - \tilde{H}_0 \leq U_{ret,0},$$

which follows from the inequalities $\tilde{G}_0$ - $\tilde{H}_0$ ≤ V_0 and $U_{ret,0}$ ≤ V_0 given in Sections 2.2.4.4 and 2.2.4.6. Even if the protection of the goal was done optimally, that is, by the means of the cheapest replicating strategy of Proposition 8, the increase in the market bucket size would still be less than the present value of savings, as a consequence of the property $\tilde{G}_0$ - $\tilde{H}_0$ ≤ V_0. This means that using income to protect the goal is not simply equivalent to transferring an amount equal to the present value of savings from the personal bucket to the market bucket. By accepting to rely on income, the investor does not free up as much wealth from this bucket as he would by selling a claim on future savings to turn it into liquid assets. It is worth noting that this is a consequence of the requirement to keep liquid wealth nonnegative: were negative values allowed, we would recover the equality $\tilde{G}_0$ - $\tilde{H}_0$ = V_0 and the increase in market bucket value would be exactly equal to the present value of savings.

The success indicators reported in Figure 32 show that the main impact of the new protection mode is on the aspirational goal. By construction, EG1 is still secured with certainty. The probability of reaching IG is not substantially modified with respect to the strategy LIQ. But the success probability for AG increases from 1.5% to 24.5% and the expected shortfall is now 47.1%, which represents a substantial reduction with respect to the 76.2% previously obtained. As a conclusion, the use of income allows to invest more in performance assets, which has a positive impact on the achievement for ambitious goals.

The second strategy that secures Essential Goal 1 with income is the strategy INC-FWD, which invests income at the forward rate. The option that secures the fraction of the goal not covered by income is worth $628,750, which is less than the option purchased with the strategy INC-ZER-RET, which is a straightforward property since forward rates are positive. Assuming that income is re-invested at the forward rate is thus less pessimistic than assuming that it will generate a zero rate of return. In terms of usage of current liquid wealth, the strategy INC-FWD is the cheapest of the three protection strategies considered.

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Thus, it is for this investment policy that the personal bucket has the smallest weight in the allocation (68.9% versus respectively 73.7% and 76.9%).

Figure 33 displays the success indicators. The essential goal is still secured with probability 1, but the protection with forward contracts has an ambiguous impact on the achievement of non-essential goals. On the one hand, the indicators for AG are improved: the success probability is significantly higher than it was with the strategy INC-ZER-RET (42% as opposed to 24.5%) and the expected shortfall has fallen down from 47.1% to 40.5%. On the other hand, the strategy performs less well with respect to IG: the success probability is now 83.9%, versus 96.9% with the strategy INC-ZER-RET, and the expected and maximum shortfalls tend to be larger. This situation may be attributed to two competing effects. First, the investor initially holds a larger fraction of assets in the form of the MSR of stocks and bonds, and this leads to shifting the distribution of wealth after five years to the right. Thus, AG is more likely to be reached. Second, there exists a competition between AG and IG: since AG is paid more often, there is less money after year 5 to finance IG in full.

4.3.2.3 Introduction of a Minimum Wealth Constraint

We now consider the introduction of a new goal, which is to secure a minimum wealth level at the 50-year horizon. This wealth-based goal models a bequest objective (the investor wants to leave a certain amount of money to his children). We refer to it as Goal 4 (in short, G4), and we take the goal threshold to be the initial liquid wealth, $940,000, capitalised at the 2.5% annual rate. This rate represents the expected annual inflation rate, so the capital is meant to be preserved in real terms:

$$G_t^4 = \begin{cases} A_0 \times 1.025^t & \text{for } t = 50 \\ 0 & \text{otherwise} \end{cases}$$

with A_0 = $940,000.

Figure 34 shows how the previous strategies perform with respect to this new goal. The success probabilities are decent (greater than 59%), but none of the strategies secures the goal with probability 1. It is therefore of interest to test strategies that secure the goal. But the first step before this is to find whether such strategies are feasible, that is, whether this goal is jointly affordable with the more priority goals. In terms of priority, we treat the wealth-based goal as a goal of intermediate priority between Goal 1 (the retirement goal) and Goal 2 (the education goal): that is, if he cannot secure G1 and G4 simultaneously, he will give up the latter goal and only secure G1, but he attaches more importance to G4 than to G2.

So we must first qualify the affordability of G4. By absence of arbitrage opportunities, if this goal and G1 are reached with certainty, then the liquid wealth just before retirement must satisfy:

$$A_{21-} \geq \tilde{G}_{21-}^1 + G_{50}^4 \times b_{21,50}, \qquad (35)$$

where $\tilde{G}_{21-}^1$ is the present value of G1 cash flows, A_0 is the initial wealth ($940,000) and $b_{21,50}$ is the price of a zero-coupon bond maturing at date 50. Conversely, if (35) holds, then the investor can afford both the bond paying G1 cash flows and a zero-coupon that will deliver A_0 on date 50. Hence, Equation (35) is a necessary and

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sufficient condition for the goals G1 and G4 to be jointly affordable.

The problem at hand is thus formally similar to the one with no minimum wealth constraint, with a higher floor at date 21. In order to secure this minimum wealth level, the investor has the choice between three approaches:

- Purchase a bond that will pay the floor on date 21: this is the strategy LIQ, which relies on liquid wealth only;
- Use income assuming a zero re-investment rate: this is the strategy INC-ZER-RET. The fraction of the goal not covered by income must be secured by purchasing an option maturing at date 21 with payoff

$$U_{ret,21} = \left(\tilde{G}^1_{21-} + G^4_{50} \times b_{21,50} - \sum_{k=1}^{20} y_k \right)^+;$$

- Re-invest income at the forward rates. This strategy, INC-FWD, has a cost equal to the price of the payoff

$$W_{21} = \left(\tilde{G}^1_{21-} + G^4_{50} \times b_{21,50} - \sum_{k=1}^{20} y_k \left(1 + f_{k,21-k}\right)^{21-k} \right)^+.$$

Table 23 shows the minimum capital required to secure the goals, individually or jointly. It appears that the investor can afford as of date 0 both the bond that delivers G1 cash flows and the zero-coupon that delivers the minimum wealth level at date 50. Hence, G4 is jointly affordable with G1 whichever mode of protection is chosen. But of course, it is less costly in terms of liquid wealth to use income. In what follows, we test strategies that secure both G1 and G4, which means that both goals are treated as essential: the retirement goal is still referred to as Essential Goal 1 (EG1) and the bequest goal as Essential Goal 2 (EG2).

It also turns out that G2 (the education goal) cannot be secured together with G1 and G4, except if the investor can fix as of date 0 the re-investment rate of future income payments. Thus, except in this situation, G2 is no longer an affordable goal and it should be regarded as an aspirational goal. In what follows, we thus refer to it as AG1, and to the home goal (previously known as AG) as AG2.

We now test the three strategies designed to secure EG1 and EG2 simultaneously. Each of them consists of a buy-and-hold position in the MSR of the stock and the bond indices and a GHP that secures the two essential goals. This GHP is a bond for the strategy LIQ and an option for the strategies INC-ZER-RET and INC-FWD. It should be noted that unlike in Case 1, where there was a clear separation between the GHPs for the two essential goals, such a separation is not possible to perform for the two strategies relying on income. Indeed, a single option is used to secure both goals.

Table 24 provides the initial risk allocation for the various strategies. Purchasing two bonds to protect the two goals proves to be very costly: this costs $916,989, which consumes about 98% of the investor's initial liquid wealth ($940,000). As a result, there remain only $23,011 available to invest in performance-seeking assets, and the market bucket represents a tiny 1.9% of the total allocation. In order to save space, we only report the success probabilities of the three strategies in the next figures. As appears from Figure 35, the strategy that only uses liquid wealth, while securing both essential goals as it should, has virtually zero chances to reach the non-essential ones. This is of course a

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consequence of the low allocation to the market assets.

If the investor relies on income to secure as much as possible of the two essential goals, he only has to purchase an option, which is worth less than the two bonds. He must liquidate $815,924 of his initial market bucket, but there remain $124,086 available to invest in the MSR portfolio. As can be seen from Figure 36, this higher allocation to performance-seeking assets translates into a huge improvement in the success probability for AG1, which grows from 0.7% to 74.8%. An increase is also observed for AG2, but it is less spectacular, with a probability that grows from 0% to 1.3%. Both probabilities are lower than with the strategy that secures only the retirement goal (see Figure 32), which reflects the opportunity cost associated with the protection of a second essential goal. Interestingly, the dollar allocation to the performance-seeking assets was higher with this strategy (see Table 21), which points to the existence of an increasing relationship between the market bucket size and the probabilities of reaching the non-essential goals.

It is with the third strategy, which uses forward contracts, that the GHP is the least expensive and that the market bucket is the largest. In this context, it comes as no surprise that the success probabilities for both AG1 and AG2 are very significantly improved with respect to the case where the totality of the protection came from liquid wealth: for AG1, the probability is a large 70.9%, and for AG2, it is 9.6%, which is still low but not negligible.

As a conclusion, the analysis of strategies protecting either EG1 or both EG1 and EG2 highlights the following points:

- A strategy that relies on liquid wealth only (LIQ) has a substantial opportunity cost in terms of the probability of reaching non-essential goals. The success probabilities for these goals are in general disappointing, especially when there are more than one essential goal to protect, since the bond that secures both goals consumes most of the initial wealth and leaves little cash available to invest in a performance-seeking portfolio;
- The success probabilities for non-essential goals are significantly improved by using income to secure as much as possible of the essential goal(s) (strategies INC-ZER-RET and INC-FWD). Indeed, it is less costly to purchase an option to secure the fraction of the goal(s) which is not covered by income than investing in a bond that delivers the cash flows of this (these) goal(s);
- There is no clear dominance of one of the strategies INC-ZER-RET and INC-FWD over the other: from the previous results, it appears that the former performs better with respect to the education goal while the latter displays better scores with respect to the home goal. This suggests that there is a form of competition between the two goals: the home goal has shorter horizon, so that paying it more often leaves less money to pay for the other goal.

4.3.3 Impact of Taxes

We now perform a robustness check of the previous results with respect to the introduction of a positive tax rate. The objective of this section is threefold. First, we present adjustments to the previous strategies intended to protect the retirement goal; second, we re-qualify the funding

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status of each goal in the presence of taxes; third, we look at the properties of these strategies with respect to the other goals (education and home goals). As explained in Section 2.3, we apply a uniform tax rate ζ, which will be taken equal to 10% or 20%, and we adopt the LIFO convention to compute the taxable gains (see Appendix 6.6.5 for mathematical details).

4.3.3.1 Adjustment to GHP for Goal 1

In the context of this case study, taxes arise from the selling operations in the stock and the bond indices held within the PSP building block and from the coupons paid by the bond that secures EG1, coupons which are equal to the consumption expenses.

As explained in Section 3.3.3.1, the effect of taxes on coupons can be virtually removed by purchasing $1/(1-\zeta)$ units of the bond. In other words, for the goal to be secured with certainty, the investor's wealth just before retirement (i.e. at date 21, just before the first payment) must satisfy:

$$A_{21-} \geq \frac{1}{1-\zeta}\tilde{G}^1_{21-},$$

where we recall that $\tilde{G}^1_{21-}$ is the present value of retirement expenses, including the one of date 21. Thus, in the presence of taxes, the GHPs for Goal 1 are the following portfolios:

- The GHP that secures the goal regardless of income is a bond of price

$$GHP_{G1,tax,t} = \frac{1}{1-\zeta}\sum_{\substack{s=21\\s>t}}^{50} G^1_s\, b_{t,s};$$

- The GHP that secures the goal when income is assumed to be re-invested at a zero rate is an option of price

$$U_{ret,tax,t} = \mathbb{E}_t\left[\frac{M_{21}}{M_t}\left(\frac{1}{1-\zeta}\tilde{G}^1_{21-} - \sum_{\substack{s=1\\s>t}}^{20} y_s\right)^+\right];$$

- The GHP that secures the goal when income is invested at the forward rates is an option of price

$$W_{tax,t} = \mathbb{E}_t\left[\frac{M_{21}}{M_t}\left(\frac{1}{1-\zeta}\tilde{G}^1_{21-} - \sum_{\substack{s=1\\s>t}}^{20} y_s\left(1+f_{s,21-s}\right)^{21-s}\right)^+\right].$$

These GHPs are clearly more expensive than the ones that secure the goal in the absence of taxes, which will have implications for the payment policy of non-essential goals: the minimum level of wealth required to pay for the education or home goal in full is higher than in the case without taxes.

A tax adjustment could also be performed for the bonds that protect the other two goals (Goals 2 and 3). If the investor was to protect these goals, he would have to purchase $\frac{1}{1-\zeta}$ units of the bonds that pay the goal cash flows. This raises the minimum capital requirement for these goals.

By taking a buy-and-hold position in one of the three GHPs that protect the retirement goal, the investor can afford the retirement expenses. But the strategies that we implement also involve as a second building block the MSR of stock and bond indices: the periodical rebalancing generates taxes which are not hedged by the adjustment to the GHP value. In other words, this adjustment provides a hedge for only a fraction of the total taxes to be paid. Being not compensated by any inflow, the taxes generated by selling operations can cause violations of the essential goal, at least in theory.

4.3.3.2 Affordability of Goals

We start by reconsidering the funding status of the various goals in the presence of taxes. Table 25 shows the minimum capital

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requirements for two values of the tax rate. For a 10% rate, the investor can purchase the bond that delivers the cash flows of Goal 1 (the retirement goal) adjusted for taxes. This does not suffice, however, to establish that Goal 1 is affordable in the sense of Section 2.3.2 because the adjustment to the GHP value does not recognise the present of taxes on rebalancing operations within the PSP. Nevertheless, we say that Goal 1 is affordable, in the sense that the bond that pays the tax-adjusted cash flows is affordable. For a 20% rate, the bond cannot be purchased with liquid wealth alone, so that Goal 1 (and subsequently the goals of lower priority) should be treated as aspirational. With such a tax rate, Goal 1 can only be protected if the investor uses income to secure as much as possible of the goal. Alternatively, Goal 1 may become affordable if the investor receives an additional endowment at date 0 to increase his liquid wealth. For instance, a 10% increase results in an initial capital

$$1.1\times\$940{,}000=\$1{,}034{,}000,$$

which is sufficient to secure Goal 1.

The other two goals are Goal 2 (the education goal) and Goal 3 (the home goal). The investor can never afford all of them. But Goal 2 becomes jointly affordable with Goal 1 if the tax rate is sufficiently low (10%) and the investor takes the approach to partly secure the goal with income invested at forward rates. Overall, taking as a reference the funding status with liquid wealth only:

- With a 10% tax rate, Goal 1 can be treated as an essential goal so that it will still be referred to as Essential Goal 1 (EG1). Goals 2 and 3 are aspirational and will be referred to as Aspirational Goals 1 and 2 (AG1 and AG2);
- With a 20% tax rate, all three goals are aspirational, except if the investor can increase the initial liquid wealth by 10%, in which case Goal 1 becomes affordable and can again be regarded as essential. Goals 2 and 3 remain aspirational in all cases.

4.3.3.3 Test of Strategies Securing Essential Goal 1

As before, we implement the three protection strategies as buy-and-hold portfolios where the "safe building block" is one of the three GHPs and the "performance building block" is the MSR of stock and bond indices.

Figure 38 shows that by purchasing $1/_{0.9}$ units of the bond that pays the retirement expenses, the investor achieves a perfect protection of EG1. As explained above, this result was not completely obvious ex-ante because the increase in the allocation to the GHP only compensates for the taxes on coupons, but does not recognise the existence of taxes on selling operations in stock and bond indices. The results suggest that these taxes can be paid by liquidating a fraction of the positions in stocks and bonds, without having to reduce the exposure to the GHP. But the probabilities of reaching the two aspirational goals are extremely low compared to the situation without taxes: they fall respectively to 4.9% for AG1 and even 0% for AG2, while they were 77.7% and 1.5% in the absence of taxes (Figure 31). This severe reduction is explained by the huge cost of the bond that delivers the tax-adjusted cash flows. This bond is worth $900,284, which represents close to 96% of the initial endowment: thus, the investor can invest hardly 4% of his wealth in performance assets.

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These disappointing scores make a strong case for the use of income to secure as much as possible of EG1. In Figure 39, we thus look at the success probabilities for the strategy INC-ZER-RET: EG1 is still attained with probability 1, but the probabilities for the other two goals are higher than with the previous strategy. The increase is spectacular for AG1, with a success probability at 80.6%: this is still less than in the absence of taxes (96.9% in Figure 32), but represents a very substantial improvement with respect to 4.9%. The increase is less marked for AG2: the success probability is only 1.9%, while it was 24.5% without taxes.

The last strategy, INC-FWD, uses income and forward contracts to partially secure the goal. The most striking element that appears from Figure 40 is that EG1 is not perfectly secured: there is a 13.4% probability of missing this goal. This is a large probability, but the examination of the shortfall indicators gives a slightly less pessimistic picture. First, it turns out that deviations from the consumption objective occur only in the last year of retirement (year 50): in other words, the investor can fully finance his consumption objectives for the first 29 years of retirement. Second, the deviations are of rather limited size, with an expected shortfall of about 1% and a worst case shortfall less than 5%. The success probabilities for the goal AG1 is 71.2%, which falls below the value achieved in the absence of taxes (83.9% in Figure 33). A stronger reduction is observed for AG2: the probability falls from 42% to 12.9%.

Although the shortfall indicators for EG1 are not extremely bad, it is undisputable that the strategy has a substantial shortfall probability with respect to this goal. It turns out that in the simulations, the paths where EG1 is missed are exactly those where the allocation to the MSR falls to zero at some point. A complete consumption of the budget allocated to the MSR can occur when a tax payment or the payment of a non-essential goal. The liquidation of the stock and bond positions gives rise to capital gains, and taxes, which will be paid at the end of the fiscal year. But because there are no more stock and bond indices in the portfolio, these taxes can only paid by decreasing the exposure to the GHP.

This mechanism is illustrated in Panel (a) of Figure 41, which shows the allocations (expressed as numbers of shares) to the MSR and the GHP on a sample path that saw a deviation from the goal.[37] At the end of year 5, the investor liquidates most of the position in the MSR in order to make the non-financial payments of this date (taxes and payment of the home goal). The taxes generated by this liquidation are paid at the end of the following year, which is year 6. Since the value of the position in MSR does not cover them, the investor not only has to liquidate the remainder of the MSR but he must also sell shares of the GHP. This is why, after year 6, he holds less than one unit of the GHP, and wealth is less than the minimum capital requirement. As a result, EG1 is no longer secured with probability 1, and after a sufficiently long period, this lack of protection results in a failure to meet the consumption objective. By contrast, Panel (b) shows the allocations on a sample path where the MSR allocation remains positive, in spite of the payments made in years 1 to 20 to finance the non-essential goals: these payments are apparent on the picture, but they never lead to a zero allocation. In this context, the taxes generated by the

37 - The number of shares of the MSR is computed as the dollar allocation to the MSR, divided by the value of the MSR, assuming an investment of $1 at date 0.

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transactions in stock and bond indices can be paid by decreasing the exposures to these assets, and there is no need to sell a fraction of the GHP: the number of shares of the GHP remains equal to 1, and the goal is fully secured.

This effect is not specific to the strategy INC-FWD. With the strategy INC-ZER-RET, the allocation to MSR can also fall to zero. But the difference is that the former strategy allows a fraction of income to be re-invested in the MSR (see Section 4.3.1.3), while the latter invests the totality of income in a separate account, earning the forward rate. As a consequence, the INC-ZER-RET strategy allows a positive allocation to the MSR to be recovered, which limits (and, in our simulations, completely avoids) the risk of having to partially liquidate the position in the GHP to pay taxes.

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5. Conclusions and Extensions

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This paper introduces a general operational framework, which formalises the goals-based risk allocation approach to wealth management proposed in Chhabra (2005), and which can be used by a financial advisor to allow individual investors to optimally allocate to categories of risks they face across all life stages and wealth segments so as to achieve personally meaningful financial goals. Through a number of realistic case study examples, we document the benefits of the approach, which respects the individual investor's essential goals with the highest degree of probability, while allowing for substantial upside potential that leads to a reasonably high probability of achieving ambitious aspirational goals.

In addition to developing and analysing optimal portfolio construction methodologies, this paper also introduces robust heuristics, which can be thought of as reasonable approximations for optimal strategies that can accommodate a variety of implementation constraints, including the presence of transaction costs, the presence of short-sale constraints, the presence of parameter estimation risk, etc. One key feature that is explicitly discussed in this paper is the constraint on limited customisation. While providing each individual investor with a dedicated investment solution precisely tailored to meet their goals and constraints would be desirable, it would not be consistent with implementation constraints faced by financial advisors. The appropriate granularity in terms of numbers and types of underlying building blocks and allocation strategies will therefore have to be carefully assessed, with a key trade-off between increasing accuracy in implementing dedicated investment solutions and increasing costs of implementation.

After developing an implementable and robust approach to allocate across risk buckets so as to solve for investor-specific goals, while accounting for key risks an investor faces, and their interplay with the investor's goals, this paper also presents a number of case studies which can be regarded as applications of the approach to various situations that are typical of individual investors' problems. In all cases, the proposed approach is shown to result in an implementable risk-based solution that dominates a standard mean-variance optimal portfolio in terms of the probability of achieving the respective goals and objectives, while taking into account the presence of a number of important practical dimensions such as taxes, illiquid assets and/or concentrated positions, health contingencies, etc.

One important dimension that is not addressed in this paper is the presence of longevity risk, which requires dedicated hedging instruments and modelling techniques, and which would be a key requirement in the context of optimal retirement investment decisions. With the need to supplement retirement savings via voluntary contributions, individuals will increasingly be responsible for their own saving and investment decisions. This global trend poses substantial challenges as individual investors not only face behavioural limitations, but also typically lack the expertise needed to make educated investment decisions. In response to these concerns, a number of new investment products have been proposed over the past few years by the

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asset management industry, both with and without protection against longevity risk. There are reasons to believe, however, that these products, known as target date funds and variable annuities, respectively, fall short of providing satisfactory solutions to the problems faced by individuals when approaching investment saving decisions. We leave for further research an in-depth analysis of the design of long-term retirement solutions in the presence of time-varying opportunity sets, multiple goals and uncertain lifetime.

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6. Technical Appendix

6. Technical Appendix

This appendix collects the proofs of the main results given in the paper. For better readability, we have divided it into sections organised around a theme.

6.1 Affordability of Goals

6.1.1 Proof of Proposition 3

Assume first that the goal is affordable and consider a strategy of value A_t that reaches it with certainty. We show by induction that $A_{T_{p-j}} \geq K_{T_{p-j}}$ for all $j = 0,\dots, p$ (this inequality, as well as the subsequent ones, are assumed to hold with probability 1).

- Since $K_{T_p} = G_{T_p}$, the property is true for $j = 0$;
- Assume that $A_{T_{p-j}} \geq K_{T_{p-j}}$ for some $j \leq p - 1$. This implies, by absence of arbitrage opportunities:

$$A_{T_{p-j-1}} \geq \mathbb{E}_{T_{p-j-1}}\left[\frac{M_{T_{p-j}}}{M_{T_{p-j-1}}} K_{T_{p-j}}\right].$$

Because the goal is secured, we thus have:

$$A_{T_{p-j-1}} \geq \max\left[G_{T_{p-j-1}}, \mathbb{E}_{T_{p-j-1}}\left[\frac{M_{T_{p-j}}}{M_{T_{p-j-1}}} K_{T_{p-j}}\right]\right].$$

The right-hand side is $K_{T_{p-j-1}}$ (we recall that $G_{T_0} = 0$ by definition).

Hence, we have $A_0 \geq K_0$. Observe that the implication "if the goal is affordable, then $A_0 \geq K_0$" does not require market completeness.

Assume now that $A_0 \geq K_0$. Because the market is complete, all the payoffs $K_{T_1},\dots,K_{T_p}$ are replicable. We let $(\underline{x}_{jt})_{0\leq t\leq T_j}$ denote the weights of the dynamic strategy that replicates K_{T_j}. We consider the following strategy, which is a roll-over of the exchange options:

$$\underline{w}_t = \underline{x}_{j+1,t},$$

for $T_j \leq t < T_{j+1}$ and $j = 0,\dots, p - 1$.

We now show by induction that $A_{T_j} \geq K_{T_j}$ for all $j = 0,\dots, p$.

- The property is true for $j = 0$ by assumption;
- If it is true for some $j < p$, then we have:

$$A_{T_j} \geq \mathbb{E}_{T_j}\left[\frac{M_{T_{j+1}}}{M_{T_j}} K_{T_{j+1}}\right].$$

The right-hand side is the price at date T_j of the payoff $K_{T_{j+1}}$ to be paid at date T_{j+1}. Over the period $[T_j, T_{j+1}[$, the portfolio is fully invested in the strategy that replicates this payoff. Hence:

$$A_{T_{j+1}} \geq K_{T_{j+1}}.$$

Eventually, we have $A_{T_j} \geq K_{T_j}$ for all $j = 1,\dots, p$, and these inequalities imply that:

$$A_{T_j} \geq G_{T_j}$$

for $j = 1,\dots, p$, so the goal is secured.

6.1.2 Proof of Corollary 1

We show that $K_{T_{p-j}} = G_{T_{p-j}}$ by recursion on j.

- The property is true for $j = 0$, by definition of K_{T_p};
- Assume that it is true for some $j \leq p-2$. Then:

$$\mathbb{E}_{T_{p-j-1}}\left[\frac{M_{T_{p-j}}}{M_{T_{p-j-1}}} K_{T_{p-j}}\right] = \mathbb{E}_{T_{p-j-1}}\left[\frac{M_{T_{p-j}}}{M_{T_{p-j-1}}} G_{T_{p-j}}\right] \leq G_{T_{p-j-1}},$$

hence:

$$K_{T_{p-j-1}} = G_{T_{p-j-1}}.$$

Hence $K_{T_{p-j}} = G_{T_{p-j}}$ for all $j = 0,\dots, p-1$. It follows that the present value of the goal is:

$$\tilde{G}_t = \mathbb{E}_t\left[\frac{M_{T_{j+1}}}{M_t} G_{T_{j+1}}\right]$$

for $T_j < t \leq T_{j+1}$ and $j = 0,\dots, p-1$,

and:

$$\tilde{G}_0 = K_0 = \mathbb{E}[M_{T_1} G_{T_1}].$$

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6.1.3 Equivalent Form of Definition 4

Consider the wealth process given by (4). We have to show that if A_T is nonnegative almost surely, then A_t is nonnegative too for all t between 0 and T.

Wealth is discontinuous (on the consumption dates), but the state-price deflator M is continuous, so the standard version of Ito's formula applies:

$$d(M_tA_t) = A_t dM_t + M_t dA_t - M_t \underline{q}_t' \underline{\sigma}_t' \underline{\lambda}_t dt.$$

Substituting the dynamics of M and A and simplifying gives:

$$d(M_tA_t) = M_t\left[\underline{\sigma}_t \underline{q}_t - A_t \underline{\lambda}_t\right]' d\underline{z}_t - M_t \sum_{j=1}^{p} c_{T_j} dJ_{T_j,t}.$$

Because $dJ_{T_j,t}$ equals 1 if $t = T_j$ and 0 otherwise, the M_T in the second term of the right-hand side can be replaced by M_{T_j}, which gives:

$$d(M_tA_t) = M_t\left[\underline{\sigma}_t \underline{q}_t - A_t \underline{\lambda}_t\right]' d\underline{z}_t - \sum_{j=1}^{p} M_{T_j} c_{T_j} dJ_{T_j,t}.$$

Define now the right-continuous process

$$\hat{A}_t = A_t + \sum_{\substack{j=1 \\ T_j \le t}}^{p} \frac{M_{T_j}}{M_t} c_{T_j}.$$

We have:

$$d\left(M_t\hat{A}_t\right) = d(M_tA_t) + \sum_{j=1}^{p} M_{T_j} c_{T_j} dJ_{T_j,t}.$$

Hence:

$$d\left(M_t\hat{A}_t\right) = M_t\left[\underline{\sigma}_t \underline{q}_t - A_t \underline{\lambda}_t\right]' d\underline{z}_t.$$

Thus, $M\hat{A}$ follows a martingale, and we have, for all t between 0 and T:

$$M_t\hat{A}_t = \mathbb{E}_t\left[M_T\hat{A}_T\right].$$

Re-arranging terms, we obtain:

$$A_t = \mathbb{E}_t\left[\frac{M_T}{M_t}A_T\right] + \mathbb{E}_t\left[\sum_{\substack{j=1 \\ T_j > t}}^{p} \frac{M_{T_j}}{M_t} c_{T_j}\right],$$

for all t in$[0, T]$. (36)

Hence, if A_T is nonnegative, A_T is nonnegative too (recall that consumption is nonnegative).

6.1.4 Proof of Proposition 4

It is shown in the text of the paper that if the goal is affordable, then the initial wealth satisfies $A_0 \geq \tilde{G}_0$.

For the converse implication, suppose that $A_0 \geq \tilde{G}_0$, and consider the strategy fully invested in the coupon-paying bond, the coupons being re-invested in the bond. The dynamics of wealth reads:

$$dA_t = A_t \frac{d\hat{G}_t}{\hat{G}_t} - \sum_{j=1}^{p} c_{T_j} dJ_{T_j,t}.$$

Moreover, the differential of the total return index is:

$$\frac{d\hat{G}_t}{\hat{G}_t} = \frac{d\tilde{G}_t}{\tilde{G}_t} + \sum_{j=1}^{p} \frac{c_{T_j}}{\tilde{G}_t} dJ_{T_j,t}.$$

Hence:

$$d\left(A_t - \tilde{G}_t\right) = \frac{A_t - \tilde{G}_t}{\hat{G}_t} d\hat{G}_t.$$

Integrating this equation, we obtain:

$$A_t = \tilde{G}_t + \left(\frac{A_0}{\tilde{G}_0} - 1\right)\hat{G}_t.$$

6.1.5 Proof of Proposition 5

Before proving the proposition, we start with a simple technical lemma, showing that the present value of a goal is not impacted if one introduces an additional horizon where the minimum wealth is zero.

Lemma. *Consider a wealth-based goal represented by the nonnegative minimum wealth levels $G_{T_1},\ldots, G_{T_p}$ on dates $T_1<\ldots< T_p$. Let τ be a date distinct from $T_1,\ldots,T_p$, and define the goal G^2 as:*

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$$G^2_{T_j} = G_{T_j},$$
for $j = 1,\dots, p$,
$$G^2_\tau = 0.$$

Then: $\tilde{G}^2_0 = \tilde{G}_0$.

Proof. There are three cases to distinguish: $\tau < T_1$, $\tau > T_p$ or τ is between T_j and T_{j+1} for some j. We only treat the last case; the first two ones are handled in a similar way. We define the payoffs K_{T_k} and $K^2_{T_k}$ and the prices $\tilde{K}_{T_k,T_{k+1}}$ and $\tilde{K}^2_{T_k,T_{k+1}}$ as is done in Proposition 3. We have:

$$K^2_{T_k} = K_{T_k}, \text{ for } k = j+1,\dots, p,$$

$$\tilde{K}^2_{T_k,T_{k+1}} = \tilde{K}_{T_k,T_{k+1}}, \text{ for } k = j,\dots, p-1.$$

Then, $K^2_\tau = \max\left[0, \tilde{K}^2_{\tau,T_{j+1}}\right] = \tilde{K}^2_{\tau,T_{j+1}}$ so that:

$$\tilde{K}^2_{T_j,\tau} = \mathbb{E}_{T_j}\left[\frac{M_{T_{j+1}}}{M_{T_{j+1}}} K^2_{T_{j+1}}\right]$$
$$= \mathbb{E}_{T_j}\left[\frac{M_{T_{j+1}}}{M_{T_{j+1}}} K_{T_{j+1}}\right] = \tilde{K}_{T_j,T_{j+1}}.$$

Hence:
$$K^2_{T_j} = \max\left[G_{T_j}, \tilde{K}_{T_j,T_{j+1}}\right] = K_{T_j}.$$

Then, we obtain $K^2_{T_k} = K_{T_k}$, for all $k = 0,1,\dots, j$.

Let us return to the framework of the proposition. The lemma implies that one can introduce additional minimum levels equal to zero to a wealth-based goal without changing its present value. Hence, without loss of generality, one can assume that all goals have the same set of horizons $\tau = \{T_1,\dots,T_p\}$. We show by induction that $K_{T_{p-j}} \geq K^l_{T_{p-j}}$:

- This is true for $j = 0$, as $K_{T_p} = G_{T_p} \geq G^l_{T_p} = K^l_{T_p}$;
- Assume that the property is true for some j between 0 and $p-1$.

Then:
$$K_{T_{p-j-1}} = \max\left[G_{T_{p-j-1}}, \tilde{K}_{T_{p-j-1},T_{p-j}}\right],$$

with:
$$G_{T_{p-j-1}} \geq G^l_{T_{p-j}},$$
$$\tilde{K}_{T_{p-j-1},T_{p-j}} = \mathbb{E}_{T_{p-j-1}}\left[\frac{M_{T_{p-j}}}{M_{T_{p-j-1}}} K_{T_{p-j}}\right]$$
$$\geq \mathbb{E}_{T_{p-j-1}}\left[\frac{M_{T_{p-j}}}{M_{T_{p-j-1}}} K^l_{T_{p-j}}\right] = \tilde{K}^l_{T_{p-j-1},T_{p-j}}.$$

Hence:
$$K_{T_{p-j-1}} \geq \max\left[G^l_{T_{p-j}}, \tilde{K}^l_{T_{p-j-1},T_{p-j}}\right] = K^l_{T_{p-j-1}}.$$

Eventually, we have $K_0 \geq K^l_0$ for $l = 1,\dots,L$. Hence:
$$K_0 \geq \max[K^1_0, \dots, K^L_0],$$
that is:
$$\tilde{G}_0 \geq \max[\tilde{G}^1_0, \dots, \tilde{G}^L_0].$$

6.1.6 Proof of Proposition 8

Assume that the goal is affordable, and take a strategy that secures it, with wealth evolving as (5). Define the payoffs $\left(V_{T_j}\right)_{j=1,\dots,p}$ as in the proposition. We let by convention $A_{0-} = A_0$, and we show by induction that $A_{T_{p-j-}} \geq V_{T_{p-j-}}$ for all $j = 0,\dots, p$.

- The property is true for j=0. Indeed, we have
$$A_{T_p} \geq c_{T_p},$$
hence:
$$A_{T_{p-}} + y_{T_p} \geq c_{T_p}.$$

Since the goal is affordable, we have also $A_{T_{p-}} \geq 0$, hence $A_{T_{p-}} \geq (c_{T_p} - y_{T_p})^+ = V_{T_p}$.

- Suppose that the property is true for some $j = 0,\dots, p-1$, that is:
$$A_{T_{p-j}} \geq V_{T_{p-j}}.$$

Taking the present values on both sides, we obtain:
$$A_{T_{p-j-1}} \geq \tilde{V}_{T_{p-j-1},T_{p-j}},$$

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that is:

$$A_{T_{p-j-1}-} \geq \tilde{V}_{T_{p-j-1},T_{p-j}} + c_{T_{p-j-1}} - y_{T_{p-j-1}}.$$

We also have $A_{T_{p-j-1}} \geq 0$, hence:

$$A_{T_{p-j-1}} \geq V_{T_{p-j-1}}.$$

Eventually, we obtain $A_0 \geq V_0$.

Conversely, suppose that $A_0 \geq V_0$, and consider a roll-over of exchange options, each of them paying off V_{T_j} at date T_j (these options exist because the market is complete). We let by convention $\tilde{V}_{T_p,T_{p+1}} = V_{T_p}$, and we show by induction that $A_{T_j} \geq \tilde{V}_{T_j,T_{j+1}}$. for all j = 0,…, p.

- The property is true for j = 0, since

$$A_0 \geq V_0 = \tilde{V}_{0,T_1};$$

- Assume that the property is true for some j = 0,…, p-1, that is:

$$A_{T_j} \geq \tilde{V}_{T_j,T_{j+1}}.$$

Then, the wealth remains nonnegative between dates T_j and T_{j+1}, and the wealth just before income at date T_{j+1} satisfies:

$$A_{T_{j+1}-} = \frac{A_{T_j}}{\tilde{V}_{T_j,T_{j+1}}} V_{T_{j+1}} \geq V_{T_{j+1}} = \left(\tilde{V}_{T_{j+1},T_{j+2}} + c_{T_{j+1}} - y_{T_{j+1}}\right)^+,$$

hence:

$$A_{T_{j+1}} \geq \tilde{V}_{T_{j+1},T_{j+2}}.$$

Eventually, we obtain $A_t \geq 0$ for all t, so the goal is secured.

Let us now establish the bounds for V_0. We show by induction that $\tilde{G}_{T_{p-j}-} - \tilde{H}_{T_{p-j}-} \leq V_{T_{p-j}} \leq \tilde{G}_{T_{p-j}-}$ for all j = 0,…, p.

- The property is true for j = 0. Indeed, we have:

$$V_{T_p} = \left(c_{T_p} - y_{T_p}\right)^+ \geq c_{T_p} - y_{T_p}, \quad V_{T_p} \leq c_{T_p},$$

and:

$$\tilde{G}_{T_{p-j}-} = c_{T_p}, \quad \tilde{H}_{T_p-} = y_{T_p}.$$

- Assume that the property is true for some j = 0,…, p-1, that is:

$$\tilde{G}_{T_{p-j}-} - \tilde{H}_{T_{p-j}-} \leq V_{T_{p-j}} \leq \tilde{G}_{T_{p-j}-}.$$

Then, taking the present values of both sides, we obtain:

$$\tilde{G}_{T_{p-j-1}} - \tilde{H}_{T_{p-j-1}} \leq \tilde{V}_{T_{p-j-1},T_{p-j}} \leq \tilde{G}_{T_{p-j-1}},$$

hence:

$$V_{T_{p-j-1}} \geq \tilde{G}_{T_{p-j-1}} - \tilde{H}_{T_{p-j-1}} + c_{T_{p-j-1}} - y_{T_{p-j-1}} = \tilde{G}_{T_{p-j-1}-} - \tilde{H}_{T_{p-j-1}-},$$

and:

$$V_{T_{p-j-1}} \leq \tilde{G}_{T_{p-j-1}} + c_{T_{p-j-1}} - y_{T_{p-j-1}} \leq \tilde{G}_{T_{p-j-1}} + c_{T_{p-j-1}} = \tilde{G}_{T_{p-j-1}-}.$$

Eventually, we obtain $\tilde{G}_0 - \tilde{H}_0 \leq V_0 \leq \tilde{G}_0$.

Suppose now that the goal and the income payments are such that $\tilde{G}_{T_j-} \geq \tilde{H}_{T_j-}$ for all j = 1,…, p. The same backward induction as before shows that $V_0 = \tilde{G}_0 - \tilde{H}_0$.

6.1.7 Proof of Proposition 9

Assume that $A_0 \geq U_0$ and purchase the p options at date 0. We recall that $e_{T_j} = y_{T_j} - c_{T_j}$ denotes the net income and we define v_{T_j} as the income payment plus the excess of date T_{j-1} invested at a zero rate:

$$v_{T_j} = u_{T_{j-1}} + e_{T_j},$$

for j =1,…, p.

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Then, we have:

$$u_{T_j} = v_{T_j}^+,$$

$$U_0 = \sum_{k=1}^{p} \mathbb{E}\left[M_{T_k}\left(-v_{T_k}\right)^+\right].$$

More generally, we also define U_{T_j} as the price at date T_j of the remaining $p - j$ options:

$$U_{T_j} = \mathbb{E}_{T_j}\left[\sum_{k=j+1}^{p} \frac{M_{T_k}}{M_{T_j}}\left(-v_{T_k}\right)^+\right].$$

Assume that at some date T_j with $j = 1,..., p-1$, we have $A_{T_j} \geq U_{T_j}$, and that U_{T_j} is invested in zero-coupons paying \$1 on date T_{j+1}. These zero coupons have price less than \$1 because nominal rates are nonnegative. Then, just before income and consumption at date T_{j+1}, we have:

$$A_{T_{j+1}-} \geq U_{T_{j+1}} + u_{T_j} = \mathbb{E}_{T_{j+1}}\left[\sum_{k=j+2}^{p} \frac{M_{T_k}}{M_{T_{j+1}}}\left(-v_{T_k}\right)^+\right] + \left(-v_{T_{j+1}}\right)^+ + u_{T_j}.$$

By definition, $v_{T_{j+1}} = u_{T_j} + e_{T_{j+1}}$, hence:

$$u_{T_j} + \left(-v_{T_{j+1}}\right)^+ + e_{T_{j+1}} \geq v_{T_j}^+ - v_{T_{j+1}} + e_{T_{j+1}} = 0.$$

Moreover, U_{T_j} and $(-v_{T_{j+1}})^+$ are nonnegative, hence:

$$u_{T_j} + \left(-v_{T_{j+1}}\right)^+ + e_{T_{j+1}} \geq e_{T_{j+1}}.$$

Hence:

$$u_{T_j} + \left(-v_{T_{j+1}}\right)^+ + e_{T_{j+1}} \geq e_{T_{j+1}}^+,$$

so that:

$$A_{T_{j+1}} \geq \mathbb{E}_{T_{j+1}}\left[\sum_{k=j+2}^{p} \frac{M_{T_k}}{M_{T_{j+1}}}\left(-v_{T_k}\right)^+\right] + e_{T_{j+1}}^+ = U_{T_{j+1}} + e_{T_{j+1}}^+.$$

This shows that $A_{T_{j+1}} \geq U_{T_{j+1}}$.

Hence, we have $A_{T_j} \geq U_{T_j}$ for all $j = 1,..., p$. In particular, A_{T_j} is nonnegative for all $j = 1, ..., p$, which implies that A_t is nonnegative for all t. Hence the goal is secured in the sense of Definition 8.

6.1.8 Proof of Proposition 10

Assume that $A_0 \geq U_{ret,0}$. Then, at date 0, the option that pays $U_{ret,T_{r+1},1}$ at date T_{r+1} is affordable. More generally, suppose that at some date T_j, where $j = 0,..., r-1$, we have $A_{T_j} \geq U_{T_j}$. Then the option with payoff $U_{ret,T_{r+1},j}$ is affordable. Purchasing this option and investing the remainder of wealth in any other portfolio strategy (with a nonnegative payoff), we have:

$$A_{T_{j+1}-} \geq U_{ret,T_{j+1},j},$$

where:

$$U_{ret,T_{j+1},j} = \mathbb{E}_{T_{j+1}}\left[\frac{M_{T_r}}{M_{T_{j+1}}} U_{ret,T_{r+1},j}\right].$$

Because the net income $e_{T_{j+1}} = y_{T_{j+1}} - c_{T_{j+1}}$ is nonnegative, we have:

$$U_{ret,T_{r+1},j} = \left[U_{ret,T_{r+1},j+1} - e_{T_{j+1}}\right]^+,$$

hence:

$$U_{ret,T_{j+1},j} \geq \mathbb{E}_{T_{j+1}}\left[\frac{M_{T_{r+1}}}{M_{T_{j+1}}}\left[U_{ret,T_{r+1},j+1} - e_{T_{j+1}}\right]\right] = U_{ret,T_{j+1},j+1} - e_{T_{j+1}} b_{T_{j+1},T_{r+1}},$$

where $b_{T_{j+1},T_{r+1}}$ is the price at date T_{j+1} of the zero-coupon bond maturing at date T_{r+1}. As long as nominal rates are nonnegative, we have $b_{T_{j+1},T_{r+1}} \leq 1$, hence:

$$U_{ret,T_{j+1},j} + e_{T_{j+1}} \geq U_{ret,T_{j+1},j+1}.$$

Hence:

$$A_{T_{j+1}} \geq U_{ret,T_{j+1},j+1}.$$

This proves that the strategy is implementable since the wealth of date T_j (after income) is sufficient to afford the

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option which pays $U_{ret,T_{r+1},j}$ on date T_{r+1}. Moreover, with this strategy, we have $A_t \geq 0$ for all t. This means that the goal is secured in the sense of Definition 8.

6.2 Probability-Maximising Strategies

6.2.1 Proof of Proposition 10

Define $N_T = G_T - F_T$, and $\tilde{N}_t = \tilde{G}_t - \tilde{F}_t$: N_T is positive by the assumption that $G_T > F_T$ almost surely, and $\tilde{N}_t$ is positive too since $\tilde{N}_t = \mathbb{E}_t\left[\frac{M_T}{M_t}N_T\right]$. Following El Karoui and Rouge (2000), we introduce the probability measure $\mathbb{Q}^N$ such that discounted prices expressed in the numeraire $\tilde{N}$ follow $\mathbb{Q}^N$-martingales. The conditional Radon-Nikodym density of $\mathbb{Q}^N$ with respect to $\mathbb{P}$ is:

$$\left.\frac{d\mathbb{Q}^N}{d\mathbb{P}}\right|_t = \frac{M_T N_T}{M_t \tilde{N}_t}.$$

We then define the set

$$\tilde{R} = \{M_T N_T \leq K\tilde{N}_0\},$$

and assume that K can be chosen in such a way that $\mathbb{E}[M_T N_T \mathbb{I}_R] = A_0 - \tilde{F}_0$, or equivalently that $\mathbb{Q}^N(\tilde{R}) = \frac{A_0 - \tilde{F}_0}{\tilde{N}_0}$.[45] We let $X^* = F_T + N_T\mathbb{I}_R$, which is a replicable payoff since the market is complete. Note that since N_T is positive, we have:

$$\{X^* \geq G_T\} = \{N_T\mathbb{I}_R \geq N_T\} = \tilde{R}.$$

Consider now any strategy with a terminal wealth A_T which satisfies $A_T \geq F_T$ almost surely, and define the "success region" $R = \{A_T \geq G_T\}$. We have:

$$\mathbb{Q}^N(R) = \frac{1}{\tilde{N}_0}\mathbb{E}[M_T N_T \mathbb{I}_R] \leq \frac{1}{\tilde{N}_0}\mathbb{E}[M_T(A_T - F_T)\mathbb{I}_R] \leq \frac{A_0 - \tilde{F}_0}{\tilde{N}_0}.$$

Hence $\mathbb{Q}^N(R) \leq \mathbb{Q}^N(\tilde{R})$. By Neyman-Pearson lemma, it follows that:

$$\mathbb{P}(R) \leq \mathbb{P}(\tilde{R}).$$

45 - A sufficient condition for the existence of K is that the cumulative distribution function of the random variable N_T under $\mathbb{Q}^N$ be continuous. This is the case if N_T has no atoms (i.e. for any real number x, the probability that $N_T = x$ is zero).

Hence X^* is the probability-maximising payoff subject to the floor constraint. Using the equality $A_{go,T} = \frac{A_0}{M_T}$, this payoff can be rewritten as:

$$A_T^* = F_T + (G_T - F_T)\times\mathbb{I}_{\left\{\frac{A_{go,T}}{N_T} \geq \frac{A_0}{K\tilde{N}_0}\right\}}.$$

The optimal wealth process is given by:

$$A_t^* = \tilde{F}_t + \tilde{N}_t\mathbb{Q}_t^N(\tilde{R}).$$

(The second term in the right-hand side is the price of the digital option which pays 1 upon realisation of the event $\tilde{R}$.) Assume that $\tilde{F}_t = \alpha\tilde{G}_t$, which implies that $\underline{w}_{GHP,s} = \underline{w}_{FHP,s}$. Then, if $\underline{\sigma}_{Fs} = \underline{\sigma}_s\underline{w}_{Gs}$ denotes the volatility vector of $\tilde{G}$:

$$N_T = \tilde{N}_t \times \exp\left[\int_t^T\left(r_s + \underline{\sigma}_{Gs}'\underline{\lambda}_s - \frac{1}{2}\|\underline{\sigma}_{Gs}\|^2\right)ds + \int_t^T \underline{\sigma}_{Gs}'d\underline{z}_s\right],$$

hence:

$$M_T N_T = M_t\tilde{N}_t \times \exp\left[-\frac{1}{2}\int_t^T \|\underline{\lambda}_s - \underline{\sigma}_{Gs}\|^2 ds - \int_t^T [\underline{\lambda}_s - \underline{\sigma}_{Gs}]'d\underline{z}_s\right].$$

By Girsanov's theorem, the process $\underline{z}_t^N = \underline{z}_t + \int_0^t[\underline{\lambda}_s - \underline{\sigma}_{Gs}]ds$ is a $\mathbb{Q}^N$-Brownian motion. Hence:

$$\frac{A_{go,T}}{N_T} = \frac{A_{go,t}}{\tilde{N}_t} \times \exp\left[-\frac{1}{2}\int_t^T \|\underline{\lambda}_s - \underline{\sigma}_{Gs}\|^2 ds + \int_t^T [\underline{\lambda}_s - \underline{\sigma}_{Gs}]'d\underline{z}_s^N\right].$$

If $\underline{\lambda}_s$ and $\underline{\sigma}_{Fs}$ are deterministic functions of time, then $A_{go,T}/N_T$ is log-normally distributed conditional on $\mathcal{F}_t$.

This implies that, if $\kappa_{t,T} = \sqrt{\int_t^T \|\underline{\lambda}_s - \underline{\sigma}_{Gs}\|^2 ds}$, we have:

$$\mathbb{Q}_t^N(\tilde{R}) = \mathcal{N}\left(\frac{\ln\frac{A_0}{K\tilde{N}_0} - \ln\frac{A_{go,t}}{\tilde{N}_t} + \frac{1}{2}\kappa_{t,T}^2}{\kappa_{t,T}}\right).$$

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Hence:

$$A_t^* = \tilde{F}_t + \tilde{N}_t \mathcal{N}(W_t),$$

with

$$W_t = \frac{\ln\frac{A_0}{K\tilde{N}_0} - \ln\frac{A_{go,t}}{\tilde{N}_t} + \frac{1}{2}\kappa_{t,T}^2}{\kappa_{t,T}}.$$

By Ito's lemma:

$$dA_t^* = d\tilde{F}_t + \mathcal{N}(W_t)d\tilde{N}_t + \tilde{N}_t n(W_t)dW_t + (\cdots)dt,$$

hence:

$$dA_t^* = \tilde{F}_t \underline{\sigma}_{Gt}' d\underline{z}_t + \mathcal{N}(W_t)\tilde{N}_t \underline{\sigma}_{Gt}' d\underline{z}_t + \frac{\tilde{N}_t n(W_t)}{\kappa_{t,T}}\left[\underline{\lambda}_t - \underline{\sigma}_{Gt}\right]' d\underline{z}_t + (\cdots)dt.$$

Matching the Brownian terms of both sides, we obtain the optimal vector of weights:

$$A_t^* \underline{w}_t^* = \tilde{F}_t \underline{w}_{GHP,t} + (A_t^* - \tilde{F}_t)\underline{w}_{GHP,t} + \frac{\tilde{N}_t n(W_t)}{\kappa_{t,T}}\left[\underline{w}_{go,t} - \underline{w}_{GHP,t}\right].$$

Let $\varphi_t = \frac{\tilde{N}_t n(W_t)}{\kappa_{t,T} A_t^*}$. Then:

$$\underline{w}_t^* = \varphi_t \underline{w}_{go,t} + (1 - \varphi_t)\underline{w}_{GHP,t}.$$

Because $\underline{w}_{GS} = \underline{w}_{FS}$, the coefficient $\kappa_{t,T}$ can be rewritten as:

$$\kappa_{t,T} = \sqrt{\int_t^T \left\|\underline{\lambda}_s - \underline{\sigma}_{Fs}\right\|^2 ds}.$$

The allocation to the growth-optimal portfolio can be rewritten as:

$$\varphi_t = \frac{1}{\kappa_{t,T}} \frac{\tilde{N}_t}{A_t^*} n\left[\mathcal{N}^{-1}\left(\frac{A_t^* - \tilde{F}_t}{\tilde{N}_t}\right)\right].$$

If A_t^* approaches $\tilde{F}_t$ from above (resp., $\tilde{G}_t$ from below), then the ratio $(A_t^* - \tilde{F}_t)/\tilde{N}_t$ approaches 0 (resp., 1). Hence, the quantity $\mathcal{N}^{-1}\left(\frac{A_t^* - \tilde{F}_t}{\tilde{N}_t}\right)$ diverges to minus infinity (resp., plus infinity). Hence, φ_t shrinks to zero.

6.3 Utility-Maximising Strategies

6.3.1 Proof of Proposition 11

6.3.1.1 Optimal Payoff

Assume that the financial market is dynamically complete, so there exists a unique state-price deflator. We follow the martingale approach of Cox and Huang (1989). Consider the following candidate optimal payoff for Program (21):

$$X^* = G_T U'^{-1}(\eta_1 M_T G_T),$$

where the constant η_1 is adjusted in such a way that the budget constraint $\mathbb{E}[M_T X^*] = A_0$ is satisfied. Such a constant does exist, because we have

$$X^* = \eta_1^{-\frac{1}{\gamma}} G_T^{1-\frac{1}{\gamma}} M_T^{-\frac{1}{\gamma}},$$

so it suffices to take:

$$\eta_1 = \frac{A_0^{-\gamma}}{\mathbb{E}\left[(M_T G_T)^{1-\frac{1}{\gamma}}\right]^{-\gamma}}.$$

(If U was not specified as the CRRA function, the constant η might not be computable in closed form, and conditions would have to be imposed on U' to ensure the existence and the uniqueness of η_1.)

Consider now any portfolio strategy that yields a positive terminal wealth A_T. Because U is concave, we have:

$$U\left(\frac{A_T}{G_T}\right) \leq U\left(\frac{X^*}{G_T}\right) + \frac{A_T - X^*}{G_T} U'\left(\frac{X^*}{G_T}\right),$$

hence:

$$U\left(\frac{A_T}{G_T}\right) \leq U\left(\frac{X^*}{G_T}\right) + (A_T - X^*)\eta M_T.$$

Because A_T is the terminal value of a self-financing portfolio strategy, we have $\mathbb{E}[M_T A_T] = A_0 = \mathbb{E}[M_T X^*]$, hence:

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$$\mathbb{E}\left[U\left(\frac{A_T}{G_T}\right)\right] \leq \mathbb{E}\left[U\left(\frac{X^*}{G_T}\right)\right].$$

This shows that X^* achieves the highest expected utility. Moreover, because the market is complete, X^* is attainable. Hence, it is the optimal terminal wealth, so we denote it with A_T^{*G}.

To solve Program (17), it suffices to note that it is equivalent to Program (21), with the goal G_T being replaced by the constant 1. Hence, the optimal terminal wealth in (17) has the form:

$$A_T^{*0} = U'^{-1}(\eta_0 M_T).$$

Using the expression for the inverse marginal utility, we get:

$$A_T^{*G} = \eta_1^{-\frac{1}{\gamma}} G_T^{1-\frac{1}{\gamma}} M_T^{-\frac{1}{\gamma}}, \quad (37)$$

$$A_T^{*0} = \eta_0^{-\frac{1}{\gamma}} M_T^{-\frac{1}{\gamma}}, \quad (38)$$

with

$$\eta_0 = \frac{A_0^{-\gamma}}{\mathbb{E}\left[M_T^{1-\frac{1}{\gamma}}\right]^{-\gamma}}.$$

Hence:

$$A_T^{*G} = \nu_1 A_T^{*0} G_T^{1-\frac{1}{\gamma}},$$

with:

$$\nu_1 = \left(\frac{\eta_1}{\eta_0}\right)^{-\frac{1}{\gamma}} = \frac{\mathbb{E}\left[M_T^{1-\frac{1}{\gamma}}\right]}{\mathbb{E}\left[(M_T G_T)^{1-\frac{1}{\gamma}}\right]}.$$

6.3.1.2 Optimal Strategy

The optimal wealth on date t for Program (21) is given by:

$$A_t^{*G} = \mathbb{E}_t\left[\frac{M_T}{M_t} A_T^{*G}\right] = \eta_1^{-\frac{1}{\gamma}} \mathbb{E}_t\left[\left(\frac{M_T G_T}{M_t \tilde{G}_t}\right)^{1-\frac{1}{\gamma}}\right] M_t^{-\frac{1}{\gamma}} \tilde{G}_t^{1-\frac{1}{\gamma}}.$$

Let Z_{Gt} denote the conditional expectation in the right-hand side, and write its dynamics as:

$$\frac{dZ_{Gt}}{Z_{Gt}} = \mu_{ZGt} dt + \underline{\sigma}'_{ZGt} d\underline{z}_t.$$

Apply now Ito's lemma to A_T^{*G} and match the diffusion terms of both sides of the equality, to obtain:

$$\underline{\sigma_t} \underline{w_t^{*G}} = \frac{1}{\gamma}\underline{\lambda}_t + \underline{\sigma}_{ZGt} + \left(1 - \frac{1}{\gamma}\right)\underline{\sigma}_{Gt},$$

so the optimal strategy is:

$$\underline{w_t^{*G}} = \frac{1}{\gamma}\underline{w}_{go,t} + \Sigma_t^{-1}\underline{\sigma_t'}\underline{\sigma}_{ZGt} + \left(1 - \frac{1}{\gamma}\right)\underline{w}_{Gt}.$$

A similar derivation can be made for Program (17), provided the present value of the goal is replaced by the zero-coupon price. The optimal strategy is:

$$\underline{w_t^{*0}} = \frac{1}{\gamma}\underline{w}_{go,t} + \Sigma_t^{-1}\underline{\sigma_t'}\underline{\sigma}_{Z0t} + \left(1 - \frac{1}{\gamma}\right)\underline{w}_{bt},$$

where Z_{0t} is the random variable

$$Z_{0t} = \mathbb{E}_t\left[\left(\frac{M_T}{M_t b_{t,T}}\right)^{1-\frac{1}{\gamma}}\right],$$

and $\underline{w}_{bt}$ is the portfolio that replicates the zero-coupon bond maturing on date T. The optimal weights in the presence of the goal can thus be written as:

$$\underline{w_t^{*G}} = \underline{w_t^{*0}} + \Sigma_t^{-1}\underline{\sigma_t'}\left[\underline{\sigma}_{ZGt} - \underline{\sigma}_{Z0t}\right] + \left(1 - \frac{1}{\gamma}\right)\left[\underline{w}_{Gt} - \underline{w}_{bt}\right].$$

The deflated goal and zero-coupon values can be expressed as:

$$M_T G_T = M_t \tilde{G}_t \times \exp\left[-\int_t^T \left\|\underline{\lambda}_s - \underline{\sigma}_{Gs}\right\|^2 ds - \int_t^T \left[\underline{\lambda}_s - \underline{\sigma}_{Gs}\right]' d\underline{z}_s\right],$$

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$$M_T = M_t b_{t,T} \times \exp\left[-\int_t^T \left\|\underline{\lambda_s} - \underline{\sigma_{b,s,T}}\right\|^2 ds - \int_t^T \left[\underline{\lambda_s} - \underline{\sigma_{b,s,T}}\right]' d\underline{z_s}\right].$$

Hence, if $\underline{\lambda_s}$, $\underline{\sigma_{Gt}}$ and $\underline{\sigma_{b,t,T}}$ are deterministic functions of time, the random variables

$$\left(\frac{M_T G_T}{M_t G_t}\right)^{1-\frac{1}{\gamma}}$$

and

$$\left(\frac{M_T}{M_t b_{t,T}}\right)^{1-\frac{1}{\gamma}}$$

are independent from $\mathcal{F}_t$, so the conditional expectations equal the unconditional expectations. Hence, Z_{Gt} and Z_{0t} are deterministic functions of time, so their volatility vectors are zero. Hence, we obtain:

$$\underline{w_t^{*G}} = \underline{w_t^{*0}} + \left(1 - \frac{1}{\gamma}\right)\left[\underline{w_{Gt}} - \underline{w_{bt}}\right].$$

6.3.2 Proofs of Proposition 12 and Corollary 2

6.3.2.1 Optimal Payoff

Consider the following candidate optimal payoff for Program (22):

$$X^* = \max(F_T, \nu_2 A_T^{*G}).$$

First, we look for a value of ν_2 such that $\mathbb{E}[M_T X^*] = A_0$. We let $f(\nu_2) = \mathbb{E}[M_T X^*]$, so that $f(0) = \tilde{F}_0$, which by assumption is less than A_0. By the dominated convergence theorem, f is continuous. We also have $f(1) \geq A_0$. Hence, by the intermediate value theorem, there exists a solution to the equation $f(\nu_2) \geq A_0$ in the range [0,1].

To see that the solution is unique when $A_0 > \tilde{F}_0$, consider two candidate solutions $\nu_{21} \leq \nu_{22}$. Because $\max(F_T, \nu_{21} A_T^{*G}) \leq \max(F_T, \nu_{22} A_T^{*G})$ and $f(\nu_{22}) = f(\nu_{21})$, we have, with probability 1:

$$\max(F_T, \nu_{21} A_T^{*G}) = \max(F_T, \nu_{22} A_T^{*G}). \quad (39)$$

The two solutions are necessarily positive since $A_0 > \tilde{F}_0$. There is a positive probability that $\nu_{21} A_T^{*G} > F_T$, otherwise we would have $f(\nu_{21}) = \tilde{F}_0$, hence $f(\nu_{21}) < A_0$. Taking a ω such that $\nu_{21} A_T^{*G}(\omega) > \tilde{F}_0$ and (39) holds, we obtain:

$$\nu_{21} A_T^{*G}(\omega) = \nu_{22} A_T^{*G}(\omega),$$

hence $\nu_{21} = \nu_{22}$ since $A_T^{*G}(\omega) > 0$.

Let A_T be the terminal value of a strategy that satisfies $A_T \geq F_T$ almost surely. By the concavity of U, we have:

$$U\left(\frac{A_T}{G_T}\right) \leq U\left(\frac{X^*}{G_T}\right) + \frac{A_T - X^*}{G_T} U'\left(\frac{X^*}{G_T}\right).$$

Because U' is strictly decreasing, we obtain:

$$U\left(\frac{A_T}{G_T}\right) \leq U\left(\frac{X^*}{G_T}\right) + \frac{A_T - X^*}{G_T} \min\left[U'\left(\frac{\nu_2 A_T^{*G}}{G_T}\right), U'\left(\frac{F_T}{G_T}\right)\right],$$

which can equivalently be written as:

$$U\left(\frac{A_T}{G_T}\right) \leq U\left(\frac{X^*}{G_T}\right) + \frac{A_T - X^*}{G_T}\left\{U'\left(\frac{\nu_2 A_T^{*G}}{G_T}\right) - \left[U'\left(\frac{\nu_2 A_T^{*G}}{G_T}\right) - U'\left(\frac{F_T}{G_T}\right)\right]^+\right\}.$$

By (37), we have

$$A_T^{*G} = \eta^{-\frac{1}{\gamma}} G_T^{1-\frac{1}{\gamma}} M_T^{-\frac{1}{\gamma}} = U'^{-1}(\eta M_T G_T) G_T$$

for some constant η. Hence:

$$U'\left(\frac{\nu_2 A_T^{*G}}{G_T}\right) = \eta \nu_2^{-\frac{1}{\gamma}} M_T G_T.$$

Because $\mathbb{E}[M_T X^*] = \mathbb{E}[M_T A_T]$ it follows that:

$$\mathbb{E}\left[U\left(\frac{A_T}{G_T}\right)\right] \leq \mathbb{E}\left[U\left(\frac{X^*}{G_T}\right)\right] - \mathbb{E}\left[(A_T - X^*)\left[U'\left(\frac{\nu_2 A_T^{*G}}{G_T}\right) - U'\left(\frac{F_T}{G_T}\right)\right]^+\right].$$

The positive part in the right-hand side is non-zero only when $\nu_2 A_T^{*G} < F_T$, in which case X^* equals F_T, and is therefore less than A_T. Hence, the expectation in the right-hand side is nonnegative, so that:

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$$\mathbb{E}\left[U\left(\frac{A_T}{G_T}\right)\right] \leq \mathbb{E}\left[U\left(\frac{X^*}{G_T}\right)\right].$$

Hence, X^* is utility-maximising. Moreover, it is an attainable payoff by the market completeness, so it is the optimal terminal wealth.

For Corollary 2, note that the optimal terminal wealth satisfies, almost surely:

$$A_T^{*F} \leq \nu_2 A_T^{*G} + F_T.$$

Taking the present values on both sides, we obtain:

$$A_0 \leq \nu_2 A_0 + \tilde{F}_0,$$

hence:

$$1 - \frac{\tilde{F}_0}{A_0} \leq \nu_2.$$

Assume now that $\nu_2 = 1$. Then, by definition of ν_2

$$\mathbb{E}\left[M_T \max\left(F_T, A_T^{*G}\right)\right] = A_0 = \mathbb{E}\left[M_T A_T^{*G}\right],$$

which implies that $M_T \max\left(F_T, A_T^{*G}\right) = M_T A_T^{*G}$ with probability 1, hence that $\max\left(F_T, A_T^{*G}\right) = A_T^{*G}$, hence that $A_T^{*G} \geq F_T$ with probability 1. By contraposition, if $\mathbb{P}\left(A_T^{*G}\right) < F_T) > 0$, then it must be the case that $\nu_2 < 1$.

6.3.2.2 Optimal Strategy

The optimal wealth process for Program (22) can be written as:

$$A_t^{*F} = \nu_2 A_t^{*G} + \tilde{F}_t \mathbb{E}_t\left[\frac{M_T F_T}{M_t \tilde{F}_t}\left[1 - \frac{\nu_2 A_T^{*G}}{F_T}\right]^+\right].$$

Let $\mathbb{Q}^F$ be the probability measure whose Radon-Nikodym density with respect to $\mathbb{P}$ is:

$$\left.\frac{d\mathbb{Q}^F}{d\mathbb{P}}\right|_t = \frac{M_T F_T}{M_t \tilde{F}_t}.$$

Under this measure, asset prices expressed in the numeraire $\tilde{F}$ follow martingales.

We have:

$$A_t^{*F} = \nu_2 A_t^{*G} + \tilde{F}_t \mathbb{E}_t^{\mathbb{Q}^F}\left[\left[1 - \frac{\nu_2 A_T^{*G}}{F_T}\right]^+\right].$$

By Girsanov's theorem, the process $\underline{z}_t^F = \underline{z}_t + \int_0^t [\underline{\lambda}_s - \underline{\sigma}_{Fs}]$ is a $\mathbb{Q}^F$-Brownian motion. The dynamics of the ratio $R^{*G} = A^{*G}/_{\tilde{F}}$ under $\mathbb{Q}^F$ can be written as:

$$\frac{dR_t^{*G}}{R_t^{*G}} = \left[\underline{\sigma}_t \underline{w}_t^{*G} - \underline{\sigma}_{Ft}\right]' d\underline{z}_t^F.$$

If the vectors $\underline{\sigma}_{Gt} = \underline{\sigma}_t \underline{w}_{Ft}$ and $\underline{\sigma}_t^{*G} = \underline{\sigma}_t \underline{w}_t^{*G}$ are deterministic functions of time, then the Black-Scholes formula implies that:

$$\mathbb{E}_t^{\mathbb{Q}^F}\left[\left[1 - \frac{\nu_2 A_T^{*G}}{F_T}\right]^+\right] = -\frac{\nu_2 A_t^{*G}}{\tilde{F}_t}\mathcal{N}(-d_{1t}) + \mathcal{N}(-d_{2t}),$$

with:

$$d_{1t} = \frac{1}{\zeta_{t,T}}\left[\ln\frac{\nu_2 A_t^{*G}}{\tilde{F}_t} + \frac{1}{2}\zeta_{t,T}^2\right],$$

$$d_{2t} = d_{1t} - \zeta_{t,T},$$

$$\zeta_{t,T} = \sqrt{\int_t^T \left\|\underline{\sigma}_s^{*G} - \underline{\sigma}_{Fs}\right\|^2 ds}.$$

Note that $\mathcal{N}(-d_{2t})$ equals $\mathbb{Q}_t^F\left(\nu_2 A_T^{*G} \leq F_T\right)$, which is the probability that the insurance put ends up in the money.

Hence:

$$A_t^{*F} = \nu_2 A_t^{*G}\mathcal{N}(d_{1t}) + \tilde{F}_t \mathcal{N}(-d_{2t}).$$

Note that this can be rewritten as

$$A_t^{*F} = \tilde{F}_t + \tilde{F}_t\left[\frac{\nu_2 A_t^{*G}}{\tilde{F}_t}\mathcal{N}(d_{1t}) - \mathcal{N}(d_{2t})\right],$$

where the term into brackets can be identified with the Black-Scholes price of a call expressed in the numeraire $\tilde{F}$:

$$\frac{\nu_2 A_t^{*G}}{\tilde{F}_t}\mathcal{N}(d_{1t}) - \mathcal{N}(d_{2t}) = \mathbb{E}_t^{\mathbb{Q}^F}\left[\left[\frac{\nu_2 A_T^{*G}}{F_T} - 1\right]^+\right].$$

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The expression for the diffusion term in the dynamics of the ratio $R^{*F} = A^{*F}/_{\tilde{F}}$ follows from the textbook expression for the delta of a call option in the Black-Scholes model. We obtain:

$$\frac{dR_t^{*F}}{R_t^{*F}} = (\cdots)dt + \nu_2 R_t^{*G}\mathcal{N}(d_{1t})\left[\underline{\sigma_t^{*G}} - \underline{\sigma_{Ft}}\right]' d\underline{z_t^F}.$$

But by Ito's lemma, this diffusion term can also be written as

$$R_t^{*F}\left[\underline{\sigma_t}\underline{w_t^{*F}} - \underline{\sigma_{Ft}}\right]' d\underline{z_t^F}.$$

Matching the two expressions, we get:

$$\underline{w_t^{*F}} = \underline{w_{FHP,t}} + \frac{\nu_2 A_t^{*G}\mathcal{N}(d_{1t})}{A_t^{*F}}\left[\underline{w_t^{*G}} - \underline{w_{FHP,t}}\right],$$

hence:

$$\underline{w_t^{*F}} = \underline{w_{FHP,t}} + \frac{A_t^{*F} - \tilde{F}_t\mathcal{N}(-d_{2t})}{A_t^{*F}}\left[\underline{w_t^{*G}} - \underline{w_{FHP,t}}\right].$$

If A_t^{*F} is close to $\tilde{F}_t$, then the call is deeply out of the money, which means that the current underlying price, $\frac{\nu_2 A_t^{*G}}{\tilde{F}_t}$, is small compared to the strike, 1. In this situation, d_{2t} goes to minus infinity, so that $\mathcal{N}(-d_{2t})$ converges to 1. Thus, the risk budget $[A_t^{*F} - \tilde{F}_t\mathcal{N}(-d_{2t})]$ shrinks to 0.

If A_t^{*F} grows to infinity, then the call is deeply in the money, so $\frac{\nu_2 A_t^{*G}}{\tilde{F}_t}$ is much larger than 1. Hence, d_{2t} diverges to plus infinity, and $\mathcal{N}(-d_{2t})$ shrinks to zero. Thus, the risk budget $[A_t^{*F} - \tilde{F}_t\mathcal{N}(-d_{2t})]$ coincides with A_t^{*F}.

6.3.3 Proofs of Proposition 13 and Corollary 3

Define:

$$X^{**} = \min\left[C_T, \max\left(F_T, \nu_3 A_T^{*G}\right)\right],$$

and let $f(\nu_3) = \mathbb{E}[M_T X^{**}]$. By the monotone convergence theorem, f is increasing and satisfies $f(+\infty) = \tilde{C}_0$.

Moreover:

$$f(\nu_2) \leq \mathbb{E}[M_T X^*] = A_0,$$

where $X^* = \max(F_T, \nu_2 A_T^{*G})$. Finally, f is continuous by the dominated convergence theorem. Because $A_0 < \tilde{C}_0$, the intermediate value theorem implies that there exists a value ν_3 in $[0, \infty[$ such that $f(\nu_3) = A_0$. The uniqueness of the solution when $A_0 > \tilde{F}_0$ is proven by the same arguments as in Appendix 6.3.2.

The proof of the optimality of X^{**} proceeds by the same concavity arguments as in Appendix 6.3.3. The derivation of the optimal strategy is also done by applying the same techniques. We do not repeat these calculations here. The reader can find a detailed proof in Deguest, Martellini and Milhau (2014).

Assume that $\nu_3 = \nu_2$. Then:

$$\mathbb{E}[M_T \min[C_T, X^*]] = A_0 = \mathbb{E}[M_T X^*],$$

which implies that, almost surely:

$$\min[C_T, X^*] = X^*,$$

hence that $X^* > C_T$ almost surely. By contraposition, if $X^* > C_T$ holds with positive probability, it must be the case that $\nu_3 > \nu_2$.

6.3.4 Proof of Proposition 14

In the presence of consumption, the budget constraint reads (see (36)):

$$\mathbb{E}[M_T A_T] = A_0 - \tilde{G}_0.$$

Suppose first that $A_0 > \tilde{G}_0$, and consider the following candidate optimal payoff for Program (18):

$$X^* = U'^{-1}(\eta_3 M_T),$$

where the constant η_3 is chosen so as to make the budget constraint hold. The

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unique possible choice is:

$$\eta_3 = \frac{(A_0 - \tilde{G}_0)^{-\gamma}}{\mathbb{E}\left[M_T^{1-\frac{1}{\gamma}}\right]^{-\gamma}}.$$

The same arguments as in Appendix 6.3.1 show that X^* is utility-maximising.

Note that by (38), X^* can be rewritten as:

$$X^* = \left(\frac{\eta_3}{\eta_0}\right)^{-\frac{1}{\gamma}} A_T^{*0} = \frac{A_0 - \tilde{G}_0}{A_0} A_T^{*0}.$$

This payoff can be replicated by taking a long position (of value $\tilde{G}_0$) in the coupon-paying bond, plus a long position (of value $[A_0 - \tilde{G}_0]$) in the strategy which is optimal for Program (17). Thus, X^* is the optimal terminal wealth, and the optimal wealth process is given by:

$$A_t^* = \mathbb{E}_t[M_T X^*] + \tilde{G}_t = \left(1 - \frac{\tilde{G}_0}{A_0}\right) A_t^{*0} + \tilde{G}_t.$$

The optimal weight vector is obtained by matching the diffusion terms in both sides of the equation:

$$A_t^* \underline{\sigma_t} \underline{w_t^*} = \left(1 - \frac{\tilde{G}_0}{A_0}\right) A_t^{*0} \underline{\sigma_t} \underline{w_t^{*0}} + \tilde{G}_t \underline{\sigma_t} \underline{w_{GHP,t}}.$$

Hence:

$$A_t^* \underline{w_t^*} = (A_t^{*c} - \tilde{G}_t) \underline{w_t^{*0}} + \tilde{G}_t \underline{w_{GHP,t}}.$$

6.4 Goals-Based Investing Strategies

6.4.1 Proof of Proposition 15

Consider a wealth-based goal with the horizons $T_1,\dots,T_p$ and the minimum wealth levels $G_{T_1},\dots, G_{T_p}$, and the strategy defined by (30). The notations K_{T_j} (option payoff) and $\tilde{K}_{T_j,T_{j+1}}$ (option price) are defined in Proposition 3. The weights (30) can be rewritten as:

$$\underline{w_t^{GBI,MH}} = m\left(1 - \frac{\tilde{G}_t}{A_t}\right)\underline{w_{PSP,t}} + \left[1 - m\left(1 - \frac{\tilde{G}_t}{A_t}\right)\right]\underline{w_{GHP,t}},$$

for $T_{j-1} < t \le T_j$ and $j = 1,\dots,p$.

We show by induction on j that $A_{T_j} \ge K_{T_j}$.

- The property is true for $j = 0$, since $A_0 \ge K_0$ and $K_0 = \tilde{K}_{0,T_1}$ by definition;

- Assume that the property is true at the rank j–1, where j is between 1 and p. We let

$$RB_t = A_t - \tilde{K}_{t,T_j}$$

denote the risk budget.

Let $A_{GHP,t}$ be the value of the GHP with an initial investment of $\tilde{G}_0$. We have, for $T_{j-1} < t \le T_j$ (see (6)):

$$A_{GHP,t} = \left[\prod_{k=1}^{j-1} \frac{\tilde{G}_{T_k}}{\tilde{G}_{T_k+}}\right] \times \tilde{K}_{t,T_j},$$

where the coefficient within the bracket is constant.

Let $A_{PSP,t}$ be the value of the PSP with an initial investment of A_0. By definition of the strategy, we have:

$$dA_t = m\frac{RB_t}{A_{PSP,t}} dA_{PSP,t} + \frac{A_t - mRB_t}{A_{GHP,t}} dA_{GHP,t},$$

hence:

$$dRB_t = m \times RB_t \frac{dA_{PSP,t}}{A_{PSP,t}} + (1-m) \times RB_t \frac{d\tilde{K}_{t,T_j}}{\tilde{K}_{t,T_j}}.$$

Let

$$df_t = \frac{m(m-1)}{2} \frac{d\langle A_{PSP,t}\rangle}{A_{PSP,t}^2} - \frac{m^2}{2} \frac{d\langle \tilde{K}_{t,T_j}\rangle}{\tilde{K}_{t,T_j}^2},$$

$$Z_t = A_{PSP,t}^m \tilde{K}_{t,T_j}^{1-m}.$$

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Applying Ito's lemma, we arrive, after a bit algebra, at:

$$RB_t = \frac{RB_{T_{j-1}}}{Z_{T_{j-1}}} \times Z_t \times \exp\left[\int_{T_{j-1}}^{t} df_s\right].$$

By assumption, $A_{T_{j-1}} \geq K_{T_{j-1}}$. Because $K_{T_{j-1}} = \max\left(F_{T_{j-1}}, \tilde{K}_{T_{j-1},T_j}\right)$, it follows that $A_{T_{j-1}} \geq \tilde{K}_{j,T_{j-1}}$, hence $RB_{T_{j-1}} \geq 0$. So RB_t is nonnegative too; in particular:

$$RB_{T_j-} = A_{T_j} - \tilde{K}_{T_j,T_j} = A_{T_j} - K_{T_j} \geq 0.$$

Hence, we have $A_{T_j} \geq K_{T_j}$ for all $j = 1,\ldots,p$. Since $K_{T_j} \geq F_{T_j}$, it follows that $A_{T_j} \geq F_{T_j}$.

6.4.2 Proof of Proposition 16

We recall that $\tilde{G}_t$ is the price of the bond whose cash flows match the consumption expenses, while $\hat{G}_t$ is the total return index, that is, the price of the bond with coupons re-invested. The dynamics of $\hat{G}$ is:

$$\frac{d\hat{G}_t}{\hat{G}_t} = \frac{d\tilde{G}_t}{\tilde{G}_t} + \sum_{j=1}^{p} \frac{c_{T_j}}{\tilde{G}_t} dJ_{jt}.$$

The dynamics of the risk budget, $(A_t - \tilde{G}_t)$, reads:

$$d(A_t - \tilde{G}_t) = A_t\left[\left(r_t + \underline{w}_t'\underline{\mu}_t\right)dt + \underline{w}_t'\underline{\sigma}_t' d\underline{z}_t\right] - \sum_{j=1}^{p} c_{T_j} dJ_{jt} - \left[\tilde{G}_t \frac{d\hat{G}_t}{\hat{G}_t} - \sum_{j=1}^{p} c_{T_j} dJ_{jt}\right],$$

hence:

$$d(A_t - \tilde{G}_t) = A_t\left[\left(r_t + \underline{w}_t'\underline{\mu}_t\right)dt + \underline{w}_t'\underline{\sigma}_t' d\underline{z}_t\right] - \tilde{G}_t \frac{d\hat{G}_t}{\hat{G}_t}.$$

The dynamics of the total return index can also be written as:

$$\frac{d\hat{G}_t}{\hat{G}_t} = \left(r_t + \underline{w}_{GHP,t}'\underline{\mu}_t\right)dt + \underline{w}_{GHP,t}'\underline{\sigma}_t' d\underline{z}_t.$$

Hence, with the Strategy (31), the risk budget evolves as:

$$d(A_t - \tilde{G}_t) = (A_t - \tilde{G}_t)\left\{r_t dt + \left[m\underline{w}_{PSP,t} + (1-m)\underline{w}_{PSP,t}\right]'\left[\underline{\mu}_t dt + \underline{\sigma}_t' d\underline{z}_t\right]\right\}.$$

Moreover, we have, by Ito's lemma:

$$\frac{d\left(A_{PSP,t}^{m}\hat{G}_t^{1-m}\right)}{A_{PSP,t}^{m}\hat{G}_t^{1-m}} = m\frac{dA_{PSP,t}}{A_{PSP,t}} + (1-m)\frac{d\hat{G}_t}{\hat{G}_t} + \frac{1}{2}m(m-1)\left\|\underline{\sigma}_{PSP,t} - \underline{\sigma}_{Gt}\right\|^2 dt,$$

with $\underline{\sigma}_{PSP,t} = \underline{\sigma}_t\, \underline{w}_{PSP,t}$ and $\underline{\sigma}_{Gt} = \underline{\sigma}_t \underline{w}_{Gt}$.

Hence:

$$A_t - \tilde{G}_t = \frac{A_0 - \tilde{G}_0}{A_{PSP,0}^{m}\hat{G}_0^{1-m}} \times A_{PSP,t}^{m}\hat{G}_t^{1-m} \times \exp\left[-\frac{1}{2}m(m-1)\int_0^t \left\|\underline{\sigma}_{PSP,s} - \underline{\sigma}_{Gs}\right\|^2 ds\right].$$

If $A_0 \geq \tilde{G}_0$, we thus have that $A_t \geq \tilde{G}_t$ for all t in $[0,T]$. In particular, wealth remains nonnegative, so the goal is affordable.

6.5 Monte-Carlo Simulation Model

This section describes the dynamics of the various stochastic processes that enter the Monte-Carlo simulation model, as well as the base case values parameter values. We also provide a detailed description of the rebalancing rules applied in our simulations, as well as the formal definition of the various success indicators for the goals and the algorithm for computing taxes.

6.5.1 Stochastic Processes and Parameter Values

6.5.1.1 Asset Prices

We model asset prices as Geometric Brownian motions. This dynamics is a special case of (1) where the drift and the volatility are constant. For instance, the stock index (S) evolves as:

$$\frac{dS_t}{S_t} = e_S dt + \underline{\sigma}_S' d\underline{z}_t,$$

where e_S is the expected arithmetic return and $\underline{\sigma}_S$ is the volatility vector. The dimension

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d of the Brownian motion is taken equal to the number of stochastic processes to simulate. We postulate similar dynamics for the bond index (*B*), the illiquid stock (*X*), the real estate (*Y*) and the alternative investment (*Z*).

We fix the expected return and the volatility of the stock and bond indices after the values estimated by Dimson, Marsh and Staunton (2002) for the US market over the 1900-2000 period (see p. 306 of their book). Such a long dataset is not available for real estate and alternative investment, so we simply set the risk and return parameters to "reasonable" values. For the illiquid stock, we take the same expected return as for the broad index, but we set volatility twice as high, in order to reflect the amount of idiosyncratic risk, which is larger in an individual stock than in an index (even if the index is not necessarily well-diversified in the sense of Modern Portfolio Theory). The risk and return parameters for the various asset classes are summarised in Table 4.

To compute the maximum Sharpe allocation to the stock and the bond indices, we need the expected excess returns of these assets, μ_S and μ_B. Since we assume constant expected returns and a stochastic interest rate (see Section 6.5.1.2 below), the actual expected excess return is stochastic:

$$\mu_{St} = e_S - r_t.$$

Nevertheless, we take constant values to compute the weights of the MSR portfolio. This is done primarily for simplicity, but it can also be noted that this apparent inconsistency reflects the situation of an investor who does not know the true parameter values. We fix the expected excess returns after the long-term values reported in p. 306 of Dimson, Marsh and Staunton (2002).

6.5.1.2 Term Structure

The nominal short-term rate follows the Vasicek model (Vasicek (1977)):

$$dr_t = a(\bar{r} - r_t)dt + \underline{\sigma}_r' d\underline{z}_t,$$

where a is the speed of mean reversion, *b* is the long-term mean and $\underline{\sigma}_r$ is the volatility vector. We assume a constant price of interest rate risk, λ_r. The price at date *t* of a zero-coupon bond paying \$1 on date *T* is then:

$$b_{t,T} = \exp[-D(T-t)r_t + E_B(T-t)],$$

with:

$$D(s) = \frac{1-e^{-as}}{a},$$

$$E_B(s) = \left(\bar{r} - \frac{\sigma_r \lambda_r}{a}\right)[D(s) - s] + \frac{\sigma_r^2}{2a^2}\left[s - 2D(s) + \frac{1-e^{-2as}}{2a}\right].$$

This expression for the zero-coupon price is used to discount cash flows that are fixed in nominal terms.

Note that in this model, the Sharpe ratio of a zero-coupon is $-\lambda_r$. For the term premium to be positive, it is necessary to take a negative value for λ_r. The parameters of the model are borrowed from Martellini, Milhau and Tarelli (2013), who estimate them from monthly observations of US sovereign zero-coupon yields for the period August 1971 – August 2012.

To discount cash flows that are fixed in real terms, we use a stochastic model for inflation. The price index is modelled as a

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Geometric Brownian motion:

$$\frac{d\Phi_t}{\Phi_t} = e_\Phi dt + \underline{\sigma}'_\Phi d\underline{z}_t,$$

where e_Φ represents expected inflation. The price of an inflation-indexed zero-coupon which pays Φ_T/Φ_0 on date T is:

$$I_{t,T} = \frac{\Phi_t}{\Phi_0}\exp[-D(T-t)r_t + E_I(T-t)],$$

with:

$$E_I(s) = E_B(s) + (\pi - \sigma_\Phi\lambda_\Phi)s + \frac{1}{2}\sigma_\Phi^2 s - \frac{\sigma_r\sigma_\Phi\rho_{r\Phi}}{a}[s - D(s)].$$

We set the expected inflation and the volatility of realised inflation to reasonable values, and we take the price of inflation risk, λ_Φ, to be zero. In theory, this parameter could be estimated from the real yield curve, but the estimation will be largely imprecise because only a relatively short dataset is available.[41] That is why we set λ_Φ to a neutral value. Note that real yields are increasing in λ_Φ: a higher value would make real yields lower, and even possibly negative.

41 - See Gürkaynak, Sack and Wright (2010) for a construction of the zero-coupon yield curve starting from 1999.

So as to avoid negative one-year real rates in the simulations for Case 1 (Section 4.1), we impose a floor equal to $E_I(1)/D(1)$ to the nominal short-term rate in the simulations. This lower bound is equal to 2.20% given our parameter values (Table 4 below), so the requirement of a nonnegative real rate is stronger than the requirement of a nonnegative nominal short-term rate. In the other two case studies (Sections 4.2 and 4.3), real interest rates are not used, so that we impose a floor equal to zero to the nominal short-term rate.

6.5.1.3 Correlations

We set the correlations between the stock, the bond and the real estate class to reasonable values. We fix the correlation between the stock index and the illiquid stock by imposing that the beta of the latter stock with respect to the index be equal to 1. This condition implies that the correlation is:

$$\rho_{SX} = \frac{\sigma_S}{\sigma_X}\times\beta_{S/X} = \frac{\sigma_S}{\sigma_X} = 0.5.$$

The correlations of the illiquid stock with the other processes are set to the same values as the correlations of the stock index.

The correlations of the various asset classes with the price index are set to reasonable values. In order to fix the correlations between the processes and the short-term rate, we use the approximation of a bond return as the negative of duration times the interest rate change. Considering a roll-over of bonds with a "short" maturity h (h must be short for the bond yield to be close to the short-term rate), we obtain the following approximation to the realised return:

$$R_{t,t+h} \approx -Dur\times(r_{t+h} - r_t).$$

Hence, the correlation between a given stochastic process and changes in the short-term rate is close to the negative of the correlation between the process and the returns on a roll-over of short bonds. Thus, we start by estimating realistic values for the correlations between the various processes and a roll-over and we take the negatives as estimates of the correlations with the short-term rate. The complete list of correlation values is given in Table 4.

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Table 4: Base case parameter values.

(i) Univariate parameters.

Asset expected returns and volatilities.	
Parameter.	Base case value.
e_S	0.12
e_B	0.051
e_X	0.012
e_Y	0.062
e_Z	0.012
μ_S	0.0782
μ_B	0.0108
σ_S	0.199
σ_B	0.083
σ_X	0.398
σ_Y	0.141
σ_Z	0.398
Continuous-time processes for short-term rate and price index.	
Parameter.	Base case value.
a	0.0668
$\bar{r}$	0.0353
σ_r	0.0168
λ_r	–0.3340
π	0.025
σ_Φ	0.0134
λ_Φ	0

(ii) Correlations.

	Stock	Bond	Illiquid stock	Real estate	Alternative investment	Short-term rate	Price index
Stock	1						
Bond	0.237	1					
Illiquid stock	0.500	0.237	1				
Real estate	0.567	0.273	0.567	1			
Alternative investment	0.132	0.226	0.132	0.513	1		
Short-term rate	0.014	–0.044	0.014	0.094	–0.132	1	
Price index	–0.09	–0.21	–0.09	–0.04	0.05	–0.44	1

This table summarises the base case parameter values assumed for the simulation of the various stochastic processes introduced in Section 4. The stock and bond expected excess returns (μ_S and μ_B) are not used for simulation purposes but to compute the weights of the MSR portfolio.

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6.5.2. Constant-Annuity Mortgage Amortisation

In this appendix, we derive the expression for the annuity in a mortgage with constant annuities and constant rate. We take the following notations:

Notation	Meaning
L_t	*Capital due at date t before annual payment.*
L	*Principal of loan.*
ℓ	*Constant annuity.*
r	*Borrowing rate.*
T	*Initial maturity of loan.*
k_t	*Capital repaid on date t.*
i_t	*Interest paid on date t.*

We assume that the annuities are paid on dates $t = 1, \ldots, T$. It follows from the above definitions that, for $t = 1, \ldots, T$:

$$i_t = rL_t,$$
$$\ell = k_t + i_t,$$

and that for $t = 1, \ldots, T-1$:

$$k_t = L_t - L_{t+1}.$$

For $t = T$, we have $k_T = L_T$, a condition which means that all the principal is redeemed after the last payment has taken place.

Hence, for $t = 1, \ldots, T-1$:

$$\ell = (1+r)L_t - L_{t+1}.$$

Dividing both sides by $(1+r)^{t+1}$, we obtain:

$$\frac{\ell}{(1+r)^{t+1}} = \frac{L_t}{(1+r)^t} - \frac{L_{t+1}}{(1+r)^{t+1}}.$$

Summing up from $t = 1$ to $t = T$, we get:

$$\sum_{t=1}^{T} \frac{\ell}{(1+r)^{t+1}} = \frac{L_1}{(1+r)^1} - 0.$$

The capital due before the last payment equals the loan principal value, so that $L_1 = L$. Hence, the present value of all annuities discounted at the borrowing rate must equal the principal:

$$L = \sum_{t=1}^{T} \frac{\ell}{(1+r)^t}.$$

Computing the geometric sum in the second term, we obtain the value of the constant annuity:

$$\ell = \frac{r}{1-(1+r)^{-T}} \times L.$$

6.6 Budget Equations and Definitions of Analytics

This technical appendix contains a description of the generic budget equations used in the simulations (for all strategies) and the detailed expressions for the weights of the GBI strategies. We next give the definitions of the success indicators that are computed for the various strategies.

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6.6.1 Notations

We take the following notations:

Notation	Meaning
Wealth processes	
$A_{btax,t}$	*Before-tax wealth at date t (i.e. before taxes and non-portfolio gains and payments).*
$A_{atax,t}$	*After-tax wealth at date t (i.e. after taxes and before non-portfolio gains and payments).*
$A_{anpf,t}$	*Wealth after non-portfolio flows (gains and payments) at date t.*
Non-portfolio inflows/outflows	
Θ_t	*Amount of taxes due at date t.*
l_t	*"Constrained consumption" stream (e.g. mortgage), net of income.*
c_t	*"Flexible consumption" stream, i.e. any consumption expenditure at date t not already included in constrained consumption.*
Asset prices and dividends	
$\underline{S}_{bdiv,t}$	*Vector of asset prices at date t, before dividend and coupon payments.*
$\underline{S}_{adiv,t}$	*Vector of asset prices at date t, after dividend and coupon payments.*
$\underline{D}_t$	*Vector of dividend and coupon payments at date t.*
Portfolio composition	
$\underline{N}_{bdiv,t}$	*Vector of numbers of shares held at date t before dividend and coupon payments, taxes and any other non-portfolio inflow or outflows.*
$\underline{N}_{btax,t}$	*Vector of numbers of shares held at date t after dividend and coupon payments, before taxes and non-portfolio gains and payments.*
$\underline{N}_{atax,t}$	*Vector of numbers of shares held at date t after dividend and coupon payments and taxes, and before non-portfolio flows (gains and payments).*
$\underline{N}_{anpf,t}$	*Vector of numbers of shares held at date t after dividend and coupon payments, taxes and non-portfolio gains and payments, and before rebalancing.*
$\underline{N}_{areb,t}$	*Vector of numbers of shares held at date t after rebalancing.*
$\underline{q}$	*Vector of sums invested in constituents (same subscripts as $\underline{N}$).*
$\underline{w}$	*Vector of constituent weights (same subscripts as $\underline{N}$).*

6.6.2. Generic Budget Equations

In this section, we write the budget equations, which apply to all case studies and strategies. The timing of payments on a given date t is as follows:

1. Get dividends and coupons, and re-invest them in constituents;
2. Pay taxes;
3. Make all other payments (mortgage, consumption) and cash in non-portfolio income;
4. Rebalance.

<u>1. Dividend and coupon payments</u>

These payments cause no discontinuity to wealth, but they imply a change in the numbers of shares held in the portfolio.

Since the dividend paid by each constituent is re-invested in the constituent itself, the number of shares held after the dividend is paid is:

$$[\underline{N}_{btax,t}]_i = [\underline{N}_{bdiv,t}]_i \times \left(1 + \frac{[\underline{D}_t]_i}{[\underline{S}_{adiv,t}]_i}\right).$$

Note that this implies no change in portfolio weights. Indeed, the weights before and

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after the dividend payments are defined as:

$$\underline{w}_{bdiv,t} = \frac{\underline{N}_{bdiv,t} \odot \underline{S}_{bdiv,t}}{A_{btax,t}},$$

$$\underline{w}_{btax,t} = \frac{\underline{N}_{btax,t} \odot \underline{S}_{adiv,t}}{A_{btax,t}},$$

and a straightforward computation shows that:

$$\underline{w}_{btax,t} = \underline{w}_{bdiv,t}.$$

2. Tax payments

These payments cause a downwards jump in wealth:

$$A_{atax,t} = A_{btax,t} - \Theta_t.$$

The number of shares in each asset is recalculated in such a way that the relative weights after taxes have been paid are the same as before (that is, the tax payment is "evenly widespread" across the constituents):

$$[\underline{N}_{atax,t}]_i = [\underline{N}_{btax,t}]_i \times \frac{A_{atax,t}}{A_{btax,t}}.$$

3. Other non-portfolio flows (gains and payments)

The effect of these payments is similar to that of taxes (except that the net effect of consumption and income on wealth may be positive):

$$A_{anpf,t} = A_{atax,t} - l_t - c_t,$$

$$[\underline{N}_{anpf,t}]_i = [\underline{N}_{atax,t}]_i \times \frac{A_{anfs,t}}{A_{atax,t}}.$$

(We recall that l_t is net of income payments.)

4. Rebalancing

This causes no discontinuity to wealth, but of course, the numbers of shares can be modified by this operation. By convention, if no rebalancing takes place on date t, we let:

$$\underline{N}_{areb,t} = \underline{N}_{anpf,t}.$$

The evolution of wealth and portfolio composition between dates t and $t + h$ is governed by the following equation:

$$A_{btax,t+h} = \underline{N}'_{areb,t}\underline{S}_{bdiv,t+h},$$

$$\underline{N}_{bdiv,t+h} = \underline{N}_{areb,t}.$$

6.6.3 Dollar Allocations and Risk Budgets for GBI Strategies

In this appendix, we give the detailed expressions for the allocations to the locally risky assets by the GBI strategies described in Section 3.3.2. These expressions differ from the theoretical equations written in Section 3.3.2 through the imposition of no short-sales constraints in the building blocks.

6.6.3.1 GBI Strategy Securing a Single Essential Goal

The essential goal can be wealth-based with a single horizon or with multiple horizons, or consumption-based. The weights of the corresponding GBI strategies without short-sales constraints are given in Equations (29), (30) and (31). Ruling out short positions, we obtain the following expressions:

GBI Strategy Securing a Single Essential Goal

Equation	Description
$\underline{q}_t^{GBI} = q_{PSP,t}\underline{w}_{PSP,t} + q_{FHP,t}\underline{w}_{FHP,t}$	Vector of dollar allocations to locally risky assets with GBI strategy.
$q_{PSP,t} = \min[A_{liq,t}, m \times RB_t]$	Dollar allocation to PSP.
$q_{FHP,t} = A_{liq,t} - q_{PSP,t}$	Dollar allocation to FHP (portfolio super-replicating the goal).
$RB_t = \max[0, A_{G,t} - F_t]$	Risk budget.
$A_{G,t}$	Reference wealth for the goal.
$A_{liq,t}$	Liquid wealth.
F_t	Floor value, equal to goal present value for exogenous goals and drawdown floor for drawdown goal.

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The reference wealth depends on the goal:
- In Case Study 1, the reference wealth for Essential Goal 1 is the sum of liquid wealth and aspirational wealth, which coincides with liquid wealth when aspirational assets are liquidated;
- In Case Study 1, the reference wealth for Essential Goal 2 is the liquid wealth;
- In Case Studies 2 and 3, the reference wealth for the consumption-based goals is the liquid wealth.

6.6.3.2. GBI Strategy Securing Two Essential Goals

The two essential goals can be wealth-based or consumption-based. The weights of the GBI strategy protecting the two goals without short-sales constraints are given in Equation (32). The following table describes the strategy with short-sales constraints applied.

6.6.3.3. GBI Strategy with a Cap

The following table adapts the definition of the GBI strategy with a cap (see (33)) to the case where short-sales constraints are imposed.

6.6.4 Success Indicators for Goals

In this section, we define the "success indicators", which measure the degree of achievement of the goals. These definitions apply to affordable and non-affordable goals, whichever portfolio strategy is implemented.

GBI Strategy Securing Two Essential Goals

Equation	Description
$\underline{q}_t^{GBI} = q_{PSP,t}\underline{w}_{PSP,t} + q_{FHP1,t}\underline{w}_{FHP1,t} + q_{FHP2,t}\underline{w}_{FHP2,t}$	Vector of dollar allocations to locally risky assets with GBI strategy.
$q_{PSP,t} = \min[A_{liq,t}, m \times RB_t]$	Dollar allocation to PSP.
$q_{FHP1,t} = (A_{liq,t} - q_{PSP,t}) \times \mathbb{I}_{\{RB_{1t} < RB_{2t}\}}$	Dollar allocation to FHP1 (portfolio super-replicating the first goal).
$q_{FHP2,t} = (A_{liq,t} - q_{PSP,t}) \times \mathbb{I}_{\{RB_{1t} \geq RB_{2t}\}}$	Dollar allocation to FHP2 (portfolio super-replicating the second goal).
$RB_t = \max[RB_{1t}, RB_{2t}]$	Risk budget.
$RB_{it} = \max[0, A_{Gi,t} - \tilde{F}_{it}]$	Risk budget associated with i^{th} goal ($i = 1,2$).
$A_{Gi,t}$	Reference wealth for i^{th} goal ($i = 1,2$).
$A_{liq,t}$	Liquid wealth.
$\tilde{F}_{it}$	Floor value associated with i^{th} goal ($i = 1,2$).

GBI Strategy with a Cap

Equation	Description
$\underline{q}_t^{GBI} = q_{PSP,t}\underline{w}_{PSP,t} + q_{Ft}\underline{w}_{FHP,t} + q_{Ct}\underline{w}_{CHP,t}$	Vector of dollar allocations to locally risky assets with GBI strategy.
$q_{PSP,t} = \min[A_{liq,t}, m \times RB_t]$	Dollar allocation to PSP.
$RB_t = RB_{Ft} \times \mathbb{I}_{\{A_t \leq \xi_t\}} + RB_{Ct} \times \mathbb{I}_{\{A_t > \xi_t\}}$	Risk budget.
$\xi_t = \frac{\tilde{F}_t + \tilde{C}_t}{2}$	Threshold.
$RB_{Ft} = \max[0, A_t - \tilde{F}_t]$	Risk budget associated with floor.
$RB_{Ct} = \max[0, \tilde{C}_t - A_t]$	Risk budget associated with cap.
$x_{Ft} = (1 - x_{PSP,t}) \times \mathbb{I}_{\{A_t \leq \xi_t\}}$	Dollar allocation to FHP.
$x_{Ct} = (1 - x_{PSP,t}) \times \mathbb{I}_{\{A_t > \xi_t\}}$	Dollar allocation to CHP.
A_t	Reference wealth for floor and cap.
$A_{liq,t}$	Liquid wealth.
$\tilde{F}_t$	Floor value.
$\tilde{C}_t$	Cap value.

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6.6.3.4. GBI Strategy with a Single Floor and a Short Position (Case Study 1)

Equation	Description
$\underline{q}_t^{GBI} = q_{PSP,t}\underline{w}_{PSP,t} + q_{Ft}\underline{w}_{FHP,t} + q_{short,t}\underline{w}_{short,t}$	Vector of dollar allocations to locally risky assets with GBI strategy.
$\underline{w}_{short,t}$	Portfolio invested at 100% in the shortable asset.
$q_{PSP,t} = \max[0, \min(A_{liq,t}, m\times RB_t - A_{asp,t})] \times \mathbb{I}_{\{m\times RB_t \geq A_{asp,t}\}}$	Dollar allocation to PSP.
$q_{short,t} = -\min[A_{X,t}, A_{asp,t} - m\times RB_t] \times \mathbb{I}_{\{A_{asp,t} > m\times RB_t\}}$	Dollar allocation to shortable asset (non-positive by definition).
$q_{Ft} = A_{liq,t} - q_{PSP,t} - q_{short,t}$	Dollar allocation to FHP.
RB_t	Risk budget.
$A_{x,t}$	Value of position in illiquid stock within the aspirational bucket.
$A_{asp,t}$	Aspirational wealth.
$A_{liq,t}$	Liquid wealth.

6.6.4.1 Wealth-Based Goals with a Single Horizon

We use the framework of Section 1.1, where the wealth-based goal is represented by a payoff G_T on date T. The success indicators must quantify the likelihood of reaching the goal, and the size of deviations from the goal in case it is mixed. For the likelihood, it is natural to look at the success probability, defined as the probability that $A_T \geq G_T$:

$$sp = \mathbb{P}(A_T \geq G_T).$$

The size of deviations from the goal can be measured as the (relative) loss:

$$Loss_T = \left[1 - \frac{A_T}{G_T}\right]^+,$$

which is zero if the goal is reached, and positive otherwise, but remains less than 100% as long as wealth remains nonnegative. The expected relative shortfall is the expectation of $Loss_T$ conditioned on the event of a loss:

$$es = \mathbb{E}[Loss_T | Loss_T > 0].$$

By Bayes' formula, we have (I denoting an indicator function):

$$es = \frac{\mathbb{E}[Loss_T \times \mathbb{I}_{\{Loss_T > 0\}}]}{\mathbb{P}(Loss_T > 0)}.$$

Because the loss is always nonnegative, the numerator equals the expected loss. Moreover, the denominator is $\mathbb{I}$ minus the success probability. Hence:

$$es = \frac{\mathbb{E}[Loss_T]}{1 - sp}$$

By convention, the expected shortfall is set to 0 if the success probability is 1.

One may also look at the worst case, that is, the worst relative loss. Mathematically, the worst shortfall is defined as the "essential supremum" of the loss. For any random variable Y, the essential supremum is defined as the infimum of the values y such that the probability that Y exceeds y is zero:

$$\text{ess sup}\, Y = \inf\{y; \mathbb{P}(Y > y) = 0\}.$$

(If Y has always a non-zero probability of exceeding any arbitrarily high threshold, the essential supremum is infinity.) Hence, the worst shortfall is defined as:

$$ws = \text{ess sup}\, Loss_T.$$

The following definition summarises the three success indicators for wealth-based goals.

Definition 11 (Success Indicators for a Wealth-Based Goal with a Single Horizon).
Consider a wealth-based goal represented by the payoff G_T. The success indicators are defined as:

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- *The success probability:* $sp = \mathbb{P}(A_T \geq G_T)$;
- *The expected shortfall:* $es = \frac{\mathbb{E}[Loss_T]}{1-sp}$;
- *The worst shortfall:* $ws = \text{ess sup } Loss_T$.

According to this terminology, a replicable goal is affordable if there exists a portfolio strategy $\underline{w}$ such that the success probability starting from the investor's initial wealth equals 1.

6.6.4.2 Wealth-Based Goals with Multiple Horizons

A wealth-based goal with multiple horizons $T_1,\ldots,T_p$ is represented by the minimum wealth levels $(G_{T_1},\ldots, G_{T_p})$. The definitions of the success indicators are similar to those for a single horizon, but an adaptation is needed to account for the fact that this goal is represented by multiple floors. The success probability is naturally defined as the probability that wealth remains above the floor on each date:

$$sp = \mathbb{P}\left(A_{T_j} \geq G_{T_j}\, ;\, \forall j = 1, \ldots, p\right).$$

A deviation from the goal can potentially arise on each of the dates $T_1,\ldots,T_p$. We define the (relative) loss on date T_j as the gap between wealth and the goal value:

$$Loss_{T_j} = \left[1 - \frac{A_{T_j}}{G_{T_j}}\right]^+.$$

The quantity of interest is the maximum of losses over all horizons:

$$\max Loss = \max\left[Loss_{T_1}, \ldots, Loss_{T_p}\right].$$

The expected maximum shortfall is then defined as the expectation of this maximum, conditioned on the event that at least one of the losses is positive (which is equivalent to saying that the maximum is positive):

$$ems = \mathbb{E}[\max Loss \,|\max Loss > 0].$$

Again, an application of Bayes' formula shows that this indicator can be expressed as a function of the expected maximum loss and the success probability:

$$ems = \frac{\mathbb{E}[\max Loss]}{1 - sp}.$$

Finally, the worst maximum shortfall is the essential supremum of maximum losses: thus, it represents the largest deviation from the consumption objective, across all dates and states of the world:

$$wms = \text{ ess sup} \max Loss\,.$$

For the drawdown goal, we will replace the expected and the worst maximum shortfalls by the expected and the worst maximum drawdowns. The drawdown is defined in terms of the current wealth (A_{T_j}) and the maximum-to-date of wealth $(\overline{A}_{T_j})$:

$$DD_{T_j} = \frac{\overline{A}_{T_j} - A_{T_j}}{\overline{A}_{T_j}},$$

with $\overline{A}_{T_j} = \max\left[A_{T_1}, \ldots, A_{T_{j-1}}\right]$. By definition, the drawdown is nonnegative, and is less than 1 (as long as wealth remains positive).

The following definition collects the mathematical expressions for the success indicators.

Definition 12 (Success Indicators for a Wealth-Based Goal with Multiple Horizons).

Consider a wealth-based goal represented by the minimum wealth levels $(G_{T_1},\ldots, G_{T_p})$. The success indicators are defined as:

- *The success probability:*
  $$sp = \mathbb{P}\left(A_{T_j} \geq G_{T_j}\, ;\, \forall j = 1, \ldots, p\right)$$
- *The expected maximum shortfall:*
  $$ems = \frac{\mathbb{E}[\max Loss]}{1-sp};$$
  or the expected maximum drawdown:
  $$emd = \mathbb{E}\left[\max DD_{T_p}\right];$$
- The worst maximum shortfall: *wms* = ess sup max *Loss*; or the worst maximum drawdown: *wmd* = ess sup max *Loss* DD_{Tp}.

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6.6.4.3 Consumption-Based Goals

A consumption-based goal is represented by a consumption stream $(c_{T_1},\dots, c_{T_p})$. The success probability is defined as the probability that on each consumption date, the available wealth covers the consumption payment:

$$sp = \mathbb{P}\left(A_{T_j-} \geq c_{T_j}\ ;\ \forall j = 1,\dots,p\right).$$

This is equivalent to computing the probability that the wealth after consumption, i.e. $A_{T_j} = A_{T_j} - c_{T_j}$, is nonnegative. The loss with respect to the goal is defined as the difference between the level of effective consumption, $c_{eff,Tj}$, and the target consumption:

$$Loss_{T_j} = \left[1 - \frac{c_{eff,T_j}}{c_{T_j}}\right]^+.$$

The expected maximum shortfall and the worst maximum shortfall are then defined as for a wealth-based goal with multiple horizons.

Definition 13 (Success Indicators for a Consumption-Based Goal).

Consider a consumption-based goal represented by the payments $(c_{T_1},\dots, c_{T_p})$. The success indicators are defined as:

- *The success probability:*

$$sp = \mathbb{P}\left(A_{T_j-} \geq c_{T_j}\ ;\ \forall j = 1,\dots,p\right);$$

- *The expected maximum shortfall:*

$$ems = \frac{\mathbb{E}[\max Loss]}{1-sp};$$

- *The worst maximum shortfall:*

$$wms = ess\ \sup \max Loss.$$

6.6.5 Taxes

The sequence of operations on a given month t is as follows:

- If t is the end of the year, pay taxes first: included are transactions and roll-over operations performed since the last tax payment date (included) and before the current date (excluded);
- Then, rebalance the portfolio: dynamic GBI weights are based on the after-tax wealth;
- Then, compute the taxes generated by operations of date t: included are the selling operations, and, if t is the end of the year, the roll-over operations.

6.6.5.1 Taxes on Selling Operations

We now give the detailed expressions for the tax generated by the selling operations on the constituents of a portfolio. We take the notations of Table 5.

Table 5: Symbols for tax computation.

Symbol	Definition
N	Number of constituents.
$S_{i,t}$	Asset price.
$N_{i,t}$	Number of shares of asset *i* held on date *t*.
h	Rebalancing period, expressed as a fraction of year.
ζ	Tax rate (20% in base case).
$\theta_{i,t}$	Amount of taxes generated by the selling operations in asset *i* on date *t*.
$\overline{\theta}_{i,t}$	Amount of taxes generated by the roll-over operation for asset *i* on date *t*.
u_{ilm}	Number of shares of asset *i* purchased at date *mh* and sold at rebalancing of date *lh*.
v_{ilm}	Number of shares of asset *i* purchased at date *mh* and sold at roll-over of date *lh*.
Θ_n	Amount of taxes to be paid at the end of year *n*.
$A_{liq,t}$	Liquid wealth of date *t*.

This table contains the definitions of the symbols that appear in the tax formulas.

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For l =1, 2, 3,..., the amount of taxes generated by the selling operations in asset i on date lh is:

$$\theta_{i,lh} = \zeta \times \left[\sum_{m=n_l}^{l-1} u_{ilm} \times (S_{i,lh} - S_{i,mh}) + \sum_{m=0}^{n_l-1} u_{ilm} \times (S_{i,lh} - S_{i,mh})^+ \right].$$

with:

$$n_l = \frac{\lfloor lh \rfloor}{h}$$

being the previous year end. Observe that $\theta_{i,lh}$ consists of two contributions: one from the transactions within the year, for which compensations between gains and losses are possible, and the other one from the older transactions, for which compensations are not possible. The positive part is taken only for price changes within the year.

According to the LIFO rule, the number of shares purchased at date mh and sold at date lh is recursively computed as:

$$u_{ilm} = \min\left[\left\{ (N_{i,(l-1)h} - N_{i,lh})^+ - \sum_{j=m+1}^{l-1} u_{ilj} \right\}^+, (N_{i,mh} - N_{i,(m-1)h})^+ \right]$$

for $1 \le m \le l-1$;

$$u_{il0} = \min\left[\left\{ (N_{i,(l-1)h} - N_{i,lh})^+ - \sum_{j=1}^{l-1} u_{ilj} \right\}^+, N_{i,0} \right]$$

for $m = 0$ or $l = 1$.

It can be mathematically checked that the sum of these numbers is equal to the number of shares to sell on date lh, as it should:

$$\sum_{m=0}^{l-1} u_{ilm} = (N_{i,(l-1)h} - N_{i,lh})^+, \text{for } l \ge 1.$$

6.6.5.2 Taxes on a Roll-Over of Bonds

Consider now an asset which is an annual roll-over of bonds (as in Case 1). The taxation mechanism is similar, with the following modifications:

- On a given date, only the selling operations done within the year are taken into account (with compensations allowed);
- At the end of each year, we force the liquidation of the position in bonds, and taxes are paid on this operation.

The amount of taxes generated by the liquidation of the portfolio at the end of year n=1,2,3,... is:

$$\bar{\theta}_{i,n} = \zeta \times \sum_{m=l-\frac{1}{h}}^{l} v_{ilm} \times (S_{i,n} - S_{i,mh}),$$

with: $$l = \frac{n}{h},$$

and v_{ilm} is recursively computed as:

$$v_{ill} = \min\left[N_{i,n}, (N_{i,n} - N_{i,n-h})^+ \right] = (N_{i,n} - N_{i,n-h})^+$$

for $m = l$

$$v_{ilm} = \min\left[\left\{ N_{i,n} - \sum_{j=m+1}^{l} v_{ilj} \right\}^+, (N_{i,mh} - N_{i,(m-1)h})^+ \right]$$

for $l + 1 - \frac{1}{h} \le m \le l - 1$,

$$v_{il,l-\frac{1}{h}} = \min\left[\left\{ N_{i,n} - \sum_{j=m+1}^{l} v_{ilj} \right\}^+, N_{i,n-1} \right]$$

for $m = l - \frac{1}{h}$.

6.6.5.3 Annual Tax Payment

Compensations across assets are possible, and the tax payment is floored at zero and capped at the liquid wealth. Thus, the

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amount of taxes to be paid at the end of year n is:

$$\Theta_n = \min\left[A_{liq,t'}, \left\{\sum_{i=1}^{N}\sum_{l=0}^{\frac{1}{h}-1}\theta_{i,n-lh} + \bar{\theta}_{i,n}\right\}^{+}\right].$$

7. Figures and Tables

7. Figures and Tables

7.1. Case Study 1

Table 6: Symbols for stochastic processes used in Case 1.

Symbol	Definition
t	Date.
S_t	Equity index value.
B_t	Bond index value.
S_{0t}	Cash value.
Y_t	Real estate value (residence or investment real estate).
X_t	Illiquid stock value (concentrated stock price and executive stock option).
Φ_t	Price index.
$A_{per,t}$	Personal wealth.
$A_{mkt,t}$	Market wealth.
$A_{asp,t}$	Aspirational wealth.
$A_{liq,t}$	Liquid wealth.
$\underline{w}_{bkt,str,t}$	Weight vector within bucket *bkt* on date *t* for strategy *str*. *bkt* = *per, mkt, asp* *str* = *CUR, SF1, GBI1, GBI2*
$\underline{w}_{liq,strat,t}$	Weight vector within liquid wealth on date *t* for strategy *str*.

This table contains the definitions of the mathematical symbols used in Case 1 (these symbols have the same meanings in Cases 2 and 3). The index t refers to the value of a process at date t.

Table 7: Descriptive statistics on risky assets.

Process	Expected return	Volatility	Maximum Drawdown
Stock	0.12	0.199	0.463
Bond	0.051	0.083	0.218
Real asset	0.063	0.141	0.402
Illiquid stock	0.12	0.398	0.861
Roll-over of 1-year indexed bonds	0.057	0.02	0.019
Price index	0.025	0.013	-

All statistics are first computed in time series on each of the 10,000 simulated paths. The 10,000 values obtained are then averaged to produce the numbers shown in the table. Statistics are computed from monthly logarithmic returns, and expected returns and volatilities are expressed in annual terms. Expected returns are corrected for Jensen's inequality (i.e. one half of the variance is added to the mean logarithmic return). The dynamics of the processes and parameter values are given in Appendix 6.5.

Table 8: Investor 1 – Funding status of exogenous goals.
(i) Values of assets (in \$).

Market wealth	*2,150,000*
Aspirational wealth	*1,450,000*
Total	*3,600,000*

(ii) Minimum capital required to secure one or more exogenous goal(s) (in \$).

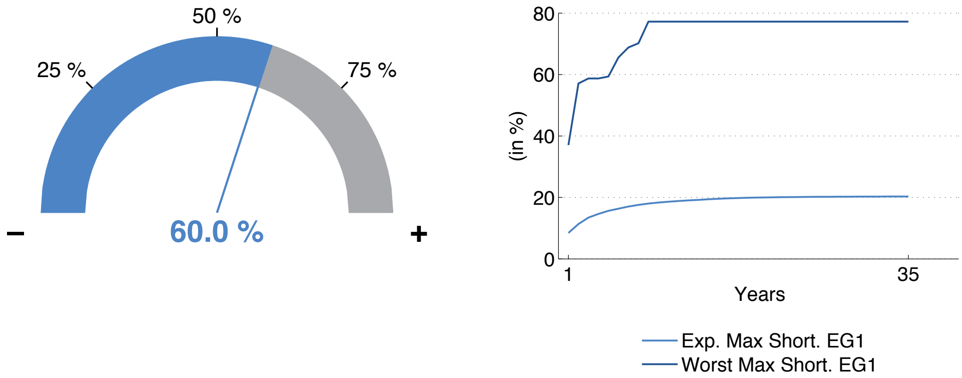
Goal 1	*3,000,000*
Goal 3	*4,810,724*
G1 and G3	*≥ 4,810,724*

The assets of Investor 1 consist of liquid market assets and less liquid aspirational assets. Panel (ii) shows the minimum capital required to secure the two exogenous goals: Goal 1 is to maintain a minimum level of wealth of \$3m plus inflation over the next 35 years, and Goal 3 is to double the sum of market wealth and aspirational wealth at the 15-year horizon. Goals are ranked by decreasing priority order.

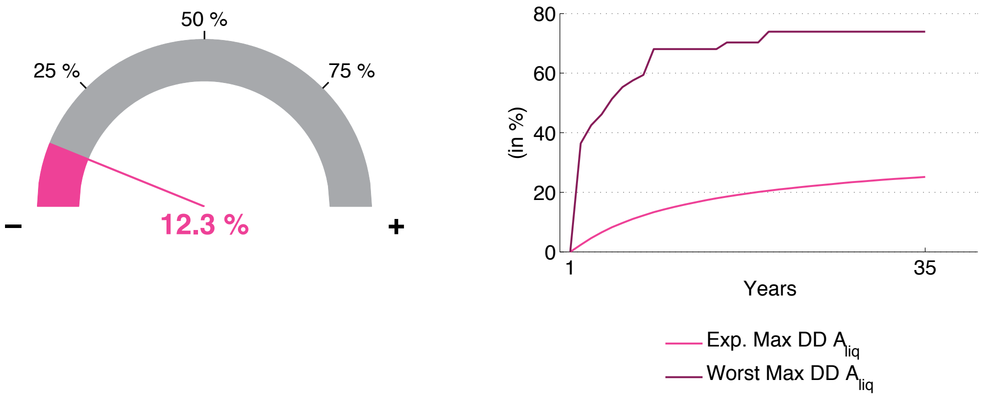
7. Figures and Tables

Figure 1: Investor 1 - Success indicators with current strategy and illiquid aspirational assets.

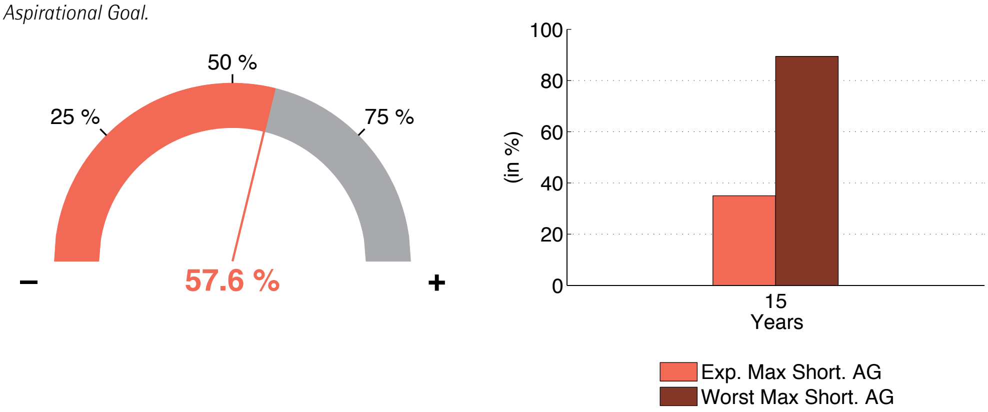
(a) Essential Goal 1.



(b) Essential Goal 2.



(c) Aspirational Goal.

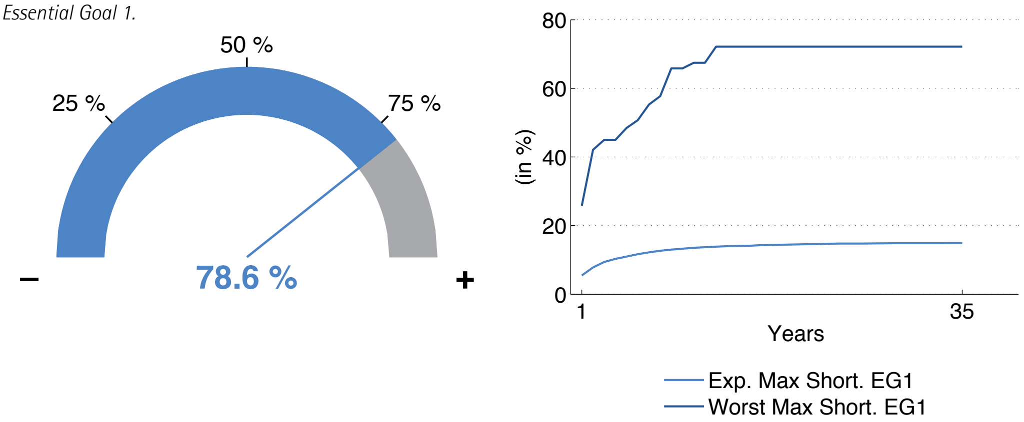


The half circles represent the success probabilities for each goal: it is estimated as the percentage of paths on which the goal was reached. For wealth-based goals (EG1 and AG), the expected maximum shortfall on date t is the expectation of the value of the maximum relative loss recorded by date t, conditional on the event of a loss. The worst maximum shortfall is defined as the worst relative loss recorded by date t across all dates and paths. For the drawdown goal (EG2), the expected maximum drawdown on date t is the expected value of the maximum drawdown recorded by date t, and the worst maximum drawdown is the worst drawdown recorded by date t across all dates and paths. All indicators are computed by taking into account only the goal horizon (years 1 to 35 for EG1 and EG2, and year 15 for AG). The "current strategy" is a fixed-mix policy with annual rebalancing towards the current market allocation. Aspirational assets are illiquid.

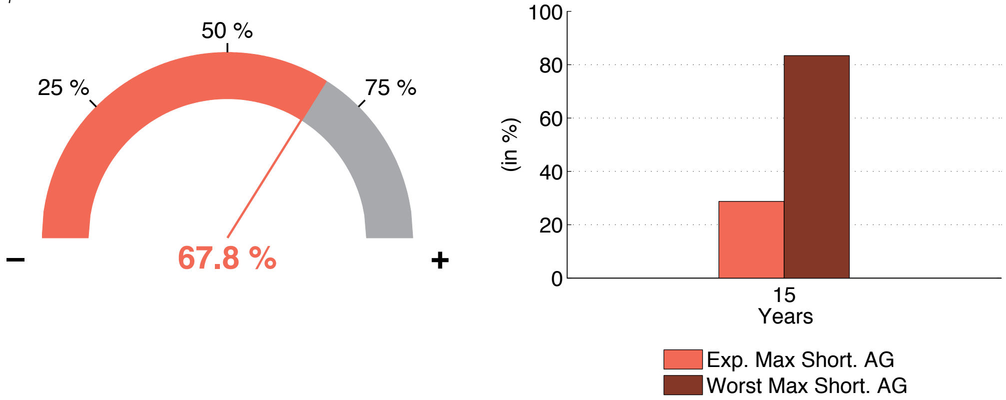
7. Figures and Tables

Figure 2: Investor 1 – Success indicators with current strategy and liquid aspirational assets.

(a) Essential Goal 1.



(b) Aspirational Goal.

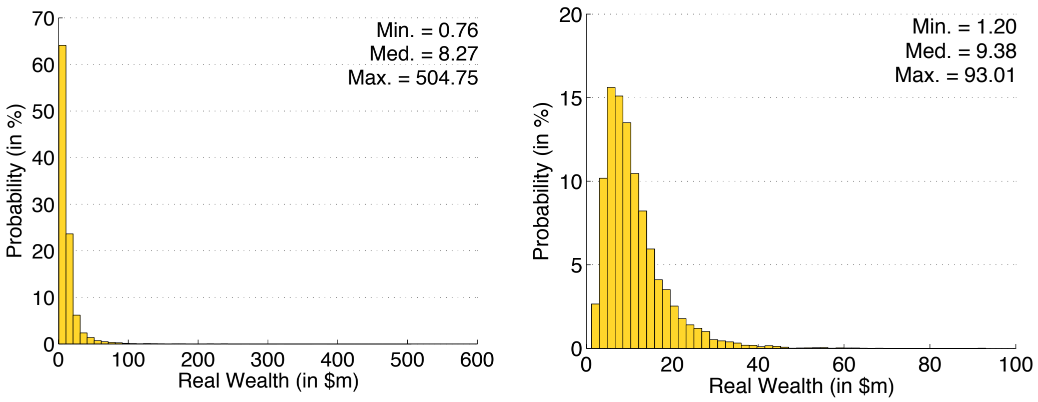


The definition of success indicators is given below Figure 1. The "current strategy" is a fixed-mix policy with annual rebalancing towards the current market allocation. Aspirational assets are liquidated at date 0, and the proceeds are re-invested in the market assets, with the same weights as in the initial market bucket. By construction, the success indicators for EG2 are the same as in the illiquid case (Figure 1), so they are not reported here.

7. Figures and Tables

Figure 3: Investor 1 – Impact of liquidity of aspirational assets on distribution of total wealth.

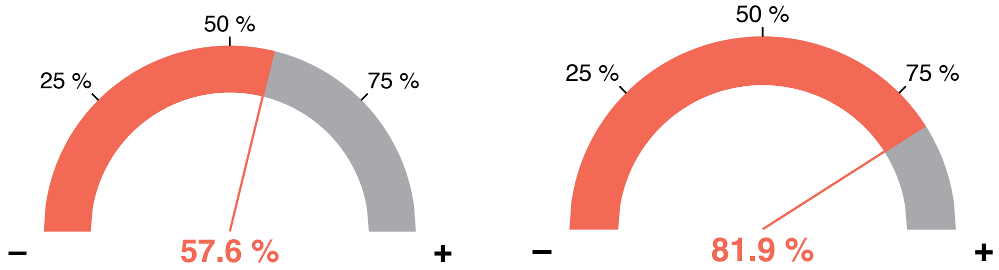
(a) Illiquid aspirational assets. *(b) Liquid aspirational assets.*



This figure shows the distribution of total wealth, which is computed as the sum of market wealth and aspirational wealth, after 15 years. This wealth is expressed in real terms, i.e. it is divided by the price index prevailing after 15 years. The indicators reported are the minimum, the median and the maximum of the distribution. In Panel (a), aspirational assets cannot be liquidated, while in Panel (b), they are liquidated at date 0, with the proceeds re-invested in the market assets. The strategy is the "current strategy", which is a fixed-mix policy with annual rebalancing towards the current market allocation.

Figure 4: Investor 1 – Impact of illiquid stock performance on success probability for aspirational goal.

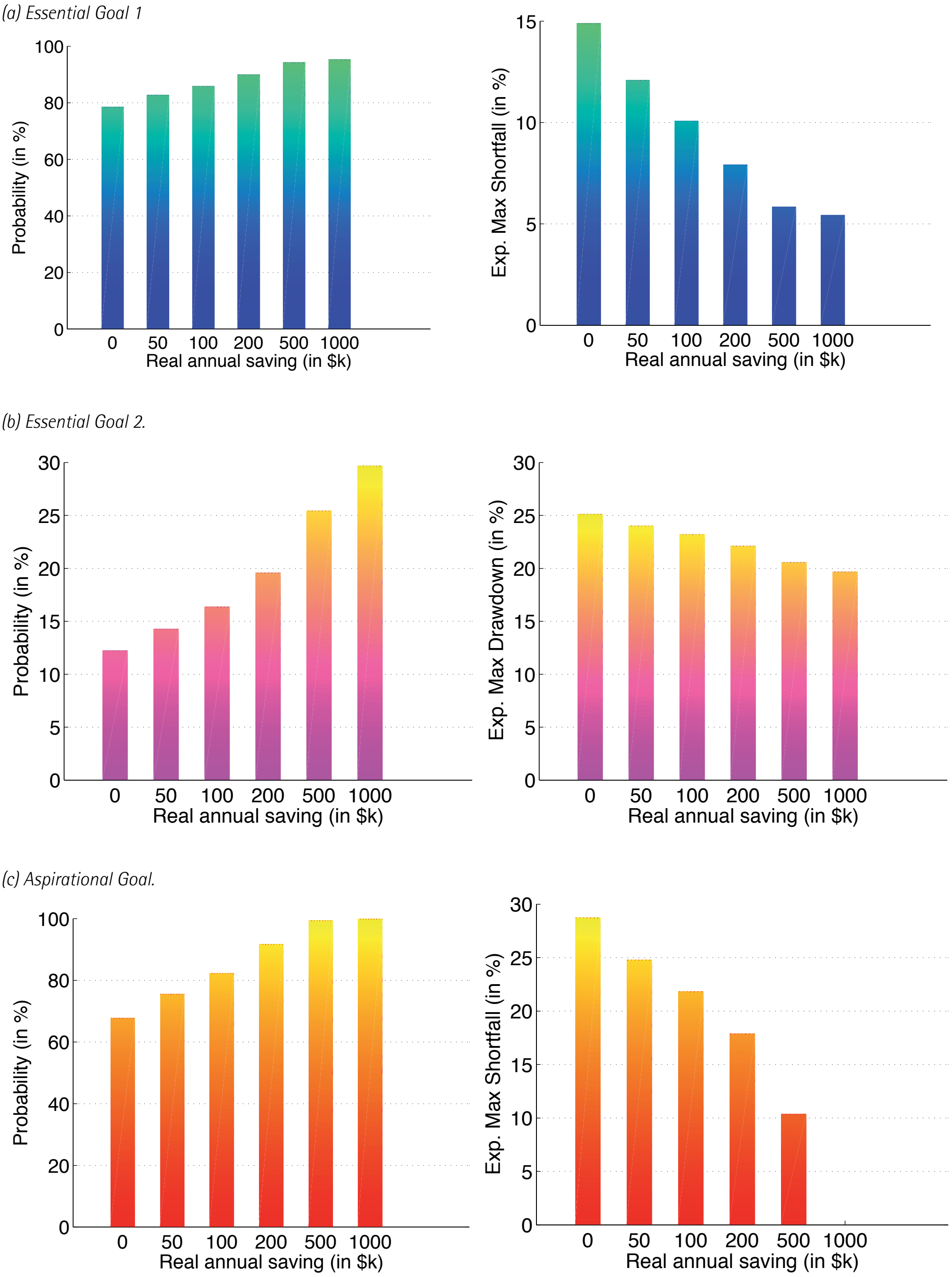
(i) Base case performance (12%/year). *(ii) Higher performance (24%/year).*



This figure shows the probability of reaching the aspirational goal for two values of the expected return on the illiquid stock (parameter μ_X): 12% (the base case value, which is a reminder of Figure 1), and 24%. The strategy is the "current strategy", which is a fixed-mix policy with annual rebalancing towards the current market allocation.

7. Figures and Tables

Figure 5: Investor 1 – Impact of annual savings on success indicators.

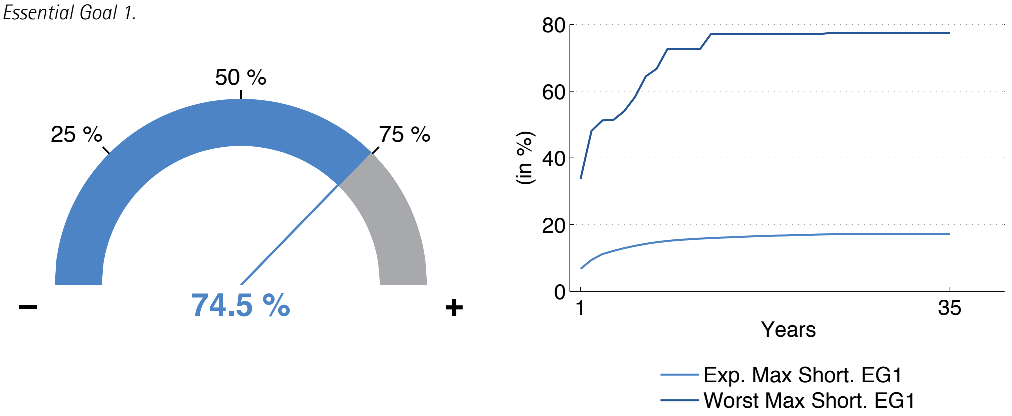


Investor 1 saves an amount of money equal to $0, 50k, 100k, 200k, 500k or 1m plus inflation at the end of each year, and these savings are re-invested in the market assets, in such a way as to leave the weights unchanged. Aspirational assets are liquidated at date 0, and the proceeds are re-invested in stocks, bonds and cash. The strategy implemented here is the "current strategy", which is a fixed-mix policy with annual rebalancing towards the current allocation. The left column shows the success probabilities for investor's goals as a function of annual savings, and the right column shows the expected maximum shortfall for goals EG1 and AG, and the expected maximum drawdown (see the caption of Figure 1 for the definition of the success indicators).

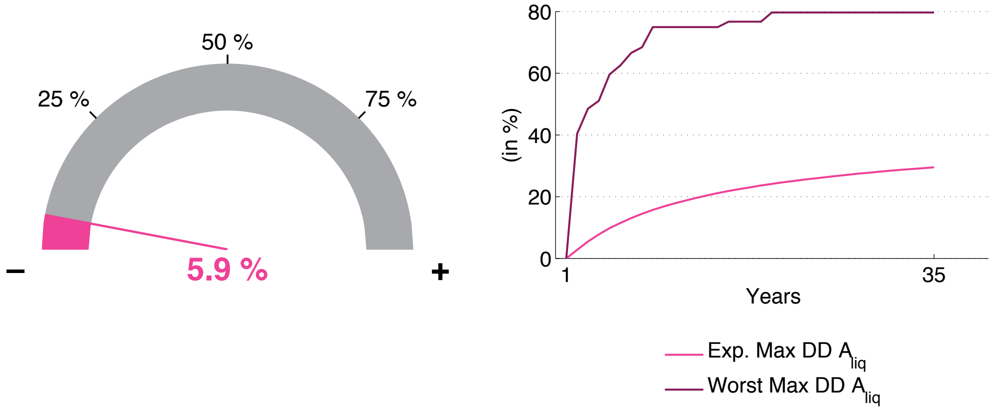
7. Figures and Tables

Figure 6: Investor 1 – Success indicators with maximum Sharpe ratio strategy.

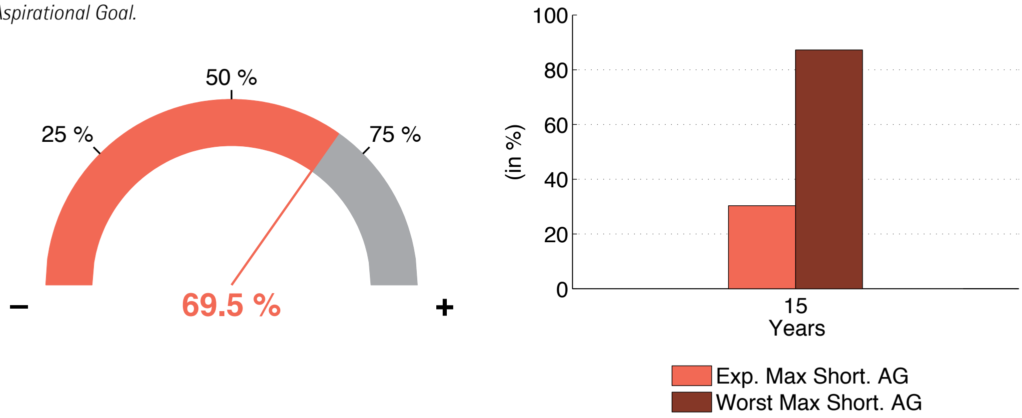
(a) Essential Goal 1.



(b) Essential Goal 2.



(c) Aspirational Goal.

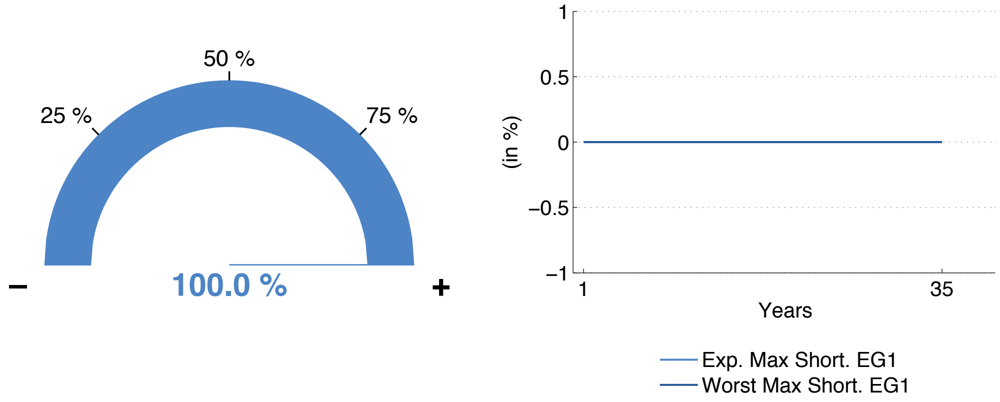


Aspirational assets are liquidated at date 0, and the proceeds are re-invested in stocks and bonds. The strategy implemented here is a fixed-mix policy with monthly rebalancing towards the maximum Sharpe ratio allocation. The left column shows the success probabilities for the goals, and the right column displays the expected maximum shortfall for goals EG1 and AG, and the expected maximum drawdown (see the caption of Figure 1 for the definition of the success indicators).

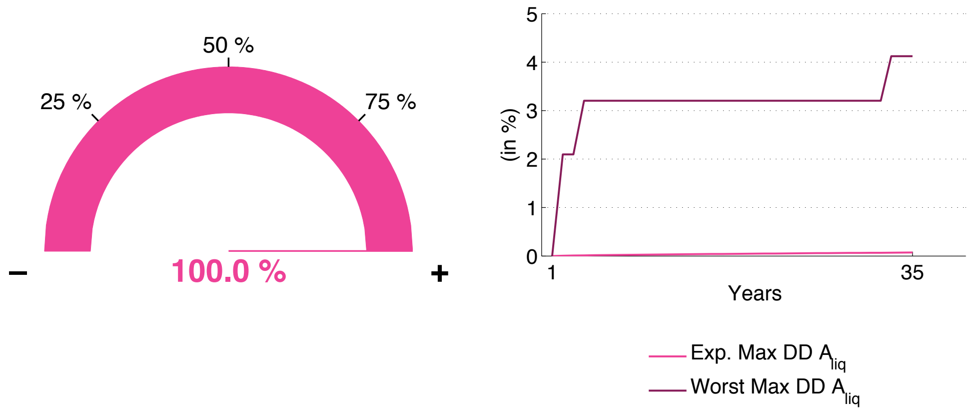
7. Figures and Tables

Figure 7: Investor 1 – Success indicators with safe strategy for Essential Goal 1.

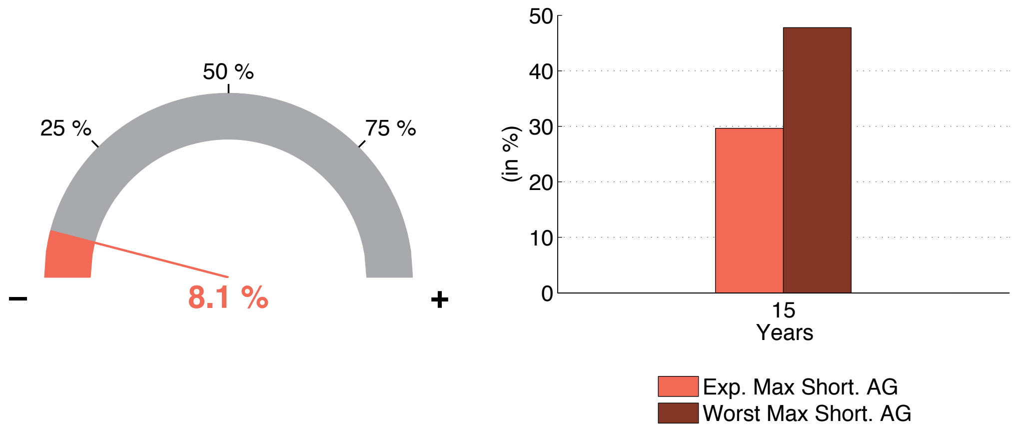
(a) Essential Goal 1.



(b) Essential Goal 2.



(c) Aspirational Goal.

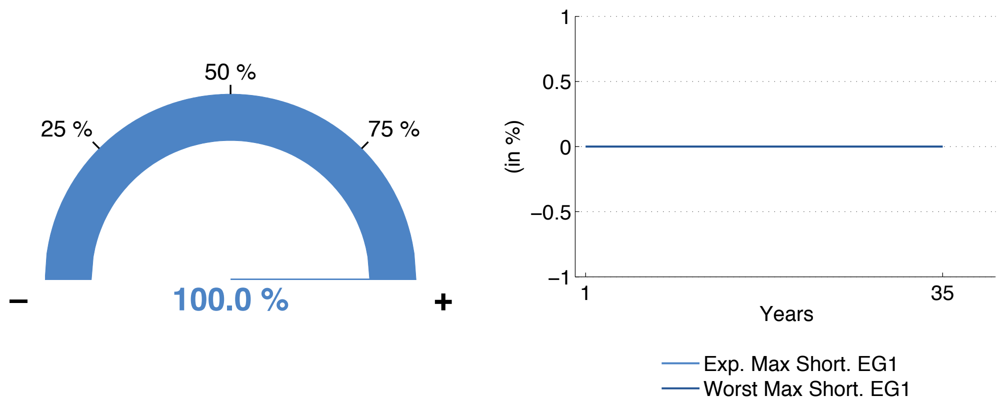


Aspirational assets and existing positions in stocks and bonds are liquidated at date 0. The safe strategy is a roll-over of 1-year inflation-indexed bonds (see description in Section 4.1.1.2). The left column shows the success probabilities for the goals, and the right column displays the expected maximum shortfall for goals EG1 and AG, and the expected maximum drawdown (see the caption of Figure 1 for the definition of the success indicators).

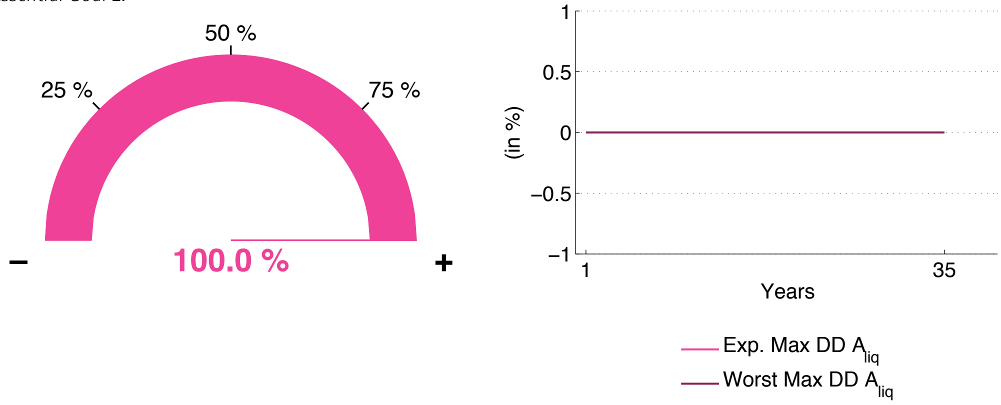
7. Figures and Tables

Figure 8: Investor 1 – Success indicators with safe strategy for Essential Goal 2.

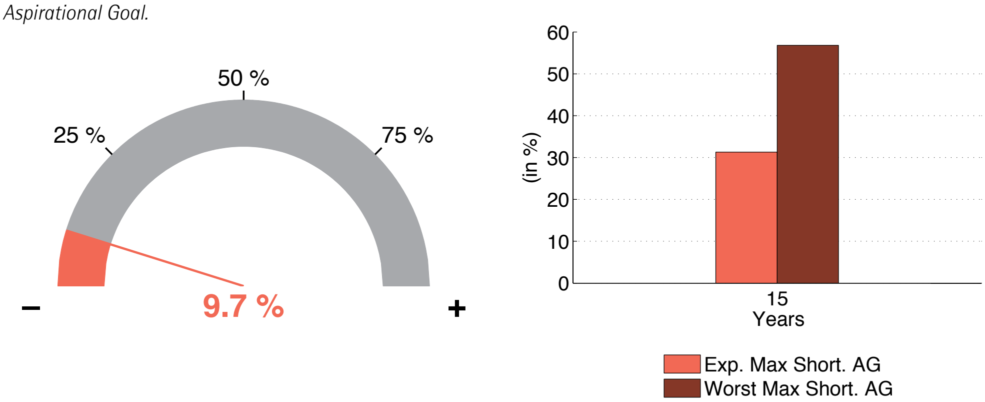
(a) Essential Goal 1.



(b) Essential Goal 2.



(c) Aspirational Goal.



Aspirational assets and existing positions in stocks and bonds are liquidated at date 0. The safe strategy is invested in cash only. The left column shows the success probabilities for the goals, and the right column displays the expected maximum shortfall for goals EG1 and AG, and the expected maximum drawdown (see the caption of Figure 1 for the definition of the success indicators).

7. Figures and Tables

Table 9: Investor 1 – Initial risk allocation with buy-and-hold strategy securing Essential Goal 1.

	Value ($)	% of Total		Value ($)	% of Total		Value ($)	% of Total
Personal Bucket	3,900,000	86.7	Market Bucket	600,000	13.3	Aspirational Bucket	0	0
Residence	1,500,000	28.3	Equity	482,534	80.4			
Cash	100,000	1.9	US Fixed Income	117,466	19.6			
GHP EG1	300,000	56.6						
Adjustable Rate Mortgage	(700,000)	13.2						

This table shows the weights at date 0 of the buy-and-hold strategy that secures Essential Goal 1. The personal risk bucket contains assets that are used to finance the investor's implicit or explicit essential goal: the residence secures the goal of not being homeless, the cash reserve secures the goal of being able to afford a minimum standard of living, and the GHP is a roll-over of 1-year indexed bonds that secures Essential Goal 1. The aspirational bucket contains in principle illiquid and concentrated positions held for wealth mobility purposes. It is empty here, as these positions have been liquidated at date 0. The market bucket contains all other assets (equities and bonds here). The table displays the weights of the various assets within each bucket, as well as the relative weights of the buckets.

Table 10: Investor 1 – Initial Risk Allocation with GBI Strategy Securing Essential Goal 1.

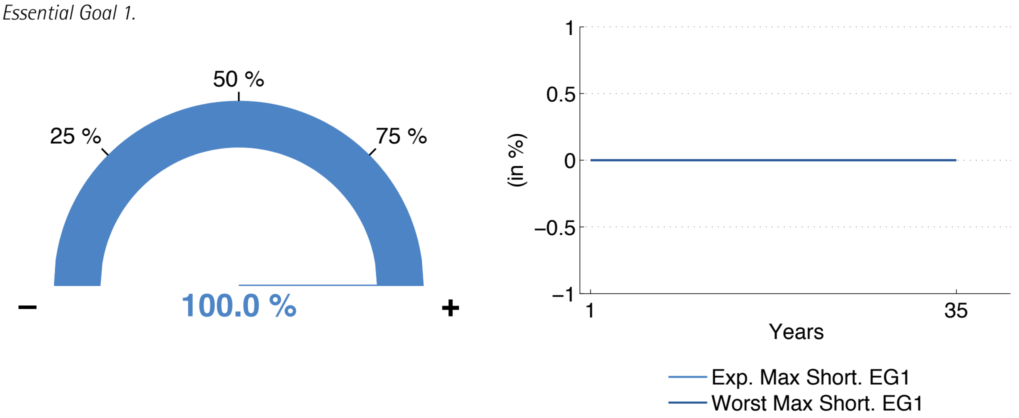
	Value ($)	% of Total		Value ($)	% of Total		Value ($)	% of Total
Personal Bucket	1,500,000	33.3	Market Bucket	3,000,000	66.7	Aspirational Bucket	0	0
Residence	1,500,000	51.7	Equity	2,412,671	80.4			
Cash	100,000	3.4	US Fixed Income	587,329	19.6			
GHP EG1	600,000	20.7						
Adjustable Rate Mortgage	(700,000)	24.1						

This table shows the risk allocation at date 0 when the investor takes a GBI strategy of the form (30) to secure Essential Goal 1, with a multiplier equal to 5. The personal risk bucket contains assets that are used to finance the investor's implicit goals and the explicitly formulated essential goal: the residence secures the goal of not being homeless, the cash reserve secures the goal of being able to afford a minimum standard of living, and the GHP is a roll-over of 1-year indexed bonds that secures Essential Goal 1. The aspirational bucket contains in principle illiquid and concentrated positions held for wealth mobility purposes. It is empty here as these positions have been liquidated at date 0. The market bucket contains all other assets (equities and bonds here). The table displays the weights of the various assets within each bucket, as well as the relative weights of the buckets.

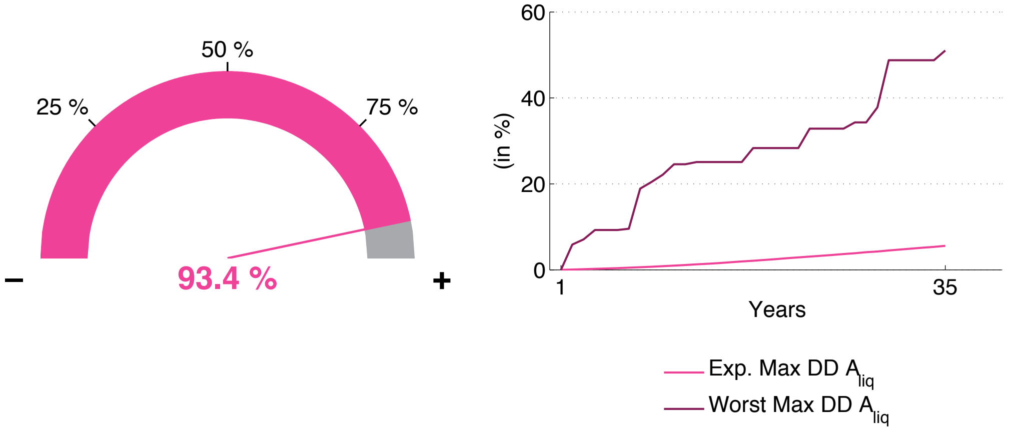
7. Figures and Tables

Figure 9: Investor 1 – Success indicators with buy-and-hold strategy securing Essential Goal 1.

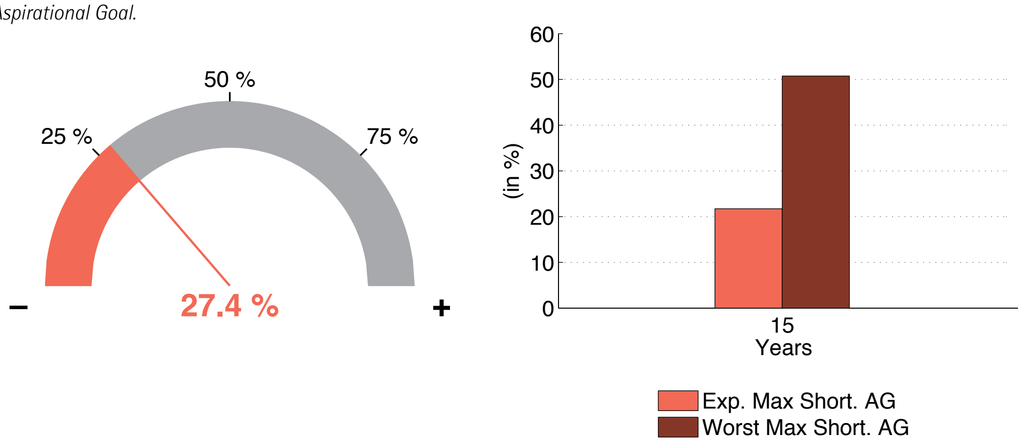
(a) Essential Goal 1.



(b) Essential Goal 2.



(c) Aspirational Goal.

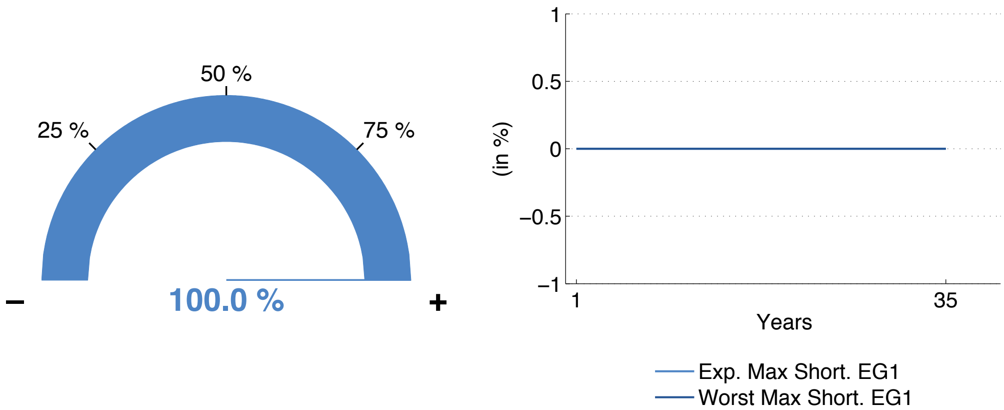


Aspirational assets and existing positions in stocks and bonds are liquidated at date 0, and the proceeds are re-invested in a buy-and-hold strategy that secures EG1. This strategy invests an amount $\widetilde{EG}_0^1$ in a roll-over of 1-year indexed bonds ($\widetilde{EG}_0^1$ being the price at time 0 of the indexed zero-coupon that pays \$3m plus realised inflation after 1 year), and the remainder of wealth in the MSR portfolio. The MSR building block is rebalanced on a monthly basis. The left column shows the success probabilities for the goals, and the right column displays the expected maximum shortfall for goals EG1 and AG, and the expected maximum drawdown (see the caption of Figure 1 for the definition of the success indicators).

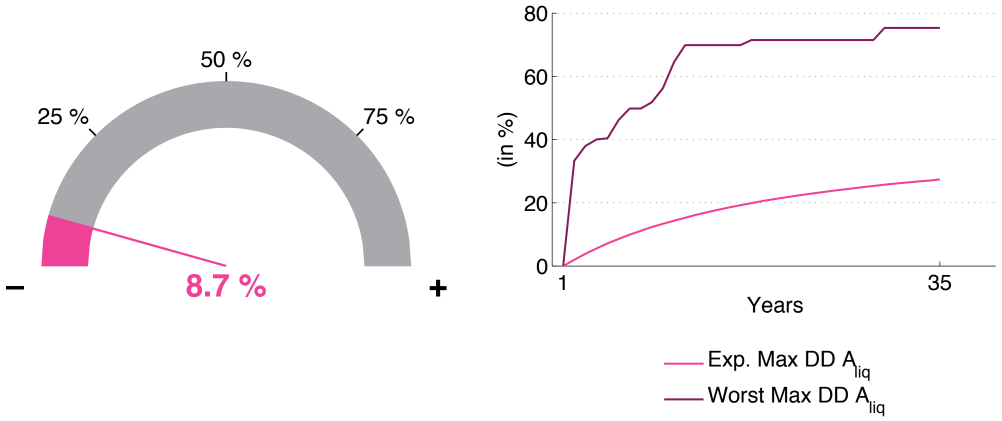
7. Figures and Tables

Figure 10: Investor 1 – Success indicators with GBI strategy securing Essential Goal 1.

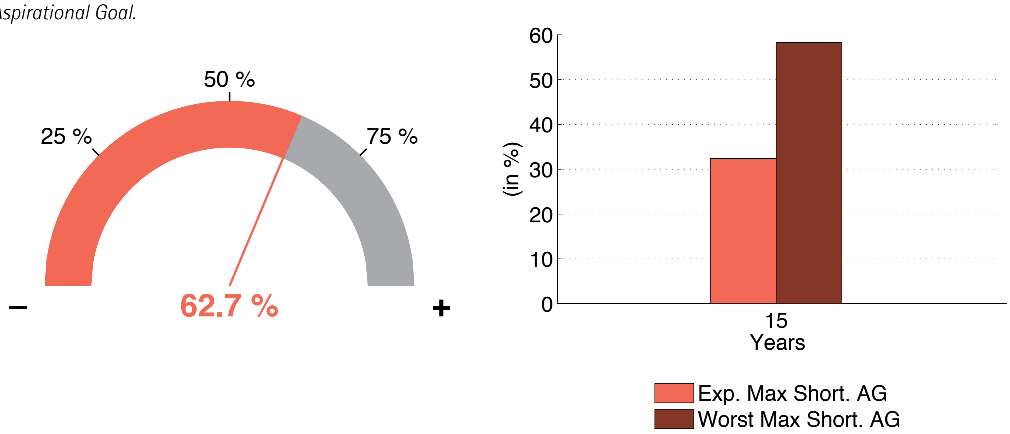
(a) Essential Goal 1.



(b) Essential Goal 2.



(c) Aspirational Goal.



Aspirational assets and existing positions in stocks and bonds are liquidated at date 0, and the proceeds are re-invested in a dynamic GBI strategy of the form (30). The performance building block is the MSR and the safe block is the GHP, which is a roll-over of 1-year indexed bonds. The floor is the present value of the minimum wealth level to achieve at the end of the current year (and is therefore discontinuous). The portfolio is rebalanced on a monthly basis, with a multiplier m = 5. The left column shows the success probabilities for the goals, and the right column displays the expected maximum shortfall for goals EG1 and AG, and the expected maximum drawdown (see the caption of Figure 1 for the definition of the success indicators).

7. Figures and Tables

Table 11: Investor 1 – Impact of multiplier on initial allocation to personal assets.

(i) m = 1

	Value ($)	% of Total
Personal Bucket	3,900,000	86.7
Residence	1,500,000	28.3
Cash	100,000	1.9
GHP EG1	3,000,000	56.6
Adjustable Rate Mortgage	(700,000)	13.2

(ii) m = 3.

	Value ($)	% of Total
Personal Bucket	2,700,000	60.0
Residence	1,500,000	36.6
Cash	100,000	2.4
GHP EG1	1,800,000	43.9
Adjustable Rate Mortgage	(700,000)	17.1

(iii) m = 5.

	Value ($)	% of Total
Personal Bucket	1,500,000	33.3
Residence	1,500,000	51.7
Cash	100,000	3.4
GHP EG1	600,000	20.7
Adjustable Rate Mortgage	(700,000)	24.1

(iv) m = 7.

	Value ($)	% of Total
Personal Bucket	900,000	20.0
Residence	1,500,000	65.2
Cash	100,000	4.3
GHP EG1	0	0
Adjustable Rate Mortgage	(700,000)	30.4

This table shows the composition at date 0 of personal risk bucket when the investor follows a GBI strategy of the form (30) to secure Essential Goal 1, as a function of the multiplier. The personal risk bucket contains the assets that are used to finance the investor's implicit goals and the explicitly formulated essential goal: the residence secures the goal of not being homeless, the cash reserve secures the goal of being able to afford a minimum standard of living, and the GHP is a roll-over of 1-year indexed bonds that secures Essential Goal 1.

Table 12: Investor 1 – Initial Risk Allocation with GBI Strategy Securing Essential Goals 1 and 2.

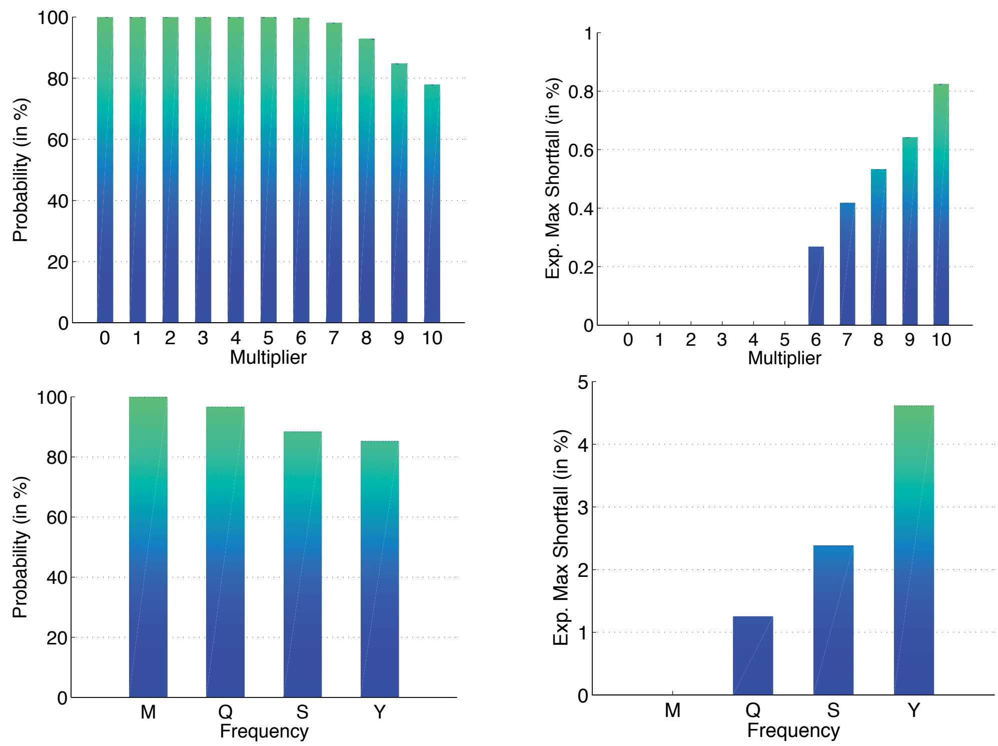
	Value ($)	% of Total		Value ($)	% of Total		Value ($)	% of Total
Personal Bucket	1,800,000	40.0	Market Bucket	2,700,000	60.0	Aspirational Bucket	0	0
Residence	1,500,000	46.9	Equity	2,171,403	80.4			
Cash	100,000	3.1	US Fixed Income	528,597	19.6			
GHP EG1	0	0						
GHP EG2	900,000	28.1						
Adjustable Rate Mortgage	(700,000)	21.9						

This table shows the risk allocation at date 0 when the investor takes a GBI strategy of the form (32) to secure Essential Goals 1 and 2, with a multiplier equal to 5. The personal risk bucket contains assets that are used to finance the investor's implicit goals and the explicitly formulated essential goal: the residence secures the goal of not being homeless, the cash reserve secures the goal of being able to afford a minimum standard of living, the GHP for Goal 1 is a roll-over of 1-year indexed bonds, and the GHP for Goal 2 is cash. The aspirational bucket contains in principle illiquid and concentrated positions held for wealth mobility purposes. It is empty here as these positions have been liquidated at date 0. The market bucket contains all other assets (equities and bonds here). The table displays the weights of the various assets within each bucket, as well as the relative weights of the buckets.

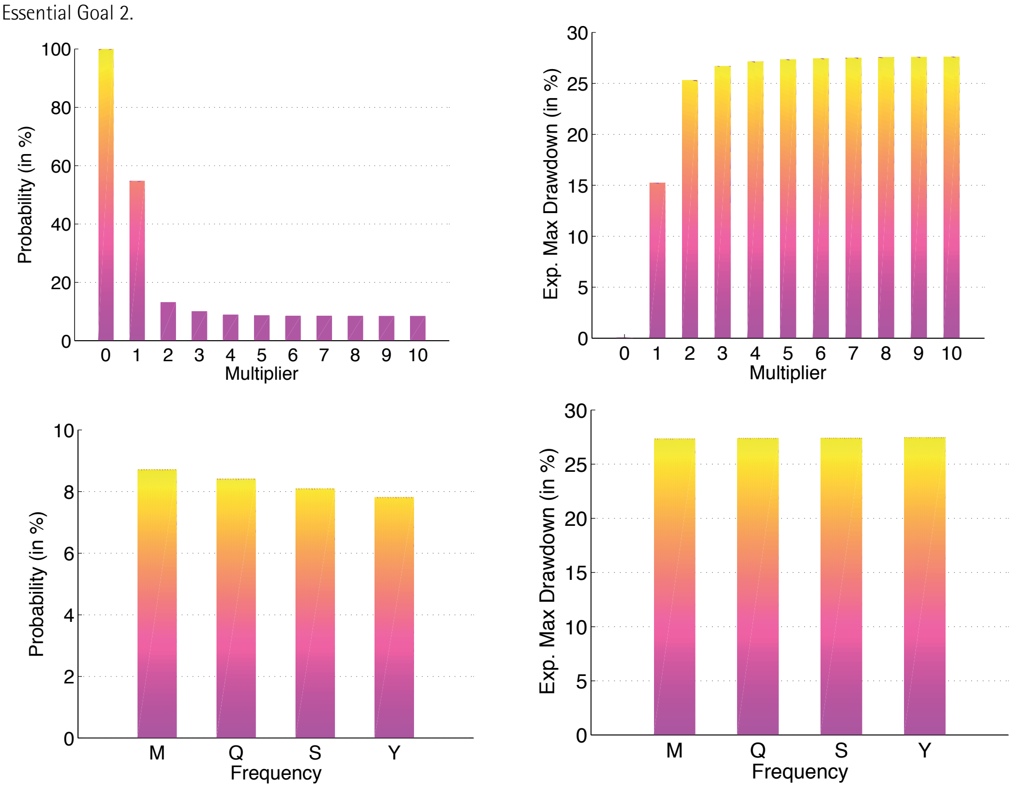
7. Figures and Tables

Figure 11: Investor 1 – Impacts of multiplier and trading frequency on success indicators with GBI strategy securing Essential Goal 1.

(a) Essential Goal 1.

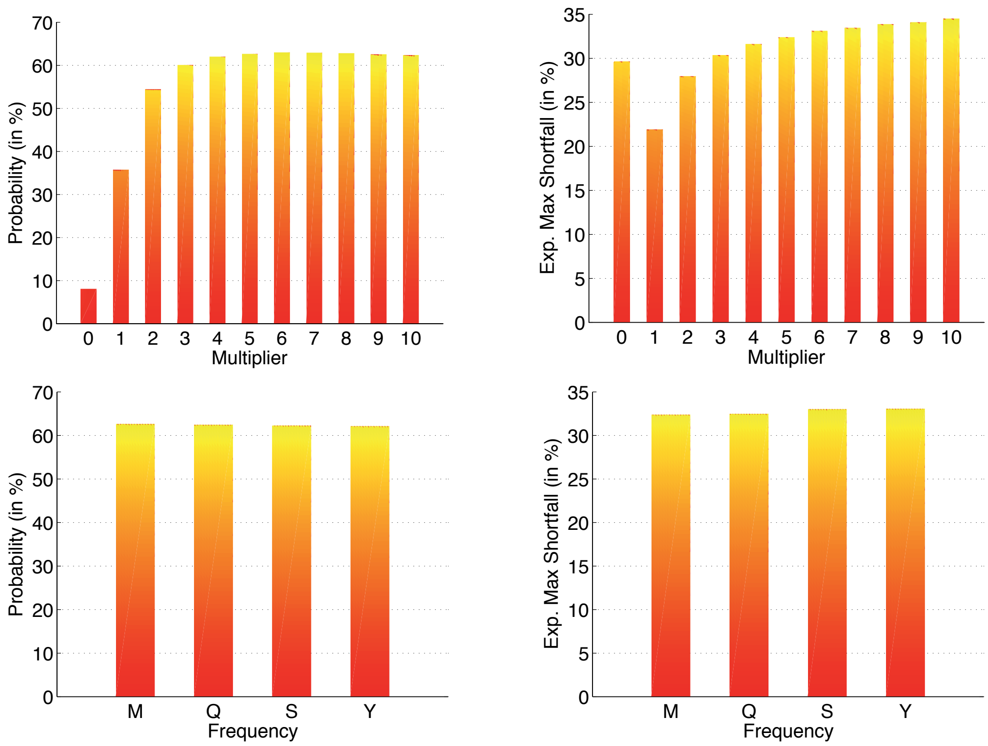


(b) Essential Goal 2.



7. Figures and Tables

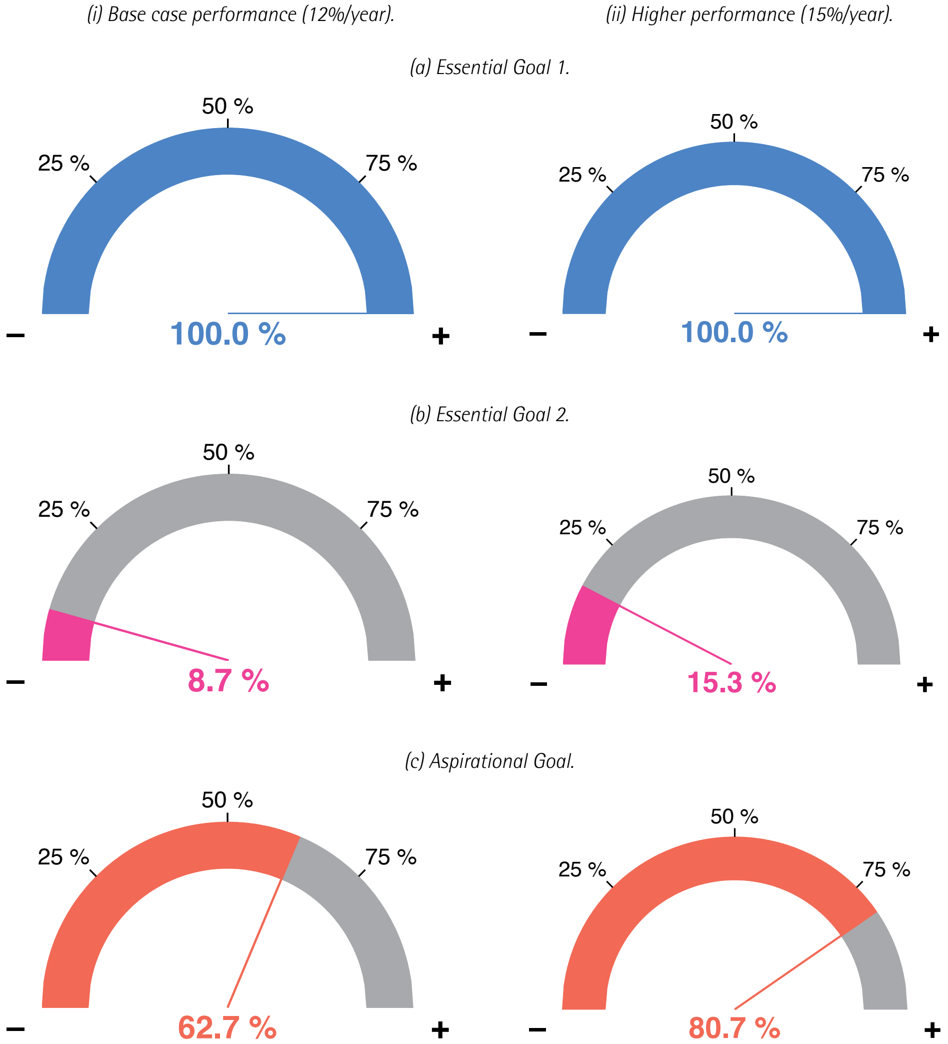
(c) Aspirational Goal.



Aspirational assets and existing positions in stocks and bonds are liquidated at date 0, and the proceeds are re-invested in a dynamic GBI strategy of the form (30). The performance building block is the MSR and the safe block is the GHP, which is a roll-over of 1-year indexed bonds. The floor is the present value of the minimum wealth level to achieve at the end of the current year. The base case multiplier is 5, and the base case rebalancing period is one month. We let the multiplier vary from 0 (portfolio fully invested in the GHP) to 10 and the rebalancing period be one month, one quarter, one semester and one year. The left column shows the success probabilities for the goals, and the right column displays the expected maximum shortfall for goals EG1 and AG, and the expected maximum drawdown (see the caption of Figure 1 for the definition of the success indicators).

7. Figures and Tables

Figure 12: Investor 1 – Impact of stock expected return on the success probabilities with GBI strategy securing Essential Goal 1.

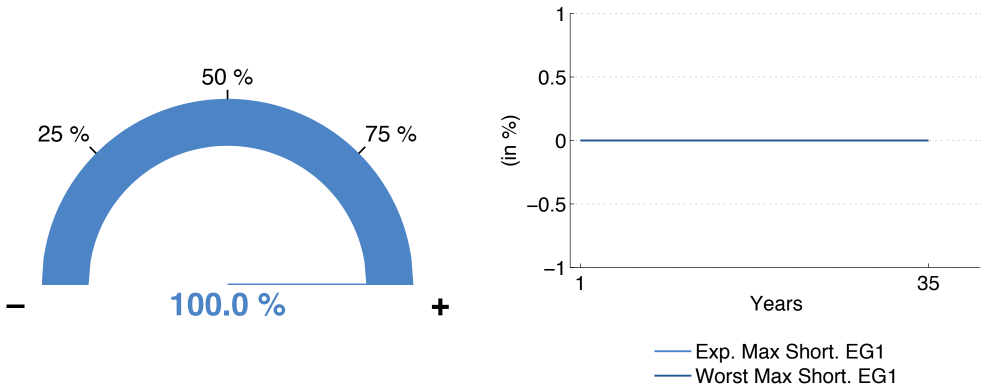


Aspirational assets and existing positions in stocks and bonds are liquidated at date 0, and the proceeds are re-invested in a dynamic GBI strategy of the form (30). The performance building block is the MSR and the safe block is the GHP, which is a roll-over of 1-year indexed bonds. The floor is the present value of the minimum wealth level to achieve at the end of the current year. The strategy is rebalanced on a monthly basis, with a multiplier m = 5. In the left column, the expected return of the stock is set to its base case value of 12%, so this column is a reminder of Figure 10. In the right column, the expected return is raised to 15%.

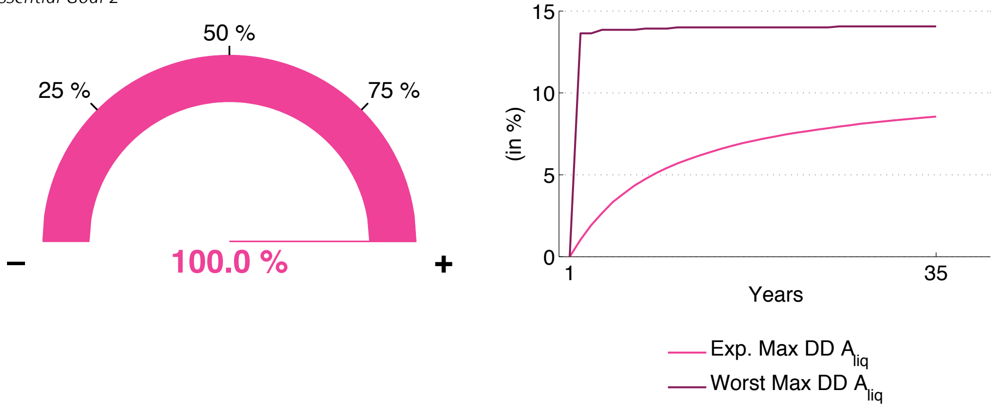
7. Figures and Tables

Figure 13: Investor 1 – Success indicators with GBI strategy securing Essential Goals 1 and 2.

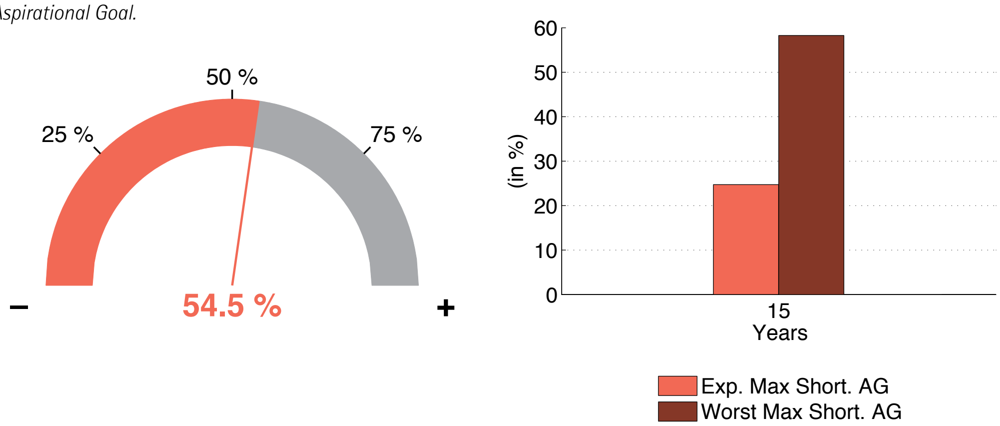
(a) Essential Goal 1.



(b) Essential Goal 2



(c) Aspirational Goal.

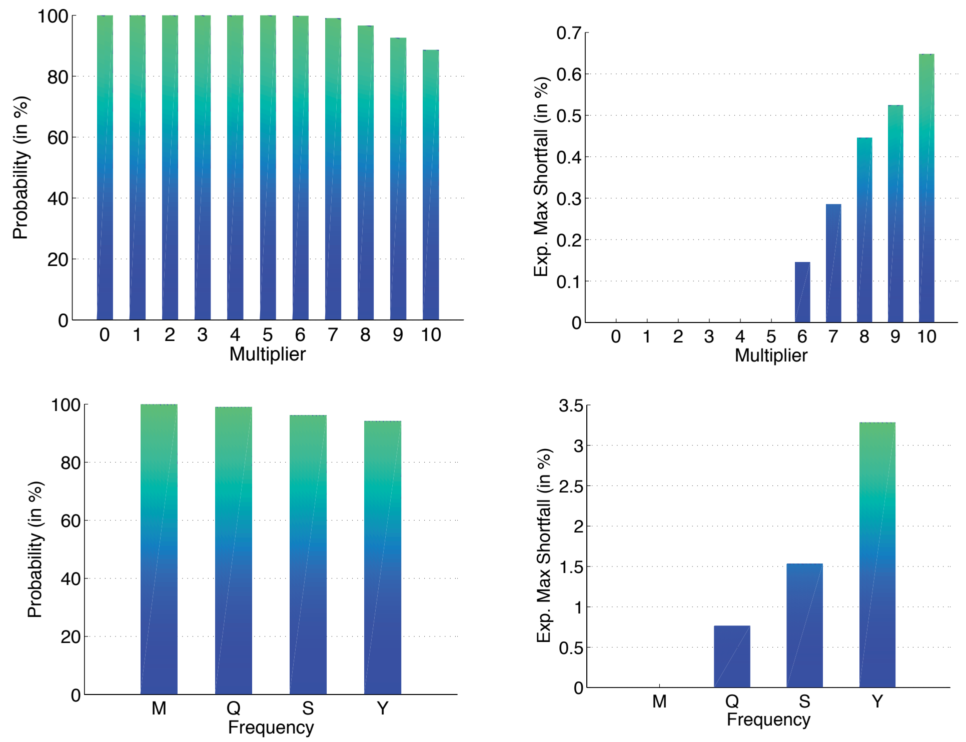


Aspirational assets and existing positions in stocks and bonds are liquidated at date 0, and the proceeds are re-invested in a dynamic GBI strategy of the form (32). The performance building block is the MSR, and the floor that appears in the risk budget is the maximum of the floors associated with the two goals (present value of minimum wealth level to attain at the end of the current year for EG1 and drawdown floor for EG2). The floor-replicating portfolio is the GHP that corresponds to the higher floor (roll-over of 1-year indexed bonds for EG1 and cash for EG2). The portfolio is rebalanced every month, with a multiplier m = 5. The left column shows the success probabilities for the goals, and the right column displays the expected maximum shortfall for goals EG1 and AG, and the expected maximum drawdown (see the caption of Figure 1 for the definition of the success indicators).

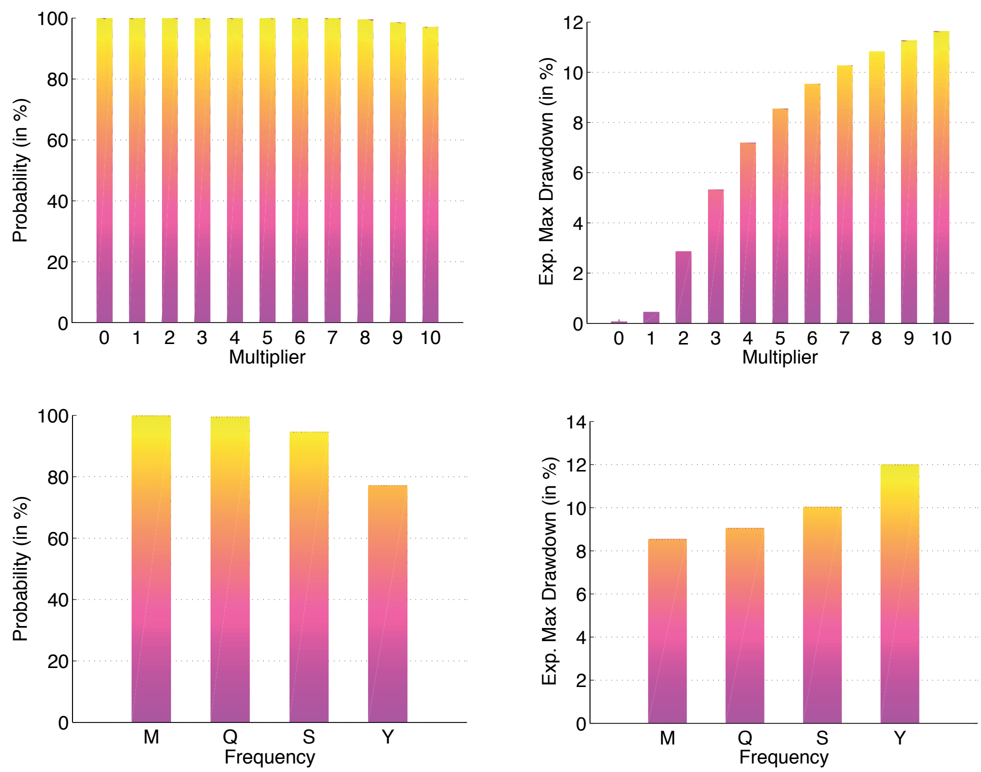
7. Figures and Tables

Figure 14: Investor 1 – Impacts of multiplier and trading frequency on success indicators with GBI strategy securing Essential Goals 1 and 2.

(a) Essential Goal 1.

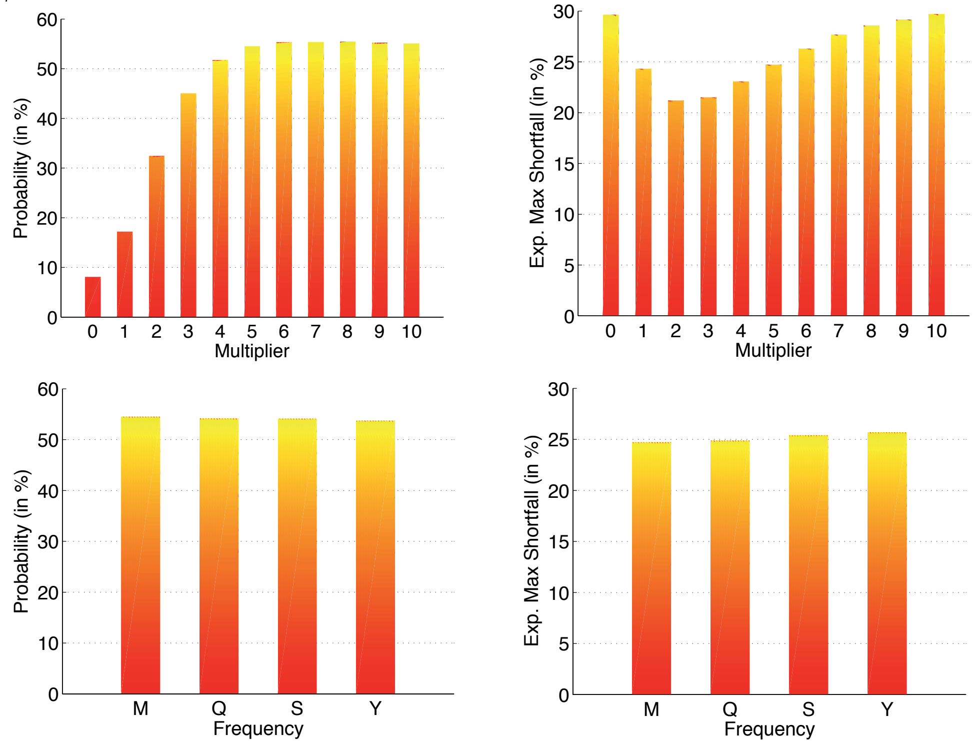


(b) Essential Goal 2.



7. Figures and Tables

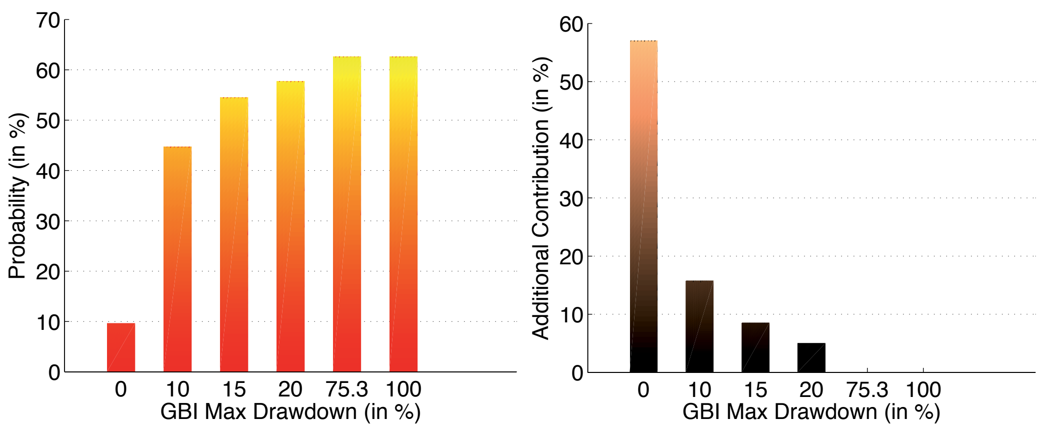
(c) Aspirational Goal.



Aspirational assets and existing positions in stocks and bonds are liquidated at date 0, and the proceeds are re-invested in a dynamic GBI strategy of the form (32). The performance building block is the MSR, the first GHP is a roll-over of 1-year indexed bonds which secures EG1, and the second GHP, which secures EG2, is fully invested in cash. The floor is the maximum of the floors associated with EG1 and EG2. The former floor is the present value of the minimum wealth level to achieve at the end of the current year, and the latter floor is 85% of the maximum to date of wealth. The base case multiplier is 5, and the base case rebalancing period is one month. We let the multiplier vary from 0 (portfolio fully invested in the roll-over or in cash) to 10 and the rebalancing period be one month, one quarter, one semester and one year. The left column shows the success probabilities for the goals, and the right column displays the expected maximum shortfall for goals EG1 and AG, and the expected maximum drawdown (see the caption of Figure 1 for the definition of the success indicators).

Figure 15: Investor 1 – Opportunity cost of drawdown constraint with GBI strategy securing Essential Goals 1 and 2.

(i) Success probabilities for Aspirational Goal. *(ii) Additional initial contribution required.*

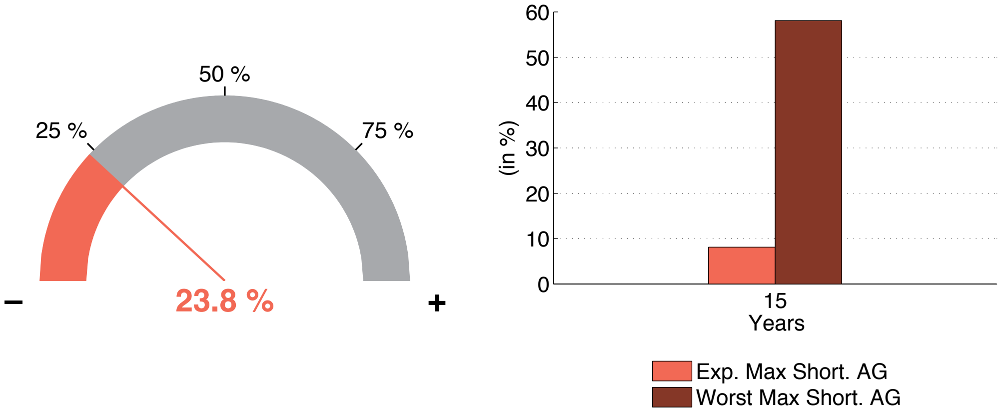


Aspirational assets and existing positions in stocks and bonds are liquidated at date 0, and the proceeds are re-invested in a dynamic GBI strategy of the form (32) which protects both Essential Goals 1 and 2. The portfolio is rebalanced every month, with a multiplier m = 5. Panel (i) shows the probability of reaching the aspirational goal as a function of the maximum drawdown imposed in the GBI strategy. Panel (ii) displays the additional initial contribution which is necessary for the GBI strategy with the drawdown constraint to have the same success probability as the otherwise equivalent GBI strategy without this constraint. The value 75.3% on the horizontal axis is the maximum drawdown of the GBI strategy without the drawdown constraint.

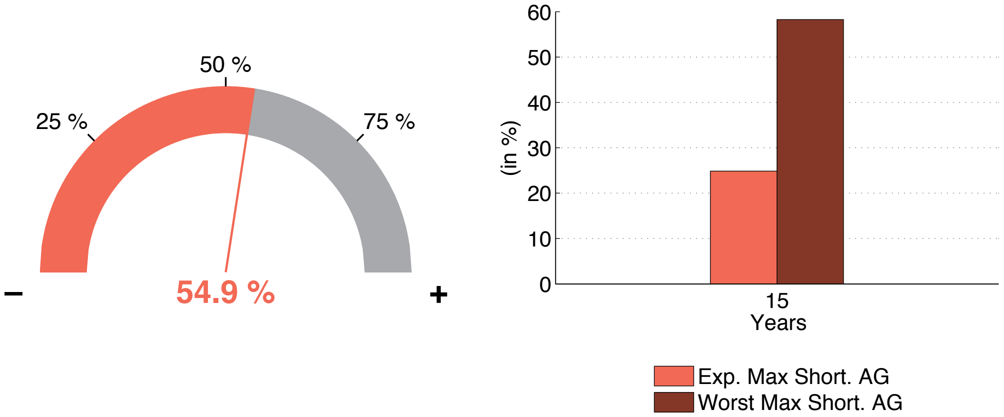
7. Figures and Tables

Figure 16: Investor 1 – Success indicators for Aspirational Goal with GBI strategy securing Essential Goals 1 and 2 and imposing a cap.

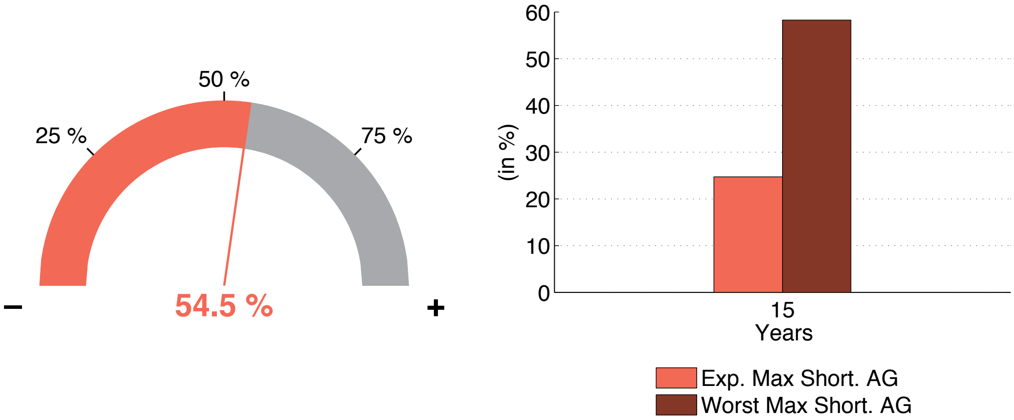
(i) Cap = present value of Aspirational Goal.



(ii) Cap = 2 × present value of Aspirational Goal.



(iii) Cap = 3 × present value of Aspirational Goal.



Aspirational assets and existing positions in stocks and bonds are liquidated at date 0, and the proceeds are re-invested in a dynamic GBI strategy of the form (33). The floor is the maximum of the two floors associated with goals EG1 and EG2 (present value of minimum wealth level to attain at the end of the current year for EG1 and drawdown floor for EG2). The cap is the present value of the aspirational goal in Panel (a), and 2 or 3 times this present value in Panels (b) and (c). The floor-replicating portfolio is the GHP that corresponds to the higher floor (roll-over of 1-year indexed bonds for EG1 and cash for EG2). The cap-replicating portfolio is the indexed zero-coupon that pays $7.2m at the end of 15 years. The portfolio is rebalanced every month, with a multiplier m = 5. The left column contains the success probabilities and the right column the expected shortfalls (see the caption of Figure 1 for the definition of the success indicators).

7. Figures and Tables

Table 13: Investor 1 – Initial Weights of GBI Strategy with Illiquid Aspirational Assets.

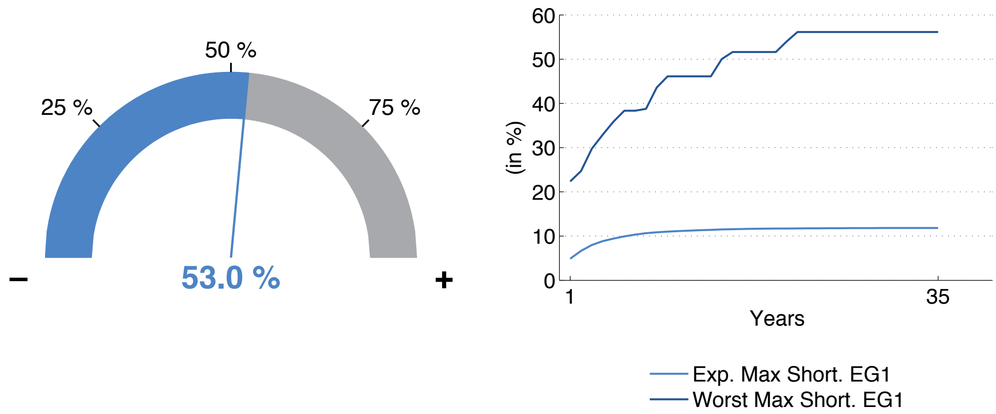
	Value ($)	% of Total		Value ($)	% of Total		Value ($)	% of Total
Personal Bucket	1,500,000	33.3	Market Bucket	1,550,000	34.5	Aspirational Bucket	1,450,000	32.2
Residence	1,500,000	51.7	Equity	1,246,546	80.4	Concentrated Stock	1,250,000	86.2
Cash	100,000	3.4	US Fixed Income	303,454	19.6	Executive Stock Option	100,000	6.9
GHP EG1	600,000	20.7				Real Asset	100,000	6.9
Adjustable Rate Mortgage	(700,000)	24.1						

This table shows the risk allocation at date 0 when the investor takes a GBI strategy of the form (30) to secure Essential Goal 1, with a multiplier equal to 5. The personal risk bucket contains assets that are used to finance the investor's implicit goals and the explicitly formulated essential goal: the residence secures the goal of not being homeless, the cash reserve secures the goal of being able to afford a minimum standard of living, and the GHP is a roll-over of 1-year indexed bonds that secures Essential Goal 1. The aspirational bucket contains illiquid and concentrated positions held for wealth mobility purposes. The market bucket contains all other assets (equities and bonds here). The table displays the weights of the various assets within each bucket, as well as the relative weights of the buckets.

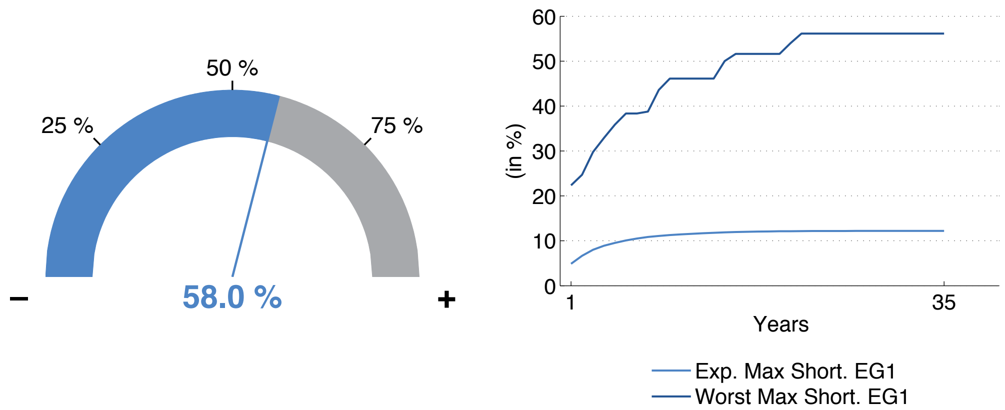
7. Figures and Tables

Figure 17: Investor 1 – Success indicators for Essential Goal 1 with GBI strategy and illiquid aspirational assets.

(i) Plain GBI strategy.



(ii) GBI strategy with ratchet effect.

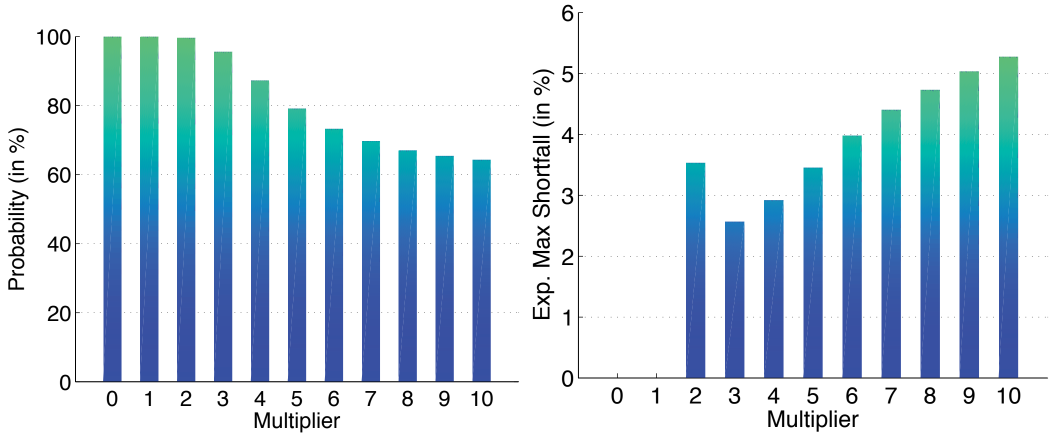


This figure shows the success indicators for Essential Goal 1 when aspirational assets cannot be liquidated, but the existing positions in stock and bond within the market bucket are liquidated and re-invested in a dynamic GBI strategy of the form (30). In Panel (ii), a ratchet effect is added to the strategy, that is, the portfolio switches to the GHP (a roll-over of 1-year indexed bonds) as soon as market wealth hits the present value of the goal. The portfolio is rebalanced on a monthly basis, with a multiplier m = 5. The left column shows the success probabilities, and the right column displays the expected maximum shortfall (see the caption of Figure 1 for the definition of the success indicators).

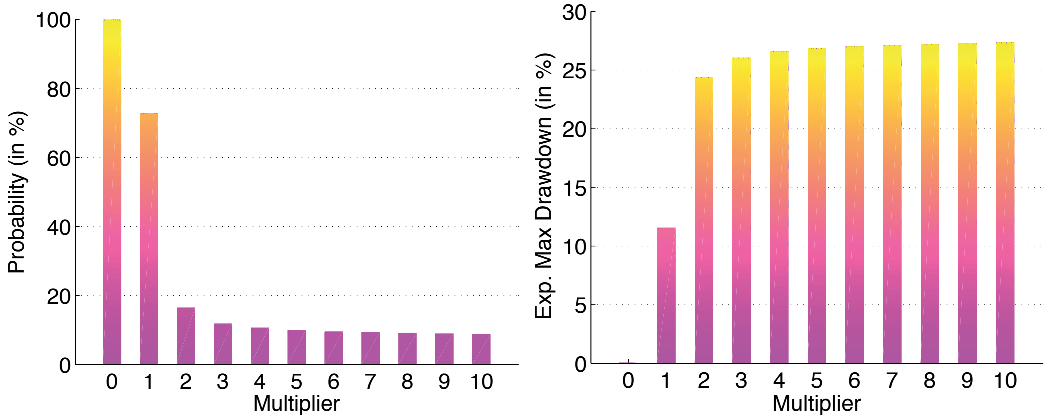
7. Figures and Tables

Figure 18: Investor 1 – Impact of multiplier on success indicators with GBI strategy and partially liquid assets.

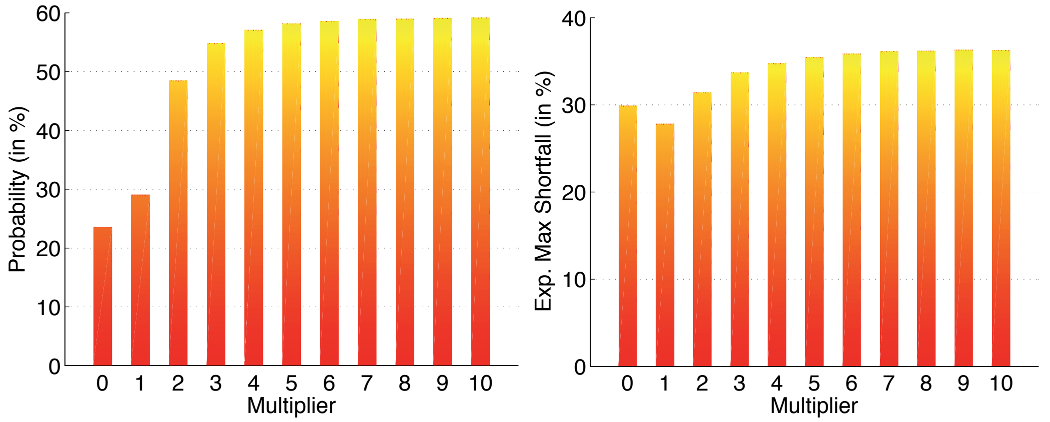
(a) Essential Goal 1.



(b) Essential Goal 2.



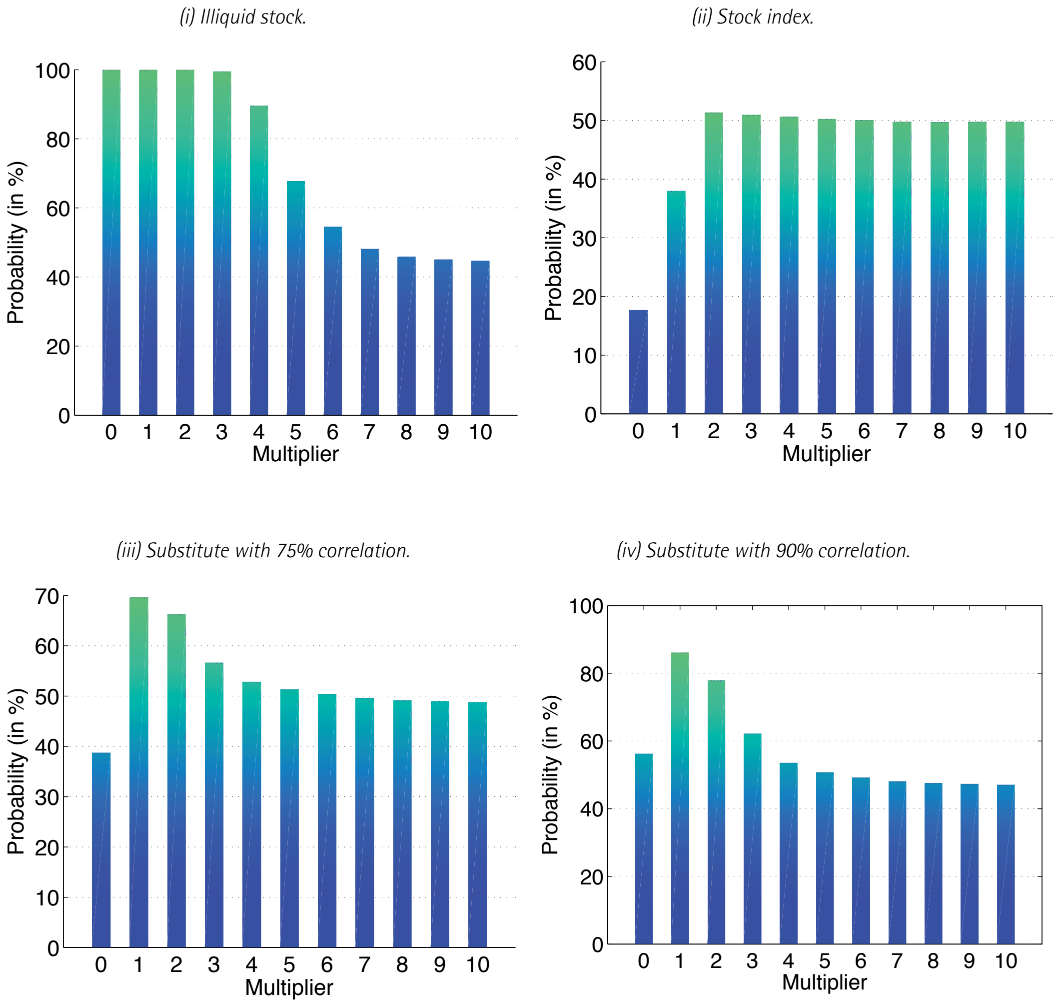
(c) Aspirational Goal.



This figure shows the impact of the multiplier on the success indicators with a dynamic GBI strategy of the form (32). Existing positions in stocks and bonds are liquidated at date 0, and aspirational assets are partially liquidated: the amount liquidated is equal to the difference between the present value of Essential Goal 1 ($3m) and the current liquid wealth ($2.15m). The proceeds of the liquidation are re-invested in the GBI strategy. The performance building block is the MSR and the GHP is a roll-over of 1-year indexed bonds which secures EG1. The floor is the present value of the minimum wealth level to achieve at the end of the current year. The portfolio is rebalanced every month. The left column shows the success probabilities for the goals, and the right column displays the expected maximum shortfall for goals EG1 and AG, and the expected maximum drawdown (see the caption of Figure 1 for the definition of the success indicators).

7. Figures and Tables

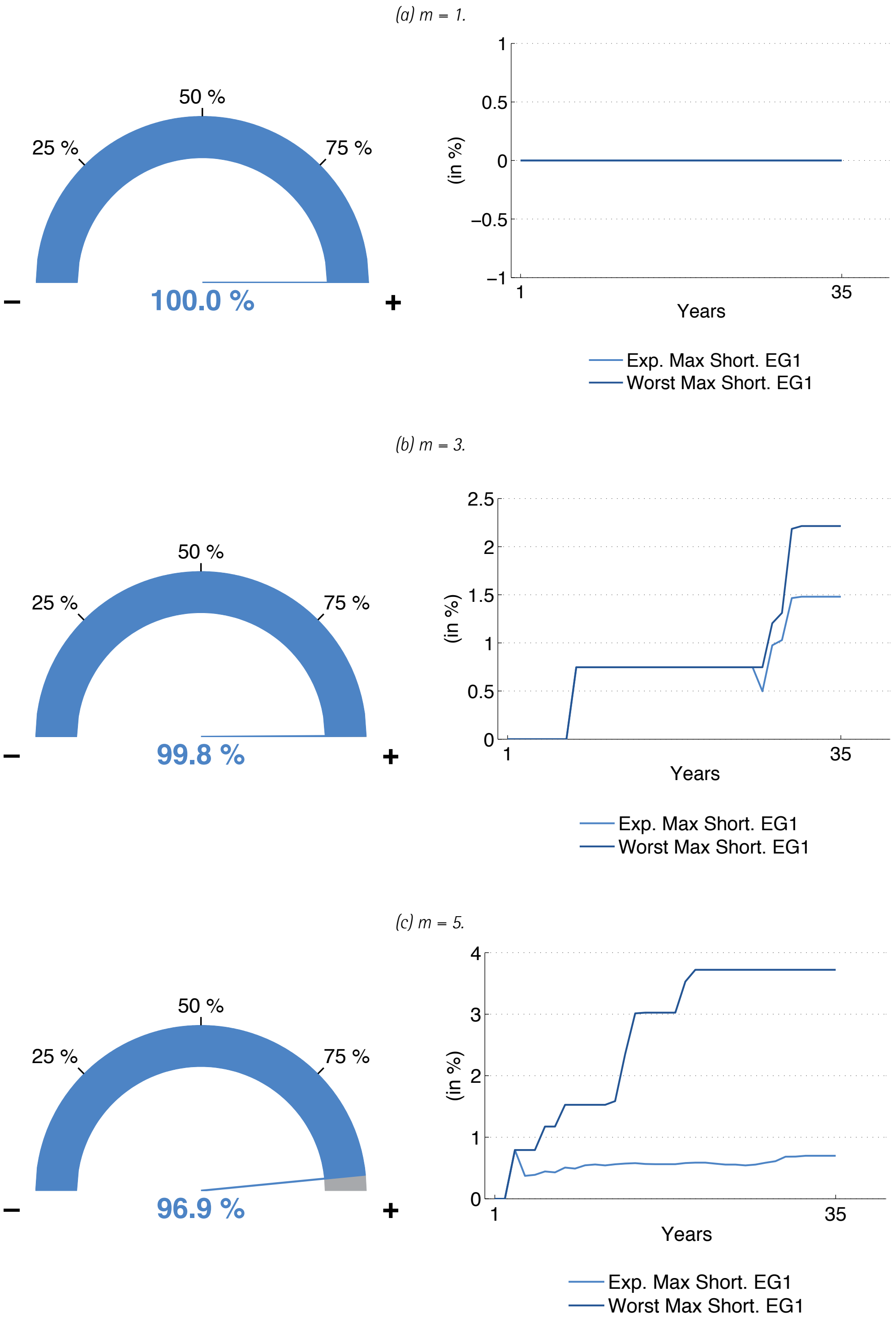
Figure 19: Investor 1 – Impact of multiplier on success probabilities for Essential Goal 1 with GBI strategy shorting the illiquid stock or a substitute.



This figure shows the impact of the multiplier on the success indicators with a dynamic GBI strategy of the form (32). Existing positions in stocks and bonds are liquidated at date 0, but aspirational assets are illiquid. The strategy shorts the illiquid stock or a substitute when its cushion is less than the aspirational wealth. In Panel (i), the shortable asset is the illiquid stock itself; in Panel (ii) it is the stock index, which has a correlation of 50% with this stock; in Panels (iii) and (iv) it is a substitute with higher correlation (75% or 90%). The portfolio is rebalanced every month.

7. Figures and Tables

Figure 20: Investor 1 – Impact of multiplier on success indicators for Essential Goal 1 with GBI strategy in the presence of taxes.

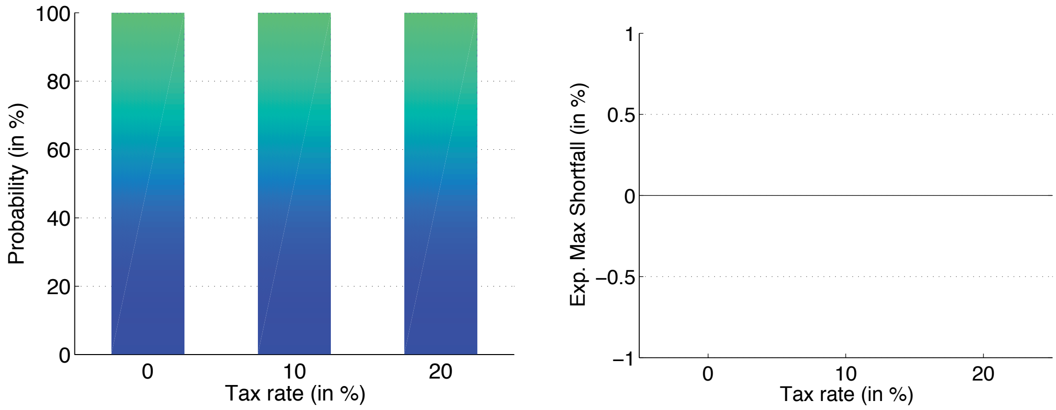


This figure shows the impact of the multiplier on the success indicators for Essential Goal 1 with a dynamic GBI strategy which aims at securing Essential Goal 1, when a 20% tax rate is applied to gains on the stock and the bond indices and gains on the roll-over operations on 1-year indexed bonds. Existing positions in stocks and bonds and aspirational assets are liquidated at date 0. The performance building block is the MSR and the GHP is a roll-over of 1-year indexed bonds with a face value of $3m. The floor is the present value of the minimum wealth level to achieve at the end of the current year, plus a tax provision. The multiplier is set to 1, 3 or 5. The left column shows the success probabilities and the right column shows the expected and worst maximum shortfalls.

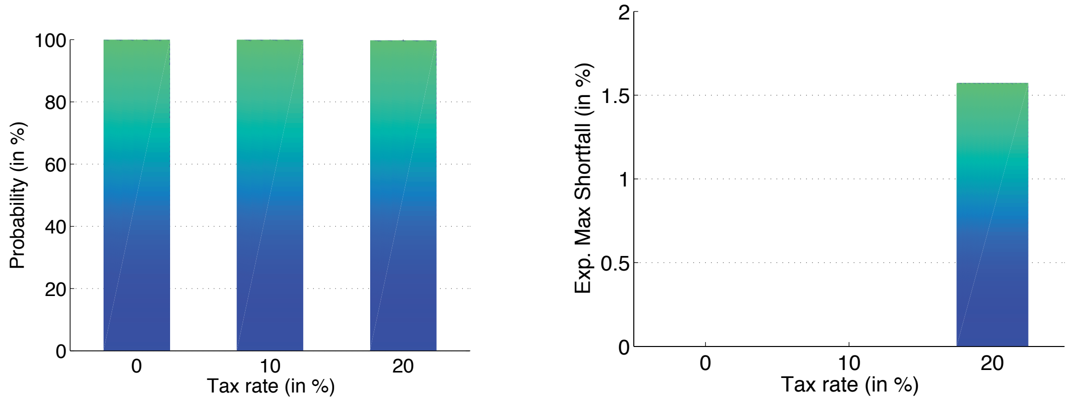
7. Figures and Tables

Figure 21: Investor 1 – Impact of tax rate on success indicators for Essential Goal 1 with GBI strategy.

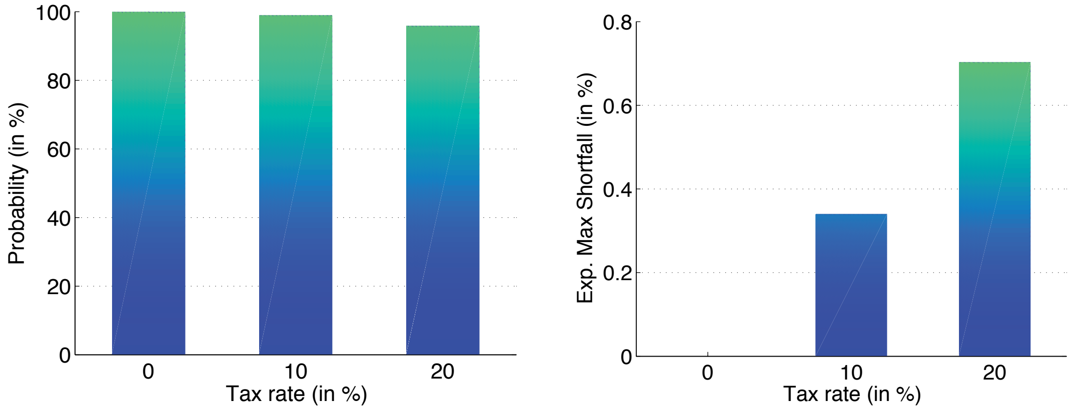
(a) m = 1.



(b) m = 3.



(c) m = 5.

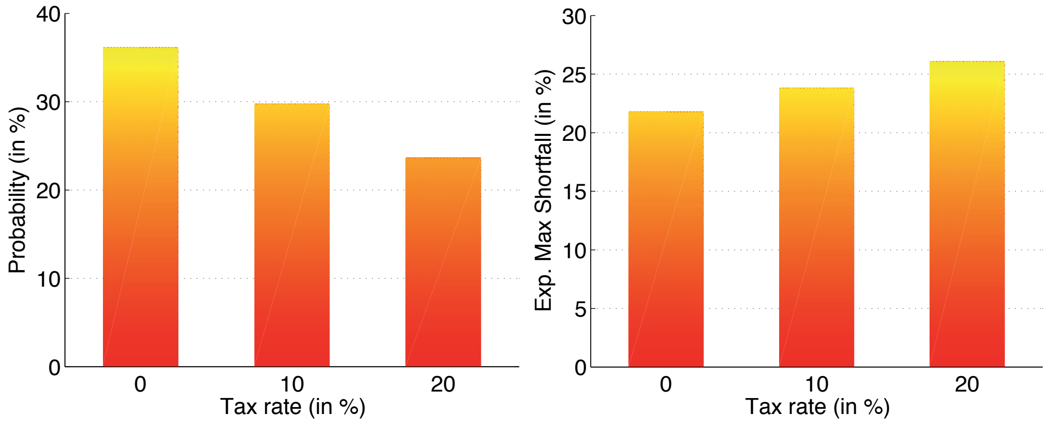


This figure shows the impact of the tax rate on the success indicators for Essential Goal 1 with a dynamic GBI strategy which aims at securing Essential Goal 1. Taxes are applied to gains on the stock and the bond indices and gains on the roll-over operations on 1-year indexed bonds. Existing positions in stocks and bonds and aspirational assets are liquidated at date 0. The performance building block is the MSR and the GHP is a roll-over of 1-year indexed bonds with a face value of $3m. The floor is the present value of the minimum wealth level to achieve at the end of the current year, plus a tax provision. The multiplier is set to 1, 3 or 5. The left column shows the success probabilities and the right column shows the expected maximum shortfalls.

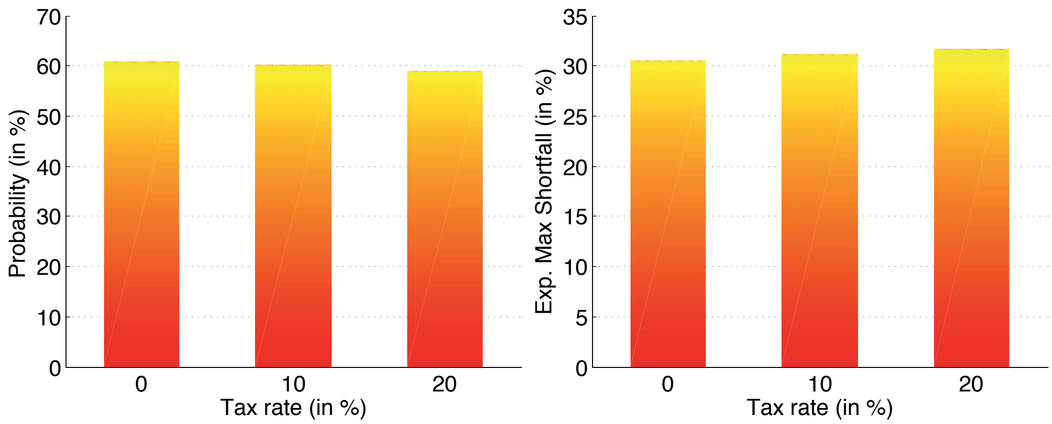
7. Figures and Tables

Figure 22: Investor 1 – Impact of tax rate on success indicators for Aspirational Goal with GBI strategy.

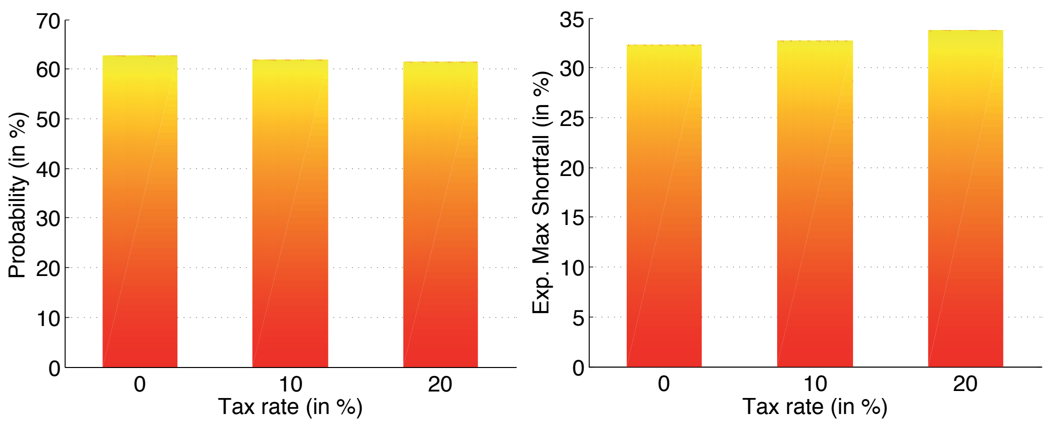
(a) m = 1.



(b) m = 3.



(c) m = 5



This figure shows the impact of the tax rate on the success indicators for Aspirational Goal with a dynamic GBI strategy which aims at securing Essential Goal 1. Taxes are applied to gains on the stock and the bond indices and gains on the roll-over operations on 1-year indexed bonds. Existing positions in stocks and bonds and aspirational assets are liquidated at date 0. The performance building block is the MSR and the GHP is a roll-over of 1-year indexed bonds with a face value of $3m. The floor is the present value of the minimum wealth level to achieve at the end of the current year, plus a tax provision. The multiplier is set to 1, 3 or 5. The left column shows the success probabilities and the right column shows the expected maximum shortfalls.

7. Figures and Tables

7.2. Case Study 2

Table 14: Investor 2 – Funding status of goals.

(i) Value of assets (in $).

Market wealth	1,400,000
Present value of guaranteed lifetime income	755,405
Total	**2,155,405**

(ii) Minimum capital required to secure one or more goal(s) (in $).

	Total Value (super-replication)	Net Value (super-replication)	Net Value (insurance)
Goal 1	1,461,467	732,093	732,093
Goal 2	362,835	336,805	309,032
G1 and G2	**1,824,302**	**1,068,898**	**1,041,125**
Goal 3	730,734	730,734	730,734
G1, G2 and G3	**2,555,036**	**1,799,632**	**1,771,859**

The assets of Investor 2 consist of liquid market assets and a claim on guaranteed lifetime income. Panel (ii) shows the minimum capital required to secure goals 1, 2 and 3 individually or jointly in three situations. In the first situation, the investor relies on both the market wealth and the income to secure the goals. In the second and third situations the investor uses the income to partially secure goals 1 and 2 (the present value of future income becomes equal to 0) but in situation two, goal 2 is fully super-replicated while in situation three, the same goal is 50% covered by insurance and 50% super-replicated. We notice that only goals 1 and 2 are affordable, hence can be considered as essential whereas goal 3 will remain aspirational. Also we notice that the minimum capital required to secure the long-term care contingencies (LTCC) which are represented as goal 2 is lower with the insurance.

Table 15: Investor 2 – Initial Risk Allocation of Strategy Securing Essential Goals 1 and 2 in the Absence of Insurance for EG2.

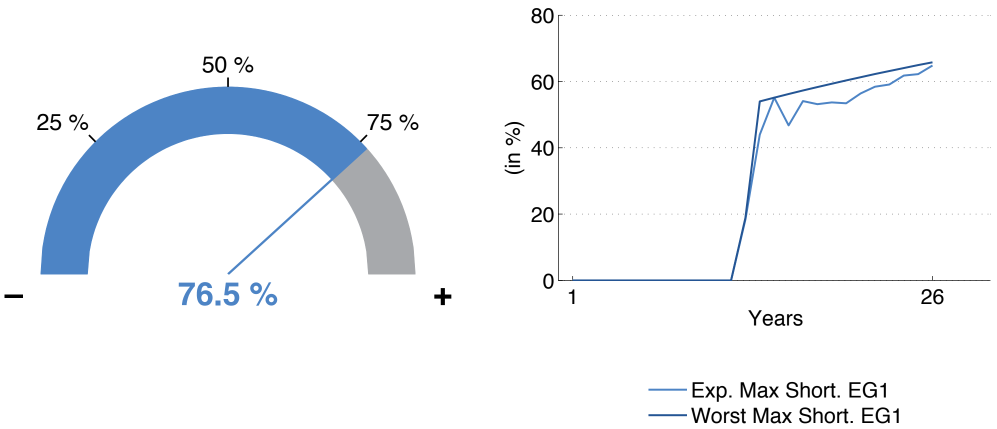
	Value ($)	% of Total		Value ($)	% of Total		Value ($)	% of Total
Personal Bucket	2,418,898	88.00	Market Bucket	331,102	12.0	Aspirational Bucket	0	0
Residence	900,000	37.2	US Equity	266,280	80.4			
Cash	450,000	18.6	US Fixed Income	64,822	19.6			
GHP: EG1+EG2	1,068,898	44.2						

This table shows the risk allocation at date 0 when the investor holds buy-and-hold positions in the MSR and the GHP for EG1+EG2, which is a bond that delivers the goal cash flows ($80,000 per year from years 1 to 26 and $100,000 per year from years 24 to 29). The personal bucket contains the assets that are used to finance investor's essential goals. The aspirational bucket is empty because income is used to secure EG1 and EG2 at the end of each year. The market bucket contains all other assets (equities and bonds here). The table displays the weights of the various assets within each bucket, as well as the relative weights of the buckets.

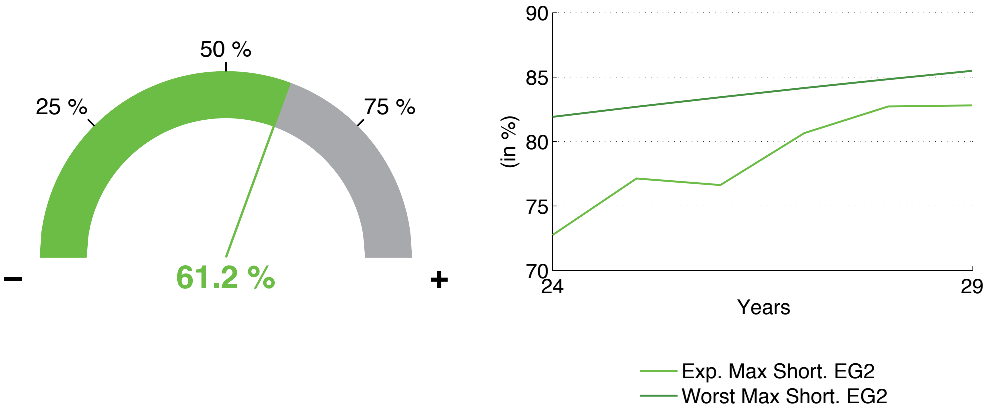
7. Figures and Tables

Figure 23: Investor 2 - Success indicators with current strategy.

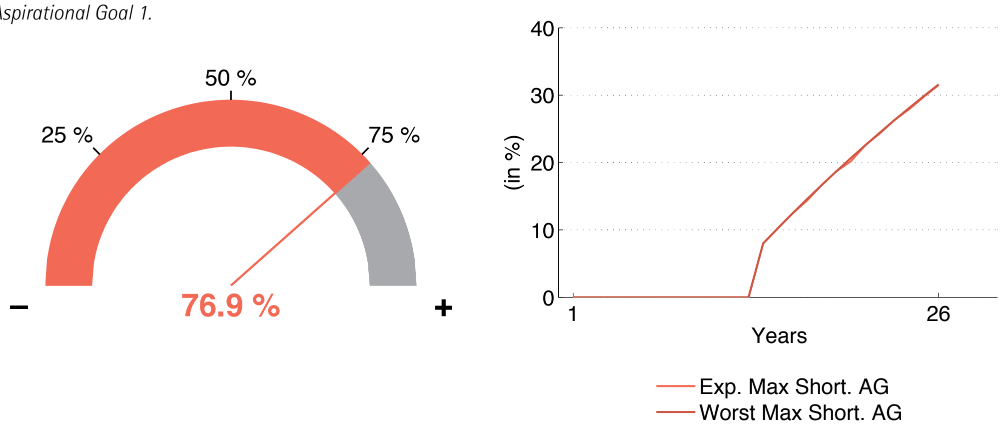
(a) Essential Goal 1.



(b) Essential Goal 2.



(c) Aspirational Goal 1.

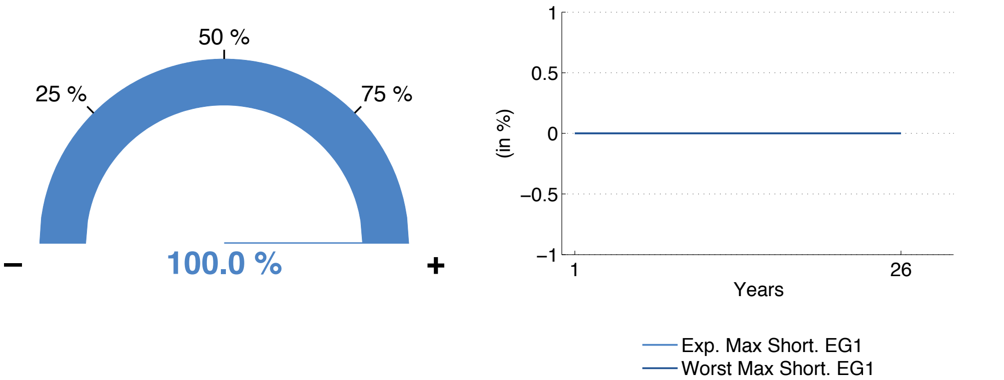


The half circles represent the success probabilities for each goal: the probability is estimated as the percentage of paths on which the goal was reached. The expected maximum shortfall on date t is the expectation of the value of the maximum relative deviation from the goal recorded by date t, conditional on the event of such a deviation. The worst maximum shortfall is defined as the worst relative loss recorded by date t across all dates and paths. The "current strategy" is a fixed-mix policy with annual rebalancing towards the current market allocation.

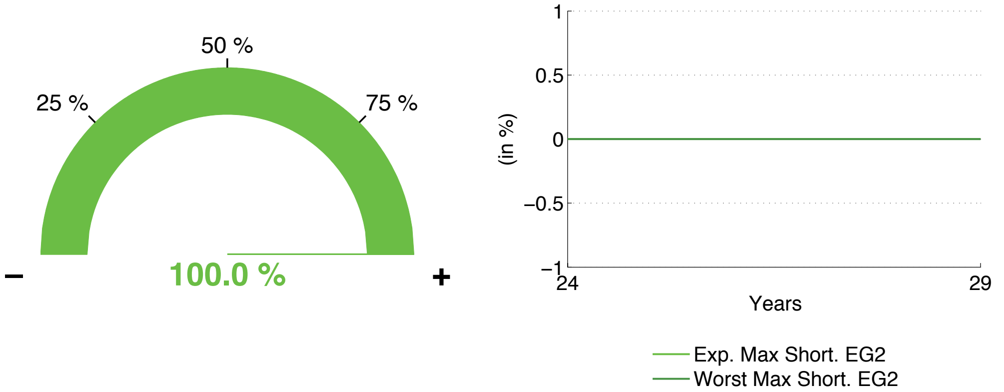
7. Figures and Tables

Figure 24: Investor 2 - Success indicators of strategy securing Essential Goals 1 and 2 in the absence of insurance for EG2.

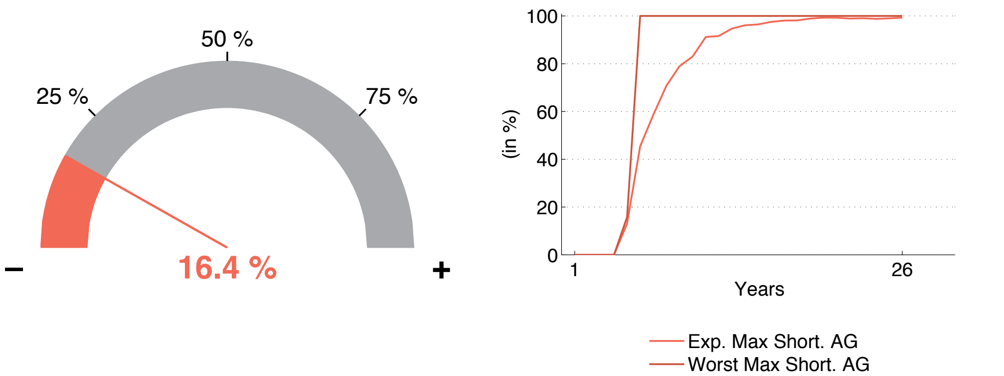
(a) Essential Goal 1.



(b) Essential Goal 2.



(c) Aspirational Goal 1.



This figure shows the success probabilities (left column) and the expected and the worst maximum shortfalls (right column) for a strategy securing Essential Goals 1 and 2 with a full super-replication of the long-term contingencies claims (EG2). The portfolio is buy-and-hold in the bond that pays the cash flows of Essential Goals 1 and 2.

7. Figures and Tables

Table 16: Investor 2 – Initial Risk Allocation of Strategy Securing Essential Goals 1 and 2 in the Presence of Insurance for EG2.

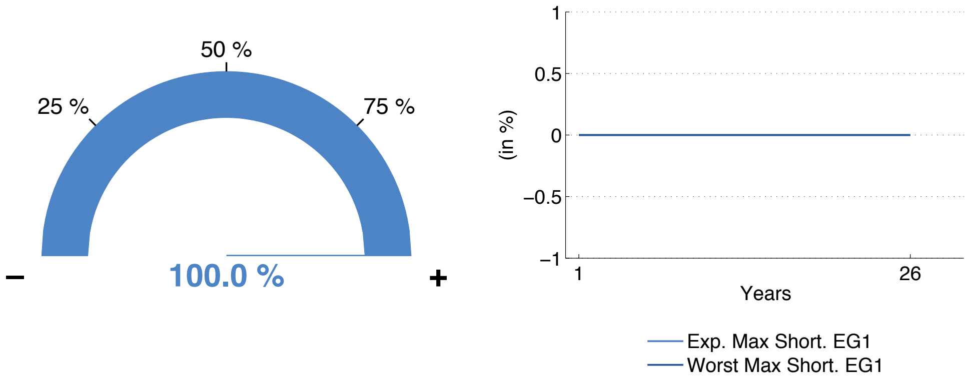
	Value ($)	% of Total		Value ($)	% of Total		Value ($)	% of Total
Personal Bucket	2,391,125	86.9	Market Bucket	358,875	13.1	Aspirational Bucket	0	0
Residence	900,000	37.7	US Equity	288,616	80.4			
Cash	450,000	18.8	US Fixed Income	70,259	19.6			
GHP: EG1+EG2	1,041,125	43.5						

This table shows the risk allocation at date 0 when the investor holds buy-and-hold positions in the MSR and the GHP for EG1+EG2, which is a bond that delivers the goal cash flows ($80,000 per year from years 1 to 26 and $50,000 per year from years 24 to 29 plus the insurance annuities). The personal bucket contains the assets that are used to finance investor's essential goals. The aspirational bucket it is empty because income is used to secure EG1 and EG2 at the end of each year. The market bucket contains all other assets (equities and bonds here). The table displays the weights of the various assets within each bucket, as well as the relative weights of the buckets.

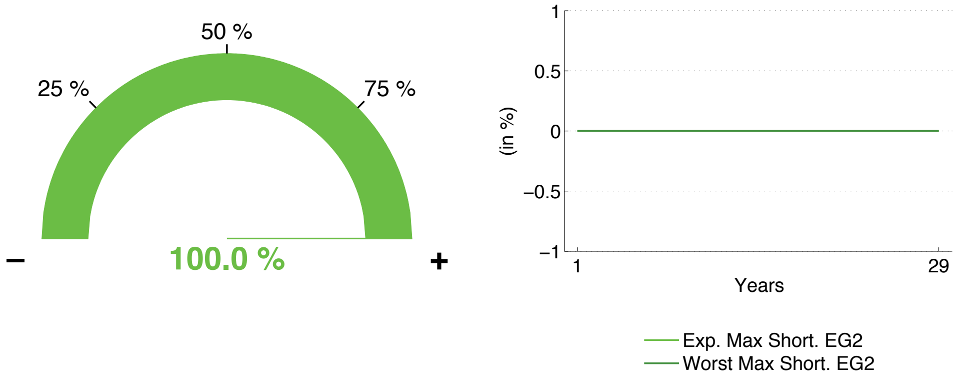
7. Figures and Tables

Figure 25: Investor 2 - Success indicators of strategy securing Essential Goals 1 and 2 in the presence of insurance for EG2.

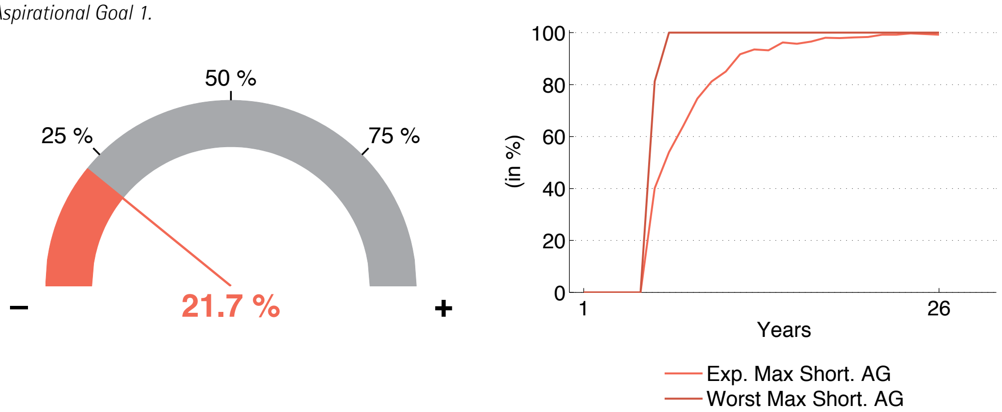
(a) Essential Goal 1.



(b) Essential Goal 2.



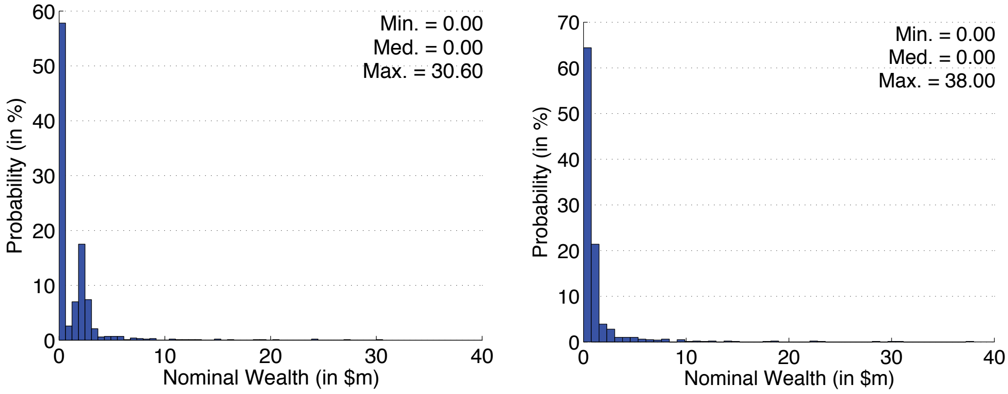
(c) Aspirational Goal 1.



This figure shows the success probabilities (left column) and the expected and the worst maximum shortfalls (right column) for a strategy securing Essential Goals 1 and 2 where the long-term contingencies claims (EG2) are 50% covered by insurance and the remaining 50% are super-replicated. The portfolio is buy-and-hold in the bond that pays the cash flows of Essential Goals 1 and 2.

7. Figures and Tables

Figure 26: Investor 2 – Terminal wealth of strategy securing Essential Goals 1 and 2 with both strategies for EG2 (no insurance in the left column and insurance in the right column).



This figure shows the distribution of the terminal wealth on year 29 considering two strategies for the long-term contingencies claims: super-replication in left column and insurance in right column. The portfolio is buy-and-hold in the bond that pays the cash flows of Essential Goals 1 and 2.

Table 17: Investor 2 – Funding status of goals in the presence of a minimum wealth goal (goal 4)

(i) Value of assets (in $).

Market wealth	1,400,000
Present value of guaranteed lifetime income	755,405
Total	**2,155,405**

(ii) Minimum capital required to secure one or more goal(s) (in $).

	Total Value	Net Value (surreplication)	Net Value (insurance)
EG1	1,461,467	732,093	732,093
EG2	362,835	336,805	309,032
Goal 4	214,792	214,792	214,792
EG1, EG2 and G4	**2,039,094**	**1,283,690**	**1,255,917**
AG1	730,734	730,734	730,734
EG1, EG2, G4 and AG1	**2,769,828**	**2,014,424**	**1,986,651**

The assets of Investor 2 consist of liquid market assets and a claim on guaranteed lifetime income. Panel (ii) shows the minimum capital required to secure each goal individually or jointly in three situations (the two last situations use the income to secure EG1 and EG2, hence reduce to zero the present value of income after netting operations). We notice that goal 4 is affordable together with EG1 and EG2 therefore it will be considered as essential goal EG3 whereas AG1 will remain aspirational.

7. Figures and Tables

Table 18: Investor 2 – Initial Risk Allocations with Strategies Securing Essential Goals 1, 2 and 3.

(a) EG2 without Insurance

	Value ($)	% of Total		Value ($)	% of Total		Value ($)	% of Total
Personal Bucket	2,633,690	95.8	Market Bucket	116,310	4.2	Aspirational Bucket	0	0
Residence	900,000	34.2	US Equity	93,539	80.4			
Cash	450,000	17.1	US Fixed Income	22,771	19.6			
GHP: EG1+EG2+EG3	1,283,690	48.7						

(b) EG2 with Insurance

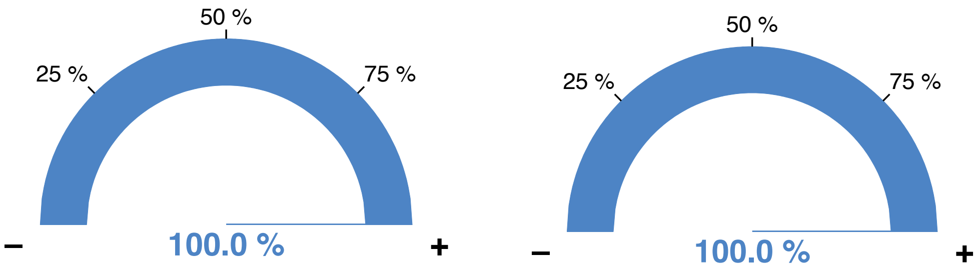
	Value ($)	% of Total		Value ($)	% of Total		Value ($)	% of Total
Personal Bucket	2,605,917	94.8	Market Bucket	144,083	5.2	Aspirational Bucket	0	0
Residence	900,000	34.6	US Equity	115,875	80.4			
Cash	450,000	17.3	US Fixed Income	28,208	19.6			
GHP: EG1+EG2+EG3	1,255,917	48.2						

This table shows the risk allocation at date 0 for two strategies securing Essential Goals 1, 2 and 3. In Panel (a), the investor purchases a bond that pays the cash flows of EG1 and EG2 and delivers the minimum wealth level of EG3 on year 29 in the absence of insurance for EG2. In Panel (b), the investor partially secures EG2 with insurance, and purchase a bond that pays the desired cash flows for EG1, EG2 and delivers the minimum wealth level of EG3 on year 29. The minimum wealth at date 29 has been set equal to the initial market wealth, i.e. equal to $1.4million.

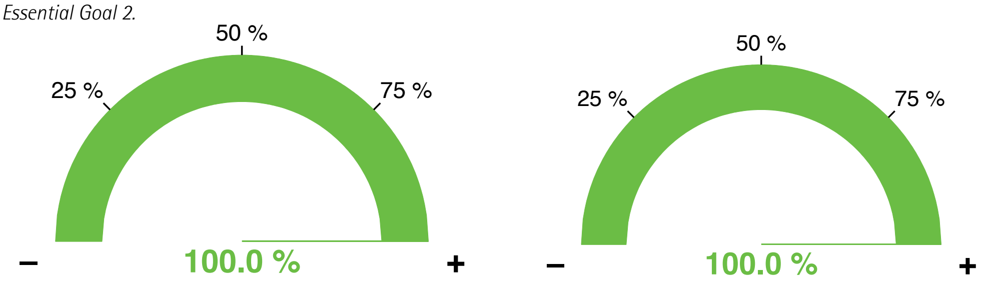
7. Figures and Tables

Figure 27: Investor 2 - Success indicators of strategy securing Essential Goals 1, 2 and 3 with both strategies for LTCC (no insurance in the left column and insurance in the right column).

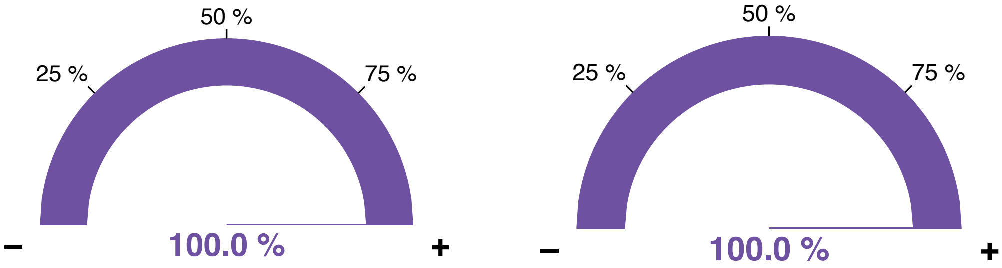
(a) Essential Goal 1.



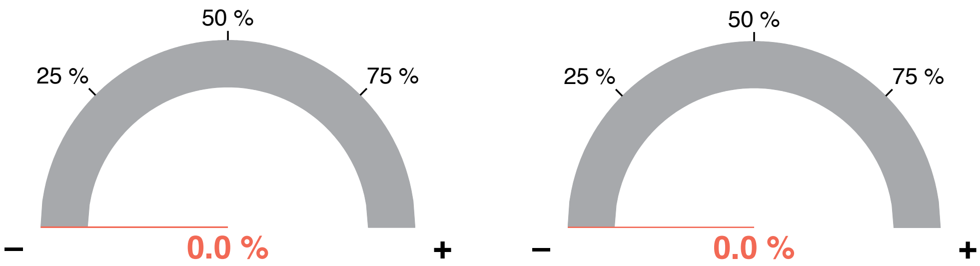
(b) Essential Goal 2.



(c) Essential Goal 3.



(d) Aspirational Goal 1.

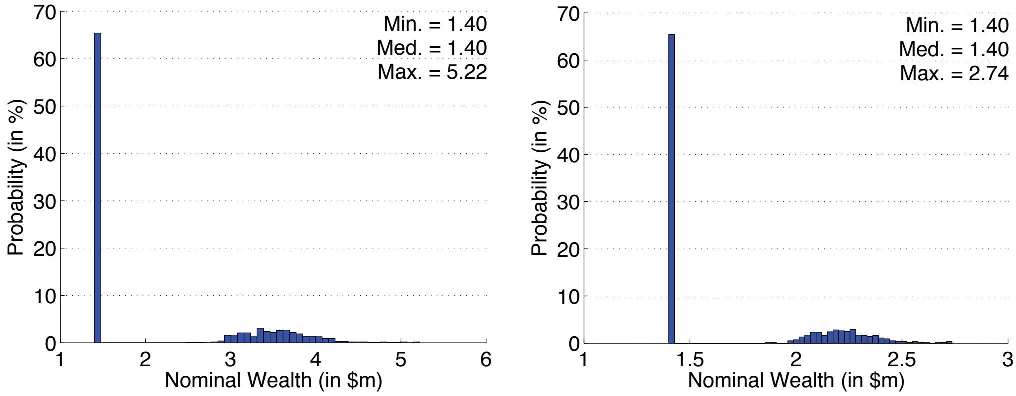


This figure shows the success probabilities of two strategies for the long-term contingencies claims: super-replication in left column and insurance in right column. The portfolio is buy-and-hold in the bond that pays the cash flows of Essential Goals 1, 2 and secures EG3 which consist of reaching a minimum wealth equal to $1.4million at year 29.

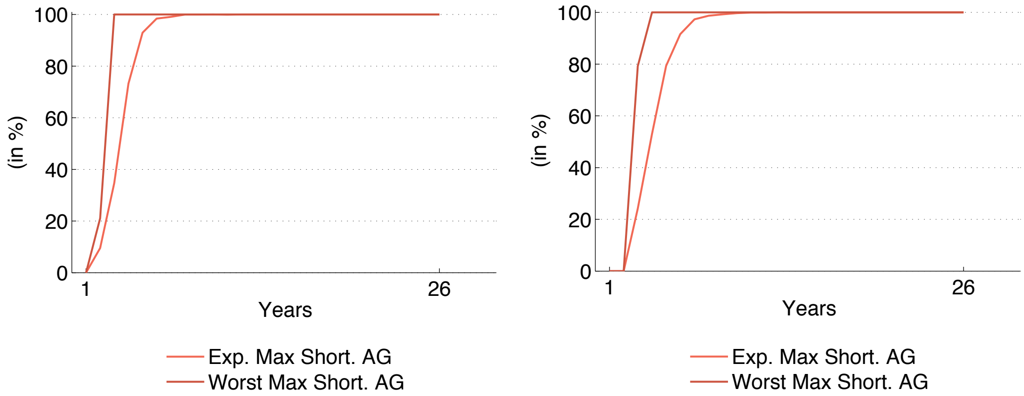
7. Figures and Tables

Figure 28: Investor 2 – Terminal wealth and shortfall indicators of strategies securing Essential Goals 1, 2 and 3 (no insurance in the left column and insurance in the right column for LTCC).

(a) Terminal Wealth.



(b) Shortfall Indicator for AG1

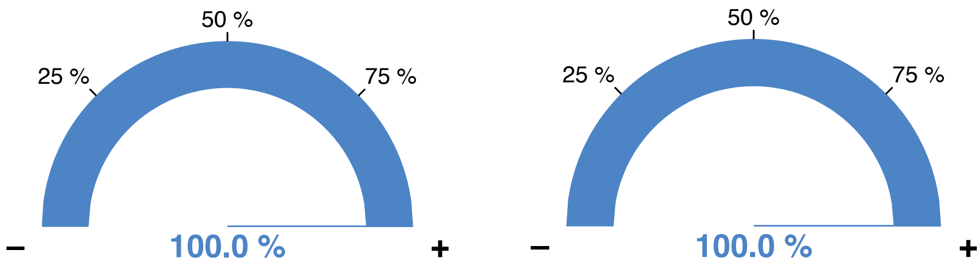


This figure shows the distribution of the terminal wealth on year 29 considering two strategies for the long-term contingencies claims: super-replication in left column and insurance in right column. It also shows the shortfall indicators for AG1 in both strategies. The portfolio is buy-and-hold in the bond that pays the cash flows of Essential Goals 1, 2 and secures EG3 which consist of reaching a minimum wealth equal to $1.4million at year 29.

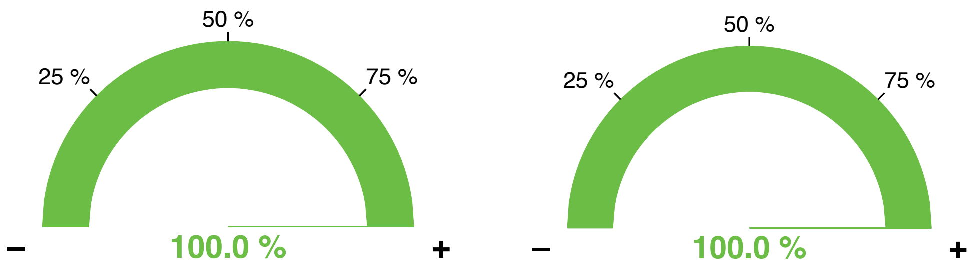
7. Figures and Tables

Figure 29: Investor 2 - Success indicators of strategy securing Essential Goals 1 and 2 with super-replication for LTCC (no tax in the left column and 20% tax rate in the right column).

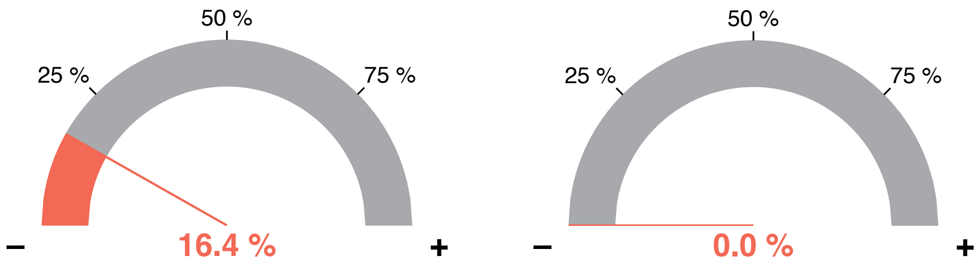
(a) Essential Goal 1.



(b) Essential Goal 2.



(c) Aspirational Goal 1.



This figure shows the success probabilities of the strategy securing EG1 and EG2 (with super-replication) in the presence of tax (right column) and in the absence of tax (left column). The portfolio is buy-and-hold in the bond that pays the cash flows of Essential Goals 1 and 2.

7. Figures and Tables

7.3. Case Study 3

Table 19: Investor 3 – Funding status of goals.

(i) Value of assets (in $).

Market wealth	940,000
Present value of future savings	181,506
Total	**1,121,506**

(ii) Minimum capital required to secure one or more goal(s) (in $).

	Liquid wealth only	Income and zero re-investment rate	Income and compound option	Income and forward contracts
Goal 1	**810,256**	**709,181**	V_0	**628,750**
Goal 2	169,893	169,893	169,893	169,893
G1 and G2	**980,149**	**879,074**	**169,893 +** V_0	**798,643**
Goal 3	305,274	305,274	305,274	305,274
G1, G2 and G3	**1,285,424**	**1,184,348**	**475,167 +** V_0	**1,103,917**

The assets of Investor 3 consist of liquid market assets and a claim on future savings net of mortgage annuities. Panel (ii) shows the minimum capital required to secure goals 1, 2 and 3 individually or jointly in four situations. In the second column, the investor relies only on liquid wealth. In the third column, he uses income to secure Goal 1, assuming a zero re-investment rate for future income payments, and purchases an option to make up for the excess of consumption over income. In the fourth column, he uses income and a compound option (see Proposition 8). In the fifth column, he uses income assuming that future income will be re-invested at the forward rate fixed at date 0, and purchases an option to make up for the excess consumption. Goals are ranked by decreasing priority order.

Table 20: Investor 3 – Initial Risk Allocation with Strategy Securing Essential Goal 1 with Liquid Wealth Only.

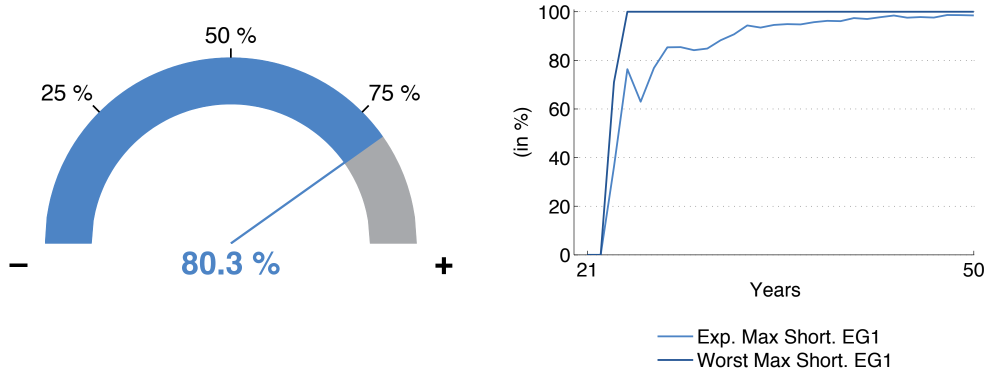
	Value ($)	% of Total		Value ($)	% of Total		Value ($)	% of Total
Personal Bucket	870,256	73.7	Market Bucket	129,744	11.0	Aspirational Bucket	181,506	15.3
Residence	300,000	21.9	Equity	104,343	80.4	Present value of savings	181,506	100.0
Cash	10,000	0.7	US Fixed Income	25,401	19.6			
GHP EG1	810,256	59.1						
Adjustable Rate Mortgage	(250,000)	18.2						

This table shows the risk allocation at date 0 when the investor holds buy-and-hold positions in the MSR and the GHP for EG1, which is a bond that delivers the goal cash flows ($90,000 per year from years 21 to 50). The personal bucket contains the assets that are used to finance investor's essential goals. The aspirational bucket contains assets held for wealth mobility purposes: it consists here of a claim on future savings net of mortgage annuities. The market bucket contains all other assets (equities and bonds here). The table displays the weights of the various assets within each bucket, as well as the relative weights of the buckets.

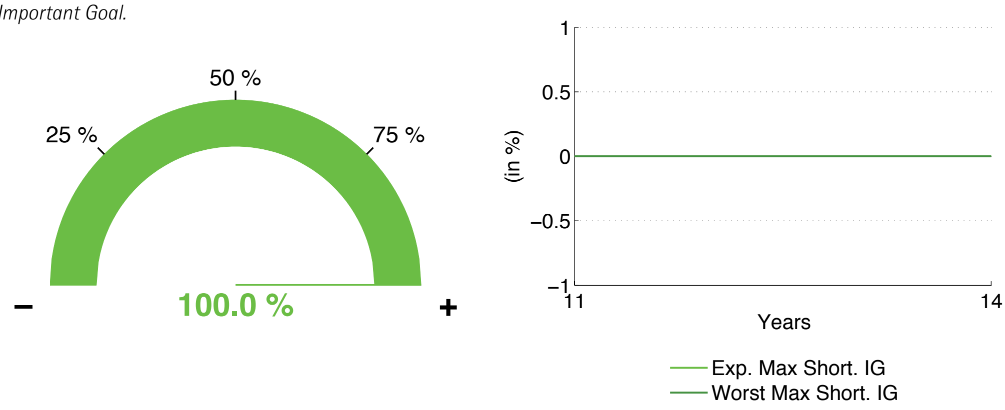
7. Figures and Tables

Figure 30: Investor 3 - Success indicators with current strategy.

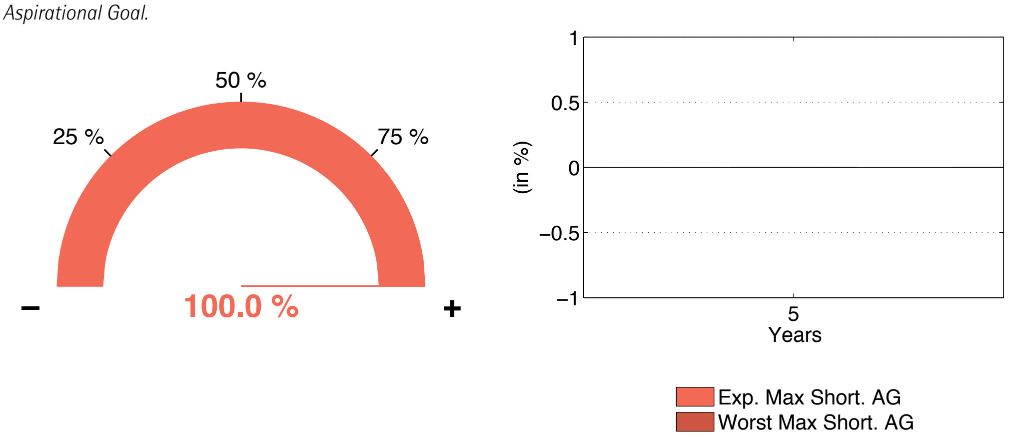
(a) Essential Goal 1.



(b) Important Goal.



(c) Aspirational Goal.

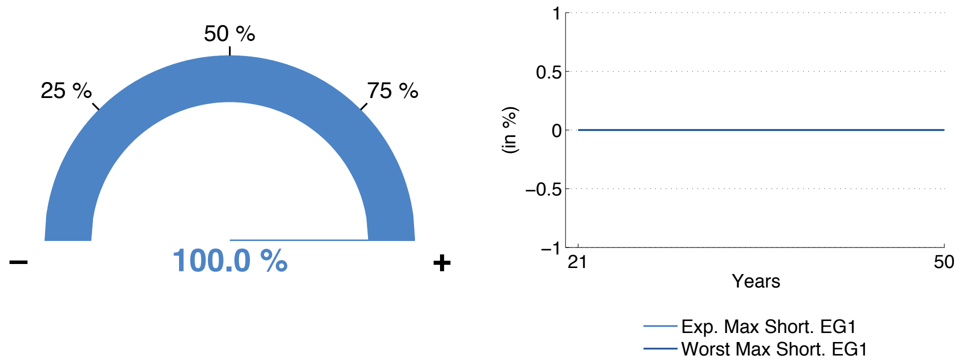


The half circles represent the success probabilities for each goal: the probability is estimated as the percentage of paths on which the goal was reached. The expected maximum shortfall on date t is the expectation of the value of the maximum relative deviation from the goal recorded by date t, conditional on the event of such a deviation. The worst maximum shortfall is defined as the worst relative loss recorded by date t across all dates and paths. The "current strategy" is a fixed-mix policy with annual rebalancing towards the current market allocation.

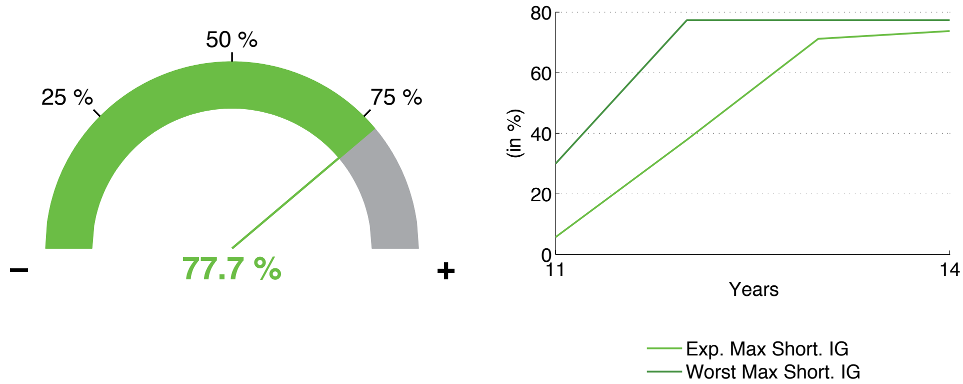
7. Figures and Tables

Figure 31: Investor 3 - Success indicators with strategy securing Essential Goal 1 with liquid wealth only.

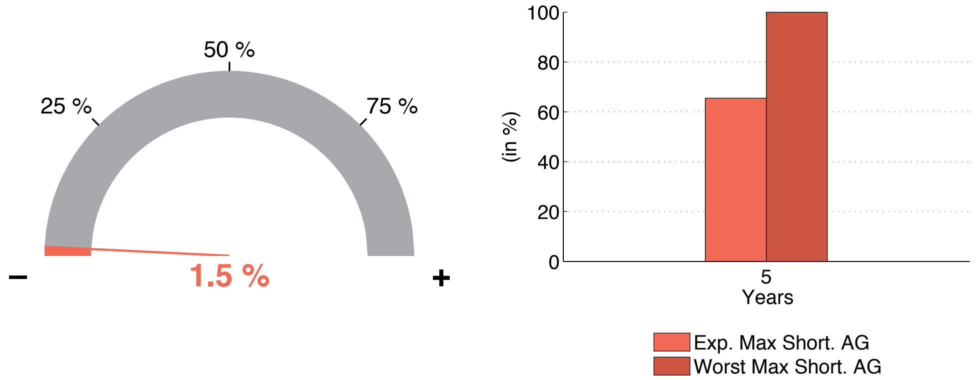
(a) Essential Goal 1.



(b) Important Goal.



(c) Aspirational Goal.



This figure shows the success probabilities (left column) and the expected and the worst maximum shortfalls (right column) for a strategy securing Essential Goal 1 with liquid wealth only. The portfolio is buy-and-hold in the MSR of stocks and bonds and the bond that pays the cash flows of Essential Goal 1. Success indicators are defined in the caption of Figure 30.

7. Figures and Tables

Table 21: Investor 3 – Initial Risk Allocation with Strategy Securing Essential Goal 1 with Income, Assuming a Zero Re-investment Rate.

	Value ($)	% of Total		Value ($)	% of Total		Value ($)	% of Total
Personal Bucket	769,181	76.9	Market Bucket	230,819	23.1	Aspirational Bucket	0	0
Residence	300,000	23.6	Equity	185,630	80.4			
Cash	10,000	0.8	US Fixed Income	45,189	19.6			
GHP EG1	709,181	55.9						
Adjustable Rate Mortgage	(250,000)	19.7						

This table shows the risk allocation at date 0 when the investor partially secures Essential Goal 1 with income, assumed to be re-invested at a zero rate. The remaining fraction of the goal is secured by purchasing an option (GHP EG1). The personal bucket contains the assets that are used to finance investor's essential goals. The aspirational bucket is empty because income is devoted to the protection an essential goal. The market bucket contains all other assets (equities and bonds here). The table displays the weights of the various assets within each bucket, as well as the relative weights of the buckets.

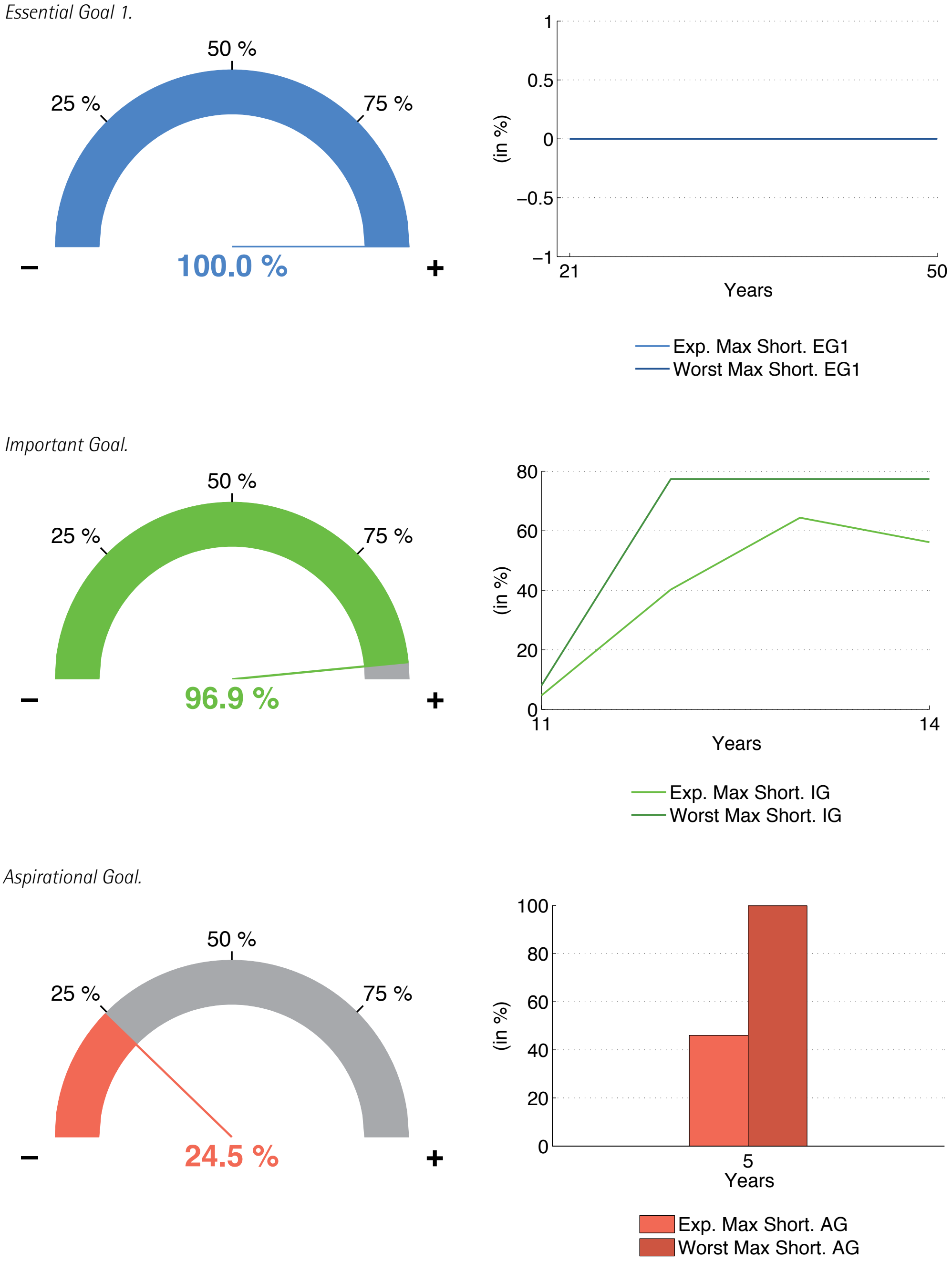
Table 22: Investor 3 – Initial Risk Allocation with Strategy Securing Essential Goal 1 with Income and Forward Contracts.

	Value ($)	% of Total		Value ($)	% of Total		Value ($)	% of Total
Personal Bucket	688,750	68.9	Market Bucket	311,250	31.1	Aspirational Bucket	0	0
Residence	300,000	25.2	Equity	250,315	80.4			
Cash	10,000	0.8	US Fixed Income	60,935	19.6			
GHP EG1	628,750	52.9						
Adjustable Rate Mortgage	(250,000)	21.0						

This table shows the risk allocation at date 0 when the investor partially secures Essential Goal 1 with income, when the income payments are re-invested at forward rates specified at time 0. The remaining fraction of the goal is secured by purchasing an option (GHP EG1). The personal bucket contains the assets that are used to finance investor's essential goals. The aspirational bucket is empty because income is devoted to the protection an essential goal. The market bucket contains all other assets (equities and bonds here). The table displays the weights of the various assets within each bucket, as well as the relative weights of the buckets.

7. Figures and Tables

Figure 32: Investor 3 - Success indicators with strategy securing Essential Goal 1 with income, assuming a zero re-investment rate for income.

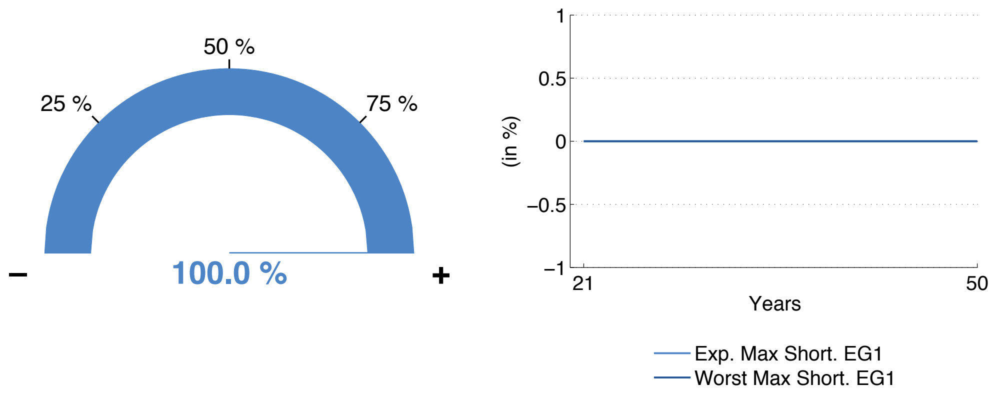


This figure shows the success probabilities (left column) and the expected and the worst maximum shortfalls (right column) for a strategy securing Essential Goal 1 with income, assuming a zero re-investment rate for income. For years 1 to 20, the portfolio is buy-and-hold in the MSR of stocks and bonds and the option that secures the fraction of Essential Goal 1 not covered by income. As of year 21, the option is replaced by the bond that pays the cash flows of Essential Goal 1. Success indicators are defined in the caption of Figure 30.

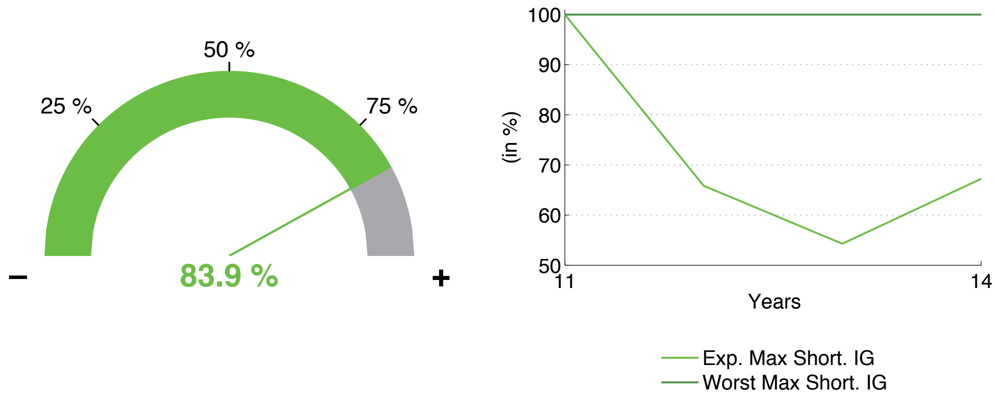
7. Figures and Tables

Figure 33: Investor 3 - Success indicators with strategy securing Essential Goal 1 with income and forward contracts.

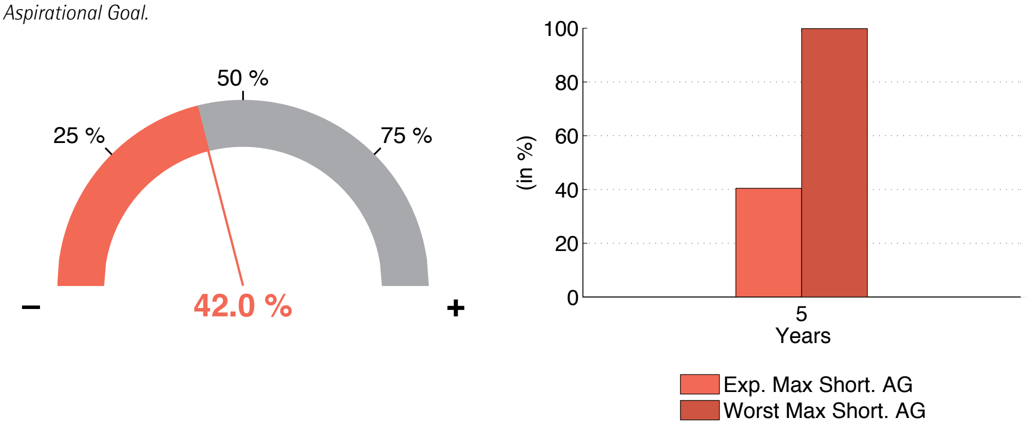
(a) Essential Goal 1.



(b) Important Goal.



(c) Aspirational Goal.

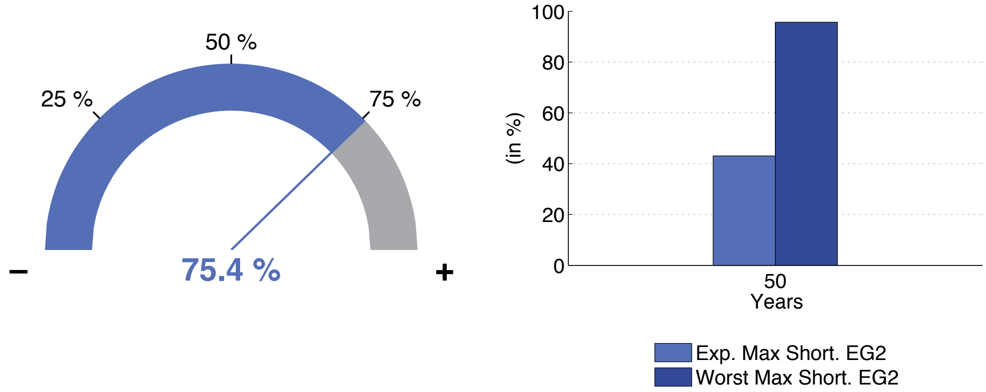


This figure shows the success probabilities (left column) and the expected and the worst maximum shortfalls (right column) for a strategy securing Essential Goal 1 with income when the income payments are re-invested at forward rates specified at date 0. For years 1 to 20, the portfolio is buy-and-hold in the MSR of stocks and bonds and the option that secures the fraction of Essential Goal 1 not covered by income. As of year 21, the option is replaced by the bond that pays the cash flows of Essential Goal 1. Success indicators are defined in the caption of Figure 30.

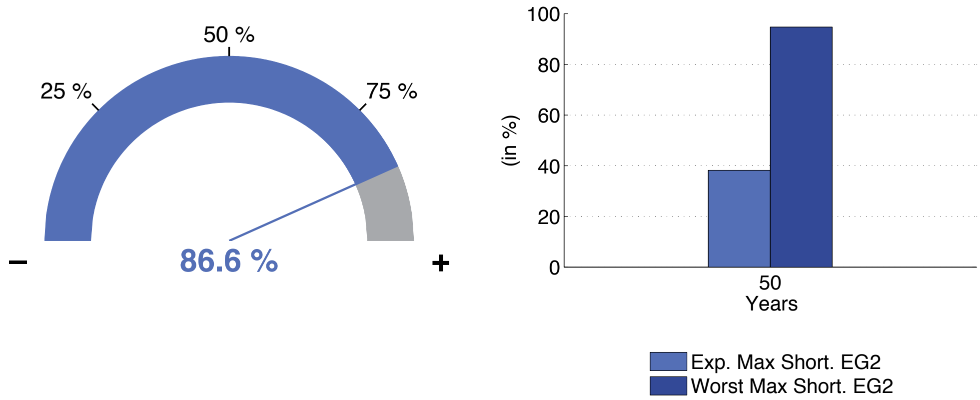
7. Figures and Tables

Figure 34: Investor 3 - Success indicators for wealth-based goal with strategies securing Essential Goal 1.

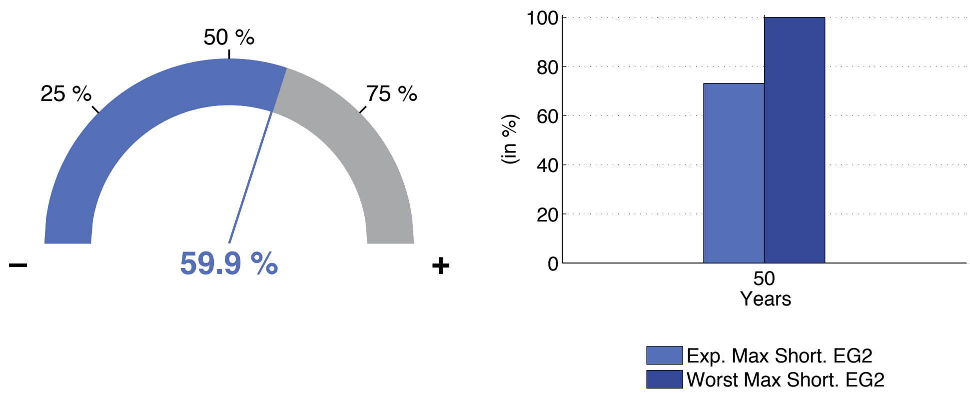
(a) Protection with liquid wealth only (LIQ).



(b) Protection with income, assuming a zero re-investment rate (INC-ZER-RET).



(c) Protection with income and forward contracts (INC-FWD).



This figure shows the success probabilities (left column) and the expected and the worst maximum shortfalls (right column) with respect to the goal of protecting the initial liquid wealth ($940,000) plus inflation at the 50-year horizon. The strategy LIQ (Panel (a)) protects Essential Goal 1 by purchasing a bond that pays the goal cash flows. The strategy INC-ZER-RET (Panel (b)) protects EG1 with income, assuming that future income payments will be re-invested at a zero rate. The fraction of EG1 not covered by income is secured by purchasing an option. The strategy INC-FWD (Panel (c)) protects EG1 with income, and the re-investment rate for the income payments is fixed at date 0 by the means of forward contracts. Again, the remaining fraction of EG1 is secured with an option.

7. Figures and Tables

Table 23: Investor 3 – Funding status of goals in the presence of a minimum wealth goal.

(i) Value of assets (in $).

Market wealth	940,000
Present value of future savings	181,506
Total	**1,121,506**

(ii) Minimum capital required to secure one or more goal(s) (in $).

	Liquid wealth only	Income and zero re-investment rate	Income and forward contracts
Goal 1	810,256	709,181	628,750
Goal 4	106,733	106,733	106,733
G1 and G4	**916,989**	**815,914**	**735,482**
Goal 2	169,893	169,893	169,893
G1, G4 and G2	**1,086,882**	**985,807**	**905,376**
Goal 3	305,274	305,274	305,274
All goals	**1,392,156**	**1,291,081**	**1,210,650**

Panel (i) shows the value of investor's assets and Panel (ii) the minimum capital required to secure the various goals when the investor expresses a new goal (Goal 4), which is to protect the initial liquid capital at the horizon of 50 years. This goal has intermediate priority rank between EG1 and IG. In the second column of Panel (ii), the investor relies only on liquid wealth. In the third column, he uses income to secure EG1 and Goal 4, assuming a zero re-investment rate for future income payments, and purchases an option to make up for the uncovered fraction of the goal. In the fourth column, he uses income assuming that future income will be re-invested at the forward rate fixed at date 0, and purchases an option to make up for the uncovered fraction. Goals are ranked by decreasing priority order.

7. Figures and Tables

Table 24: Investor 3 – Initial Risk Allocations with Strategies Securing Essential Goals 1 and 2.

(a) Protection with liquid wealth only.

	Value ($)	% of Total		Value ($)	% of Total		Value ($)	% of Total
Personal Bucket	976,989	82.7	Market Bucket	23,011	1.9	Aspirational Bucket	181,506	15.4
Residence	300,000	20.3	Equity	18,506	80.4	Present value of savings	181,506	100
Cash	10,000	0.7	US Fixed Income	4,505	19.6			
GHP EG1 and EG2	916,989	62.1						
Adjustable Rate Mortgage	(250,000)	16.9						

(b) Protection with income re-invested at zero rate.

	Value ($)	% of Total		Value ($)	% of Total		Value ($)	% of Total
Personal Bucket	875,914	87.6	Market Bucket	124,086	12.4	Aspirational Bucket	0	0
Residence	300,000	21.8	Equity	99,793	80.4			
Cash	10,000	0.7	US Fixed Income	24,293	19.6			
GHP EG1 and EG2	815,914	59.3						
Adjustable Rate Mortgage	(250,000)	18.2						

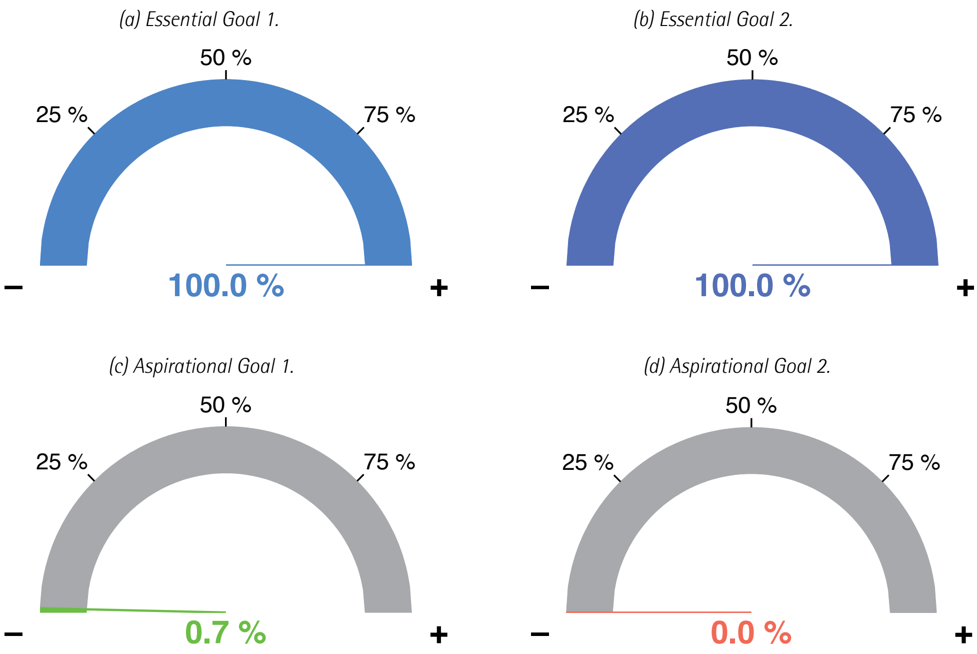
(c) Protection with income and forward contracts.

	Value ($)	% of Total		Value ($)	% of Total		Value ($)	% of Total
Personal Bucket	795,482	79.5	Market Bucket	204,518	20.5	Aspirational Bucket	0	0
Residence	300,000	23.2	Equity	164,478	80.4			
Cash	10,000	0.8	US Fixed Income	40,040	19.6			
GHP EG1 and EG2	735,482	56.8						
Adjustable Rate Mortgage	(250,000)	19.2						

This table shows the risk allocation at date 0 for three strategies securing Essential Goals 1 and 2. In Panel (a), the investor purchases a bond that pays EG1 cash flows for years 21 to 50 and delivers the minimum wealth level of EG2 on year 50. This bond is denoted "GHP EG1 and EG2". In Panel (b), the investor partially secures the two essential goals with income, assuming a zero re-investment rate for future income, and secures the uncovered fraction of the goals by purchasing an option. In Panel (c), he fixes at date 0 the re-investment rate by entering forward contracts and he purchases another option to complete the protection. The personal bucket contains the assets that are used to finance investor's essential goals. The aspirational bucket contains a claim on future savings except when income is devoted to the protection of such essential goals. The market bucket contains all other assets (equities and bonds here). The table displays the weights of the various assets within each bucket, as well as the relative weights of the buckets.

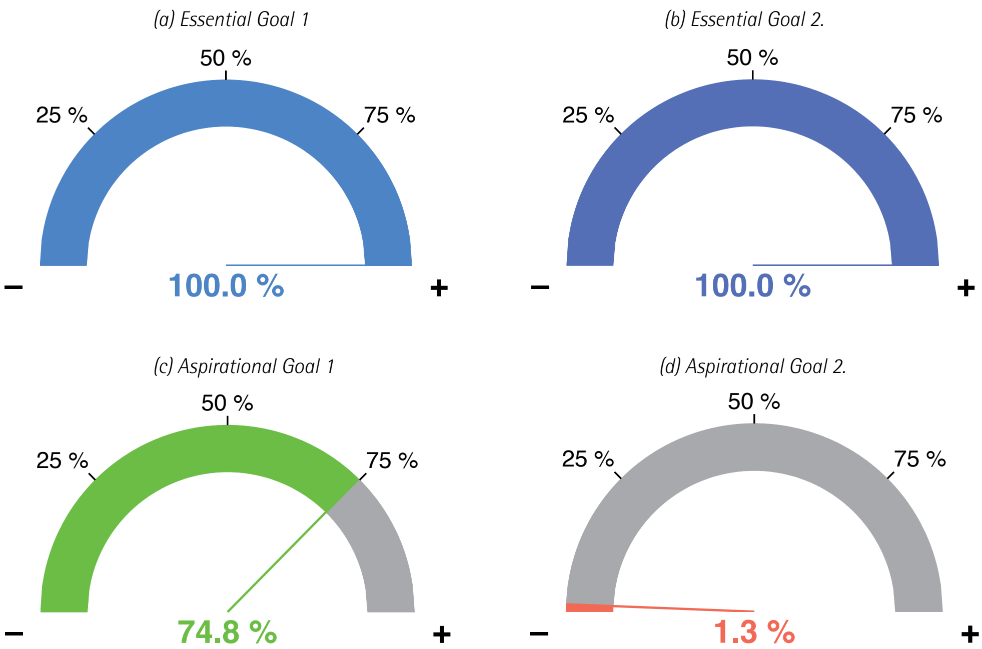
7. Figures and Tables

Figure 35: Investor 3 - Success probabilities with strategy securing Essential Goals 1 and 2 with liquid wealth only.



This figure shows the success probabilities for a strategy securing Essential Goals 1 and 2 with liquid wealth only. The portfolio is buy-and-hold in the MSR of stocks and bonds and the bond that pays the cash flows of Essential Goal 1 for years 21 to 50 plus the minimum wealth level in year 50.

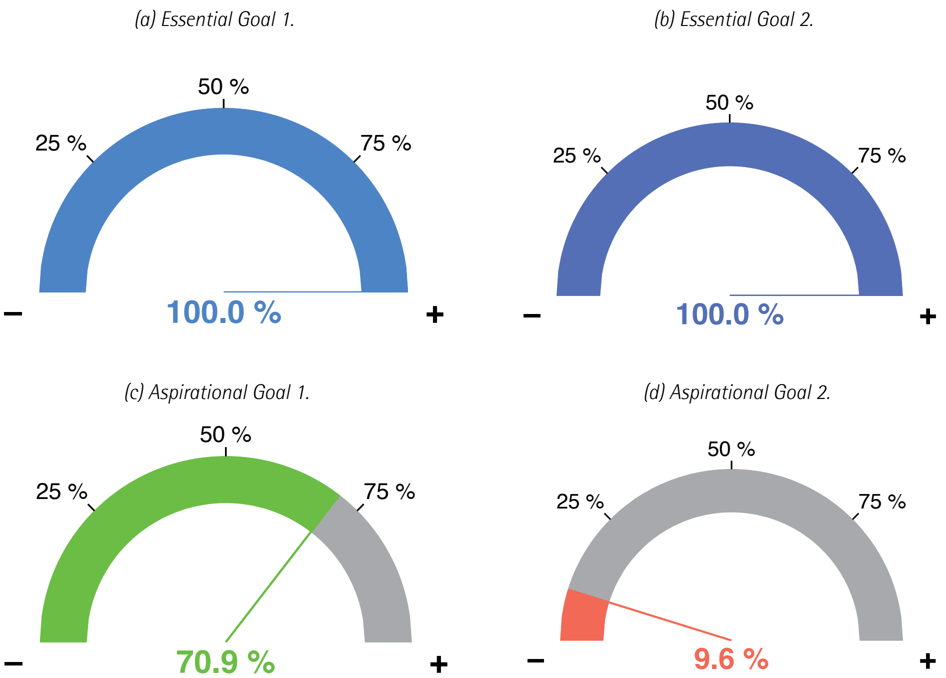
Figure 36: Investor 3 - Success probabilities with strategy securing Essential Goals 1 and 2 with income, assuming a zero re-investment rate for income.



This figure shows the success probabilities for a strategy securing Essential Goals 1 and 2 with income, assuming a zero re-investment rate for future income. The fraction of the goals not covered by income is protected with an option maturing at date 21 (the retirement date). The portfolio is buy-and-hold in the MSR of stocks and bonds and the option.

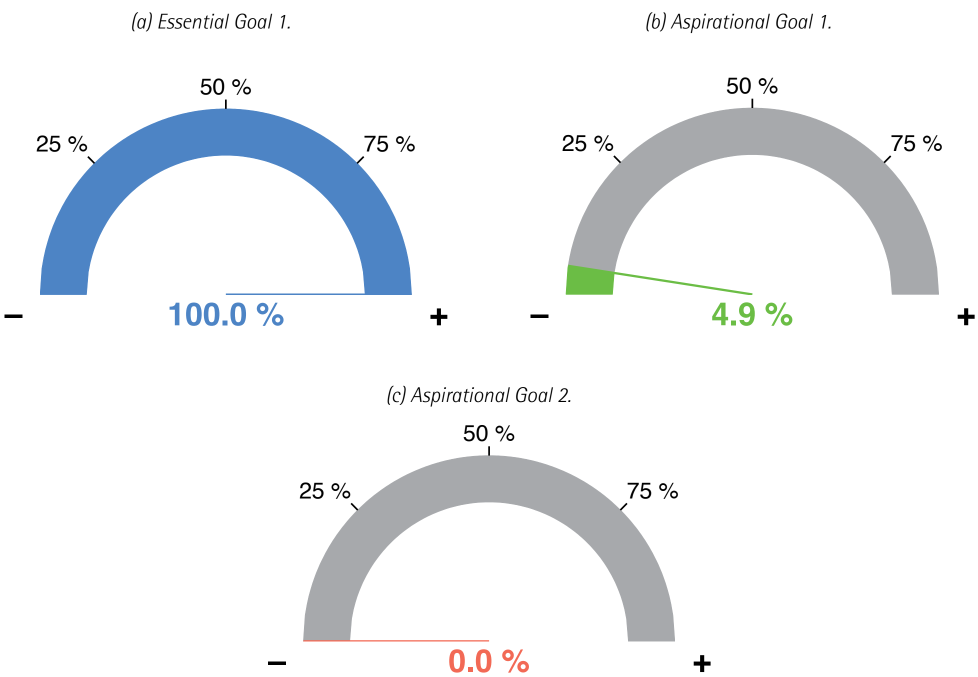
7. Figures and Tables

Figure 37: Investor 3 - Success probabilities with strategy securing Essential Goals 1 and 2 with income and forward contracts.



This figure shows the success probabilities for a strategy securing Essential Goals 1 and 2 with income, when the re-investment rate of income is fixed at date 0. The fraction of the goals not covered by income is protected with an option maturing at date 21 (the retirement date). The portfolio is buy-and-hold in the MSR of stocks and bonds and the option.

Figure 38: Investor 3 - Success probabilities with strategy securing Essential Goal 1 with liquid wealth only; 10% tax rate.



This figure shows the success probabilities for a strategy securing Essential Goal 1 with liquid wealth only when a 10% tax rate is applied to bond coupons and to selling operations in stock and bond indices. The portfolio is buy-and-hold in the MSR of stock and bond indices and the bond that pays the cash flows of Essential Goal 1 for years 21 to 50, adjusted for the tax rate.

7. Figures and Tables

Table 25: Investor 3 – Funding status of goals in the presence of taxes.

(i) Value of assets (in $).

Market wealth	940,000
Present value of future savings	181,506
Total	**1,121,506**

(ii) Minimum capital required to secure one or more goal(s) (in $) – 10% tax rate.

	Liquid wealth only	Income and zero re-investment rate	Income and forward contracts
Goal 1	900,284	799,209	718,778
Goal 2	188,770	188,770	188,770
G1 and G2	**1,089,054**	**987,979**	**907,548**
Goal 3	339,194	339,194	339,194
G1, G2 and G3	**1,428,248**	**1,237,173**	**1,246,742**

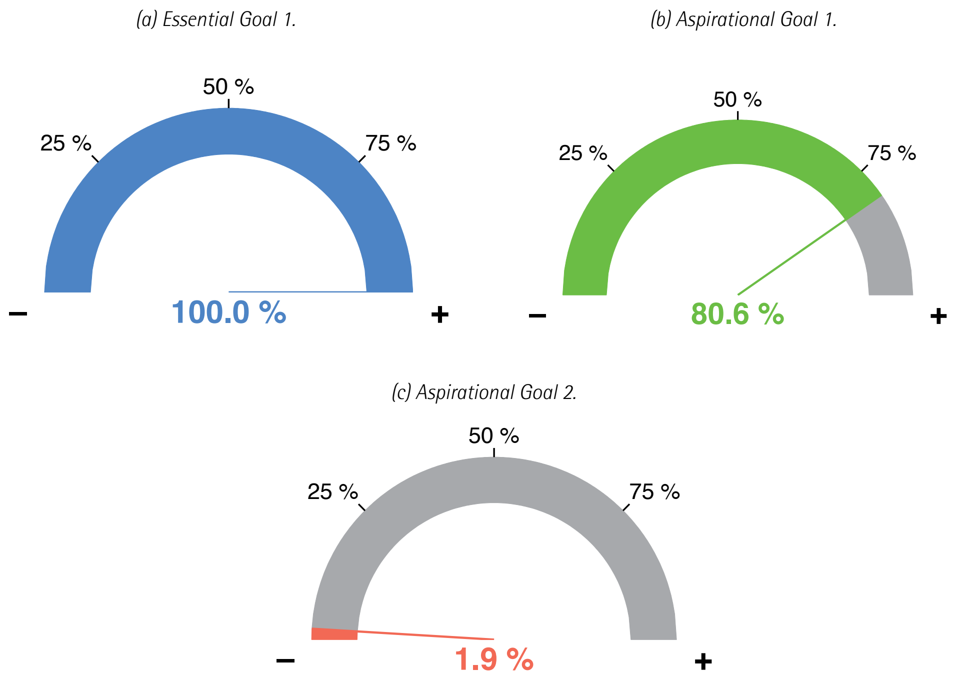
(iii) Minimum capital required to secure one or more goal(s) (in $) – 20% tax rate.

	Liquid wealth only	Income and zero re-investment rate	Income and forward contracts
Goal 1	1,012,820	911,745	831,314
Goal 2	212,366	212,366	212,366
G1 and G2	**1,225,186**	**1,124,111**	**1,043,680**
Goal 3	381,593	381,593	381,593
G1, G2 and G3	**1,606,779**	**1,505,704**	**1,425,273**

Panel (i) shows the assets of the investor, which consist of liquid market assets and a claim on future savings. Panels (ii) and (iii) show the minimum liquid wealth required in order to secure one or more goal(s) for three modes of protection of Goal 1 (the retirement goal): the investor relies only on liquid wealth; he uses income assuming a zero re-investment rate; he uses income re-invested at forward rates specified at date 0. Goal 2 is the education goal and Goal 3 is the home goal. The difference between Panels (ii) and (iii) is the tax rate applied to the coupons paid by bonds (10% or 20%). Goals are ranked by decreasing priority order.

7. Figures and Tables

Figure 39: Investor 3 - Success probabilities with strategy securing Essential Goal 1 with income re-invested at a zero rate; 10% tax rate.

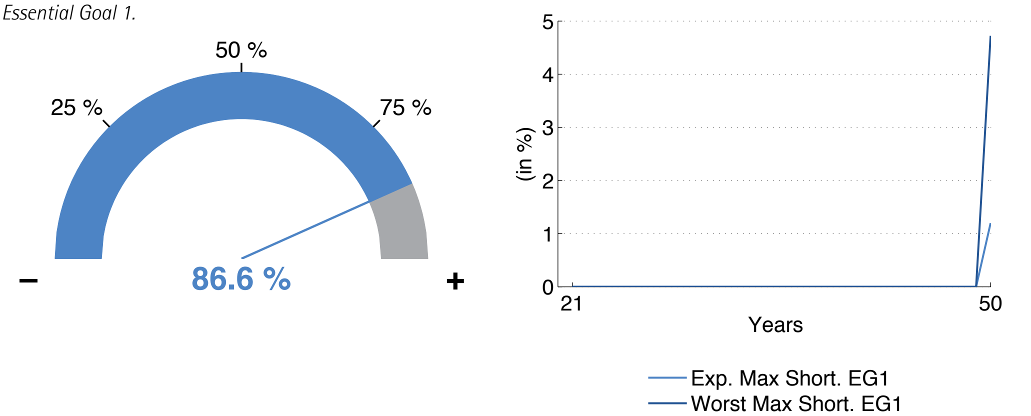


This figure shows the success probabilities for a strategy securing Essential Goal 1 with income assumed to be re-invested at a zero rate when a 10% tax rate is applied to bond coupons and to selling operations in stock and bond indices. The portfolio is buy-and-hold in the MSR of stock and bond indices and an option that secures the fraction of the goal value which is not covered by income.

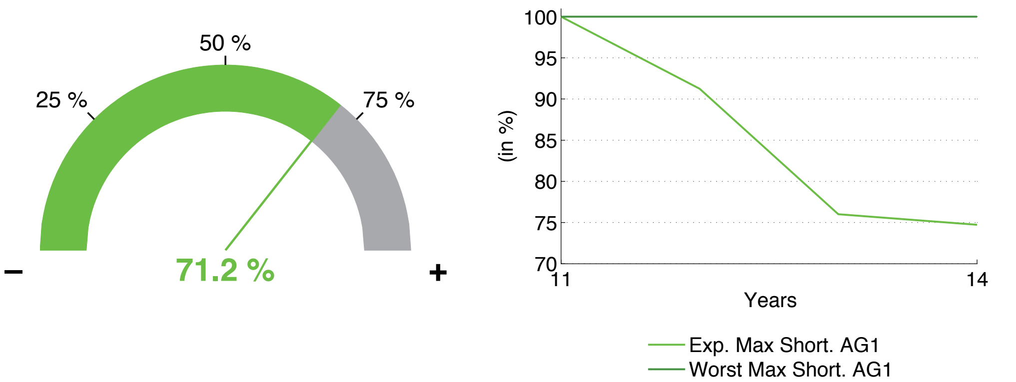
7. Figures and Tables

Figure 40: Investor 3 - Success indicators with strategy aiming to secure Essential Goal 1 with income re-invested at forward rates; 10% tax rate.

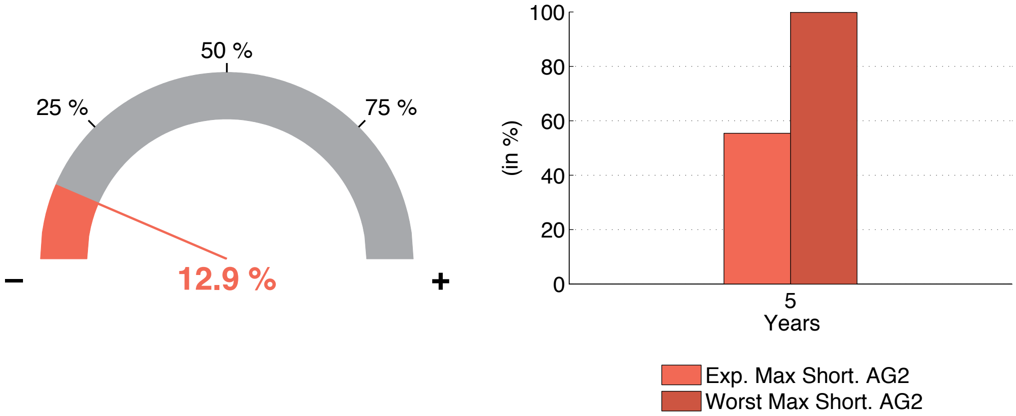
(a) Essential Goal 1.



(b) Aspirational Goal 1.



(c) Aspirational Goal 2.

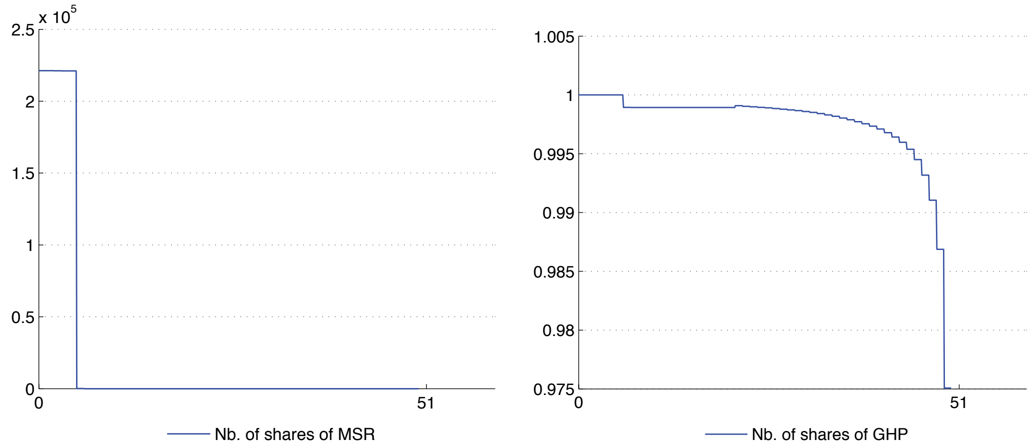


This figure shows the success probabilities (left column) and the shortfall indicators (right column) for a strategy aiming to secure Essential Goal 1 with income re-invested at forward rates when a 10% tax rate is applied to bond coupons and to selling operations in stock and bond indices. The portfolio is buy-and-hold in the MSR of stock and bond indices and an option that secures the fraction of the goal value which is not covered by income.

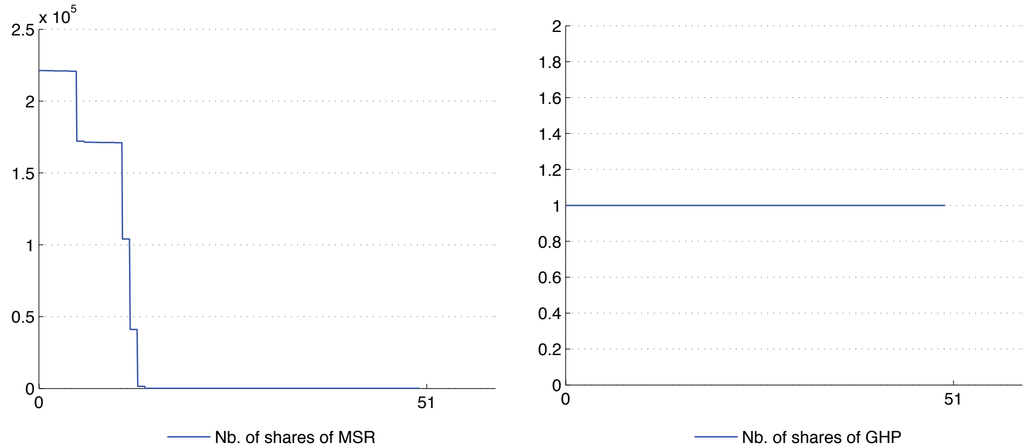
7. Figures and Tables

Figure 41: Investor 3 – Sample paths of allocations to MSR and GHP with strategy protecting Essential Goal 1 with income and forward contracts.

(a) Path with deviation from Essential Goal 1.



(b) Path with success for Essential Goal 1.



Panel (a) shows the time series of allocations to the MSR and the GHP on a path where Essential Goal 1 is not reached. Panel (b) shows the allocations on a path where the goal is reached. Each allocation is expressed as a number of shares. The strategy implemented here uses income to secure the largest possible fraction of Essential Goal 1, the re-investment rate of income being locked up at date 0 by the means of forward contracts. The fraction of the goal that is not covered by income is secured with an option.

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- ALM and asset management

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Nbr of permanent staff	97
Nbr of research associates	26
Nbr of affiliate professors	28
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External financing	€10,100,000
Nbr of conference delegates	1,782
Nbr of participants at EDHEC-Risk Institute Executive Education seminars	1,576

About EDHEC-Risk Institute

The EDHEC-Risk Institute PhD in Finance

The EDHEC-Risk Institute PhD in Finance is designed for professionals who aspire to higher intellectual levels and aim to redefine the investment banking and asset management industries. It is offered in two tracks: a residential track for high-potential graduate students, who hold part-time positions at EDHEC, and an executive track for practitioners who keep their full-time jobs. Drawing its faculty from the world's best universities, such as Princeton, Wharton, Oxford, Chicago and CalTech, and enjoying the support of the research centre with the greatest impact on the financial industry, the EDHEC-Risk Institute PhD in Finance creates an extraordinary platform for professional development and industry innovation.

Research for Business

The Institute's activities have also given rise to executive education and research service offshoots. EDHEC-Risk's executive education programmes help investment professionals to upgrade their skills with advanced risk and asset management training across traditional and alternative classes. In partnership with CFA Institute, it has developed advanced seminars based on its research which are available to CFA charterholders and have been taking place since 2008 in New York, Singapore and London.

In 2012, EDHEC-Risk Institute signed two strategic partnership agreements with the Operations Research and Financial Engineering department of Princeton University to set up a joint research programme in the area of risk and investment management, and with Yale School of Management to set up joint certified executive training courses in North America and Europe in the area of investment management.

As part of its policy of transferring know-how to the industry, EDHEC-Risk Institute has also set up ERI Scientific Beta. ERI Scientific Beta is an original initiative which aims to favour the adoption of the latest advances in smart beta design and implementation by the whole investment industry. Its academic origin provides the foundation for its strategy: offer, in the best economic conditions possible, the smart beta solutions that are most proven scientifically with full transparency in both the methods and the associated risks.

EDHEC-Risk Institute Publications and Position Papers (2012-2015)

EDHEC-Risk Institute Publications (2012-2015)

2015

• Blanc-Brude, F., and M. Hasan. The Valuation of Privately-Held Infrastructure Equity Investments (January).

2014

• Coqueret, G., R. Deguest, L. Martellini, and V. Milhau. Equity Portfolios with Improved Liability-Hedging Benefits (December).

• Blanc-Brude, F., and D. Makovsek. How Much Construction Risk do Sponsors take in Project Finance. (August).

• Loh, L., and S. Stoyanov. The Impact of Risk Controls and Strategy-Specific Risk Diversification on Extreme Risk (August).

• Blanc-Brude, F., and F. Ducoulombier. Superannuation v2.0 (July).

• Loh, L., and S. Stoyanov. Tail Risk of Smart Beta Portfolios: An Extreme Value Theory Approach (July).

• Foulquier, P. M. Arouri and A. Le Maistre. P. A Proposal for an Interest Rate Dampener for Solvency II to Manage Pro-Cyclical Effects and Improve Asset-Liability Management (June).

• Amenc, N., R. Deguest, F. Goltz, A. Lodh, L. Martellini and E.Schirbini. Risk Allocation, Factor Investing and Smart Beta: Reconciling Innovations in Equity Portfolio Construction (June).

• Martellini, L., V. Milhau and A. Tarelli. Towards Conditional Risk Parity — Improving Risk Budgeting Techniques in Changing Economic Environments (April).

• Amenc, N., and F. Ducoulombier. Index Transparency – A Survey of European Investors Perceptions, Needs and Expectations (March).

• Ducoulombier, F., F. Goltz, V. Le Sourd, and A. Lodh. The EDHEC European ETF Survey 2013 (March).

• Badaoui, S., Deguest, R., L. Martellini and V. Milhau. Dynamic Liability-Driven Investing Strategies: The Emergence of a New Investment Paradigm for Pension Funds? (February).

• Deguest, R., and L. Martellini. Improved Risk Reporting with Factor-Based Diversification Measures (February).

• Loh, L., and S. Stoyanov. Tail Risk of Equity Market Indices: An Extreme Value Theory Approach (February).

2013

• Lixia, L., and S. Stoyanov. Tail Risk of Asian Markets: An Extreme Value Theory Approach (August).

• Goltz, F., L. Martellini, and S. Stoyanov. Analysing statistical robustness of cross-sectional volatility. (August).

EDHEC-Risk Institute Publications (2012-2015)

• Lixia, L., L. Martellini, and S. Stoyanov. The local volatility factor for asian stock markets. (August).

• Martellini, L., and V. Milhau. Analysing and decomposing the sources of added-value of corporate bonds within institutional investors' portfolios (August).

• Deguest, R., L. Martellini, and A. Meucci. Risk parity and beyond - From asset allocation to risk allocation decisions (June).

• Blanc-Brude, F., Cocquemas, F., Georgieva, A. Investment Solutions for East Asia's Pension Savings - Financing lifecycle deficits today and tomorrow (May)

• Blanc-Brude, F. and O.R.H. Ismail. Who is afraid of construction risk? (March)

• Lixia, L., L. Martellini, and S. Stoyanov. The relevance of country- and sector-specific model-free volatility indicators (March).

• Calamia, A., L. Deville, and F. Riva. Liquidity in european equity ETFs: What really matters? (March).

• Deguest, R., L. Martellini, and V. Milhau. The benefits of sovereign, municipal and corporate inflation-linked bonds in long-term investment decisions (February).

• Deguest, R., L. Martellini, and V. Milhau. Hedging versus insurance: Long-horizon investing with short-term constraints (February).

• Amenc, N., F. Goltz, N. Gonzalez, N. Shah, E. Shirbini and N. Tessaromatis. The EDHEC european ETF survey 2012 (February).

• Padmanaban, N., M. Mukai, L . Tang, and V. Le Sourd. Assessing the quality of asian stock market indices (February).

• Goltz, F., V. Le Sourd, M. Mukai, and F. Rachidy. Reactions to "A review of corporate bond indices: Construction principles, return heterogeneity, and fluctuations in risk exposures" (January).

• Joenväärä, J., and R. Kosowski. An analysis of the convergence between mainstream and alternative asset management (January).

• Cocquemas, F. Towar¬ds better consideration of pension liabilities in european union countries (January).

• Blanc-Brude, F. Towards efficient benchmarks for infrastructure equity investments (January).

2012

• Arias, L., P. Foulquier and A. Le Maistre. Les impacts de Solvabilité II sur la gestion obligataire (December).

• Arias, L., P. Foulquier and A. Le Maistre. The Impact of Solvency II on Bond Management (December).

• Amenc, N., and F. Ducoulombier. Proposals for better management of non-financial risks within the european fund management industry (December).

EDHEC-Risk Institute Publications (2012-2015)

- Cocquemas, F. Improving Risk Management in DC and Hybrid Pension Plans (November).
- Amenc, N., F. Cocquemas, L. Martellini, and S. Sender. Response to the european commission white paper "An agenda for adequate, safe and sustainable pensions" (October).
- La gestion indicielle dans l'immobilier et l'indice EDHEC IEIF Immobilier d'Entreprise France (September).
- Real estate indexing and the EDHEC IEIF commercial property (France) index (September).
- Goltz, F., S. Stoyanov. The risks of volatility ETNs: A recent incident and underlying issues (September).
- Almeida, C., and R. Garcia. Robust assessment of hedge fund performance through nonparametric discounting (June).
- Amenc, N., F. Goltz, V. Milhau, and M. Mukai. Reactions to the EDHEC study "Optimal design of corporate market debt programmes in the presence of interest-rate and inflation risks" (May).
- Goltz, F., L. Martellini, and S. Stoyanov. EDHEC-Risk equity volatility index: Methodology (May).
- Amenc, N., F. Goltz, M. Masayoshi, P. Narasimhan and L. Tang. EDHEC-Risk Asian index survey 2011 (May).
- Guobuzaite, R., and L. Martellini. The benefits of volatility derivatives in equity portfolio management (April).
- Amenc, N., F. Goltz, L. Tang, and V. Vaidyanathan. EDHEC-Risk North American index survey 2011 (March).
- Amenc, N., F. Cocquemas, R. Deguest, P. Foulquier, L. Martellini, and S. Sender. Introducing the EDHEC-Risk Solvency II Benchmarks – maximising the benefits of equity investments for insurance companies facing Solvency II constraints - Summary - (March).
- Schoeffler, P. Optimal market estimates of French office property performance (March).
- Le Sourd, V. Performance of socially responsible investment funds against an efficient SRI Index: The impact of benchmark choice when evaluating active managers – an update (March).
- Martellini, L., V. Milhau, and A.Tarelli. Dynamic investment strategies for corporate pension funds in the presence of sponsor risk (March).
- Goltz, F., and L. Tang. The EDHEC European ETF survey 2011 (March).
- Sender, S. Shifting towards hybrid pension systems: A European perspective (March).
- Blanc-Brude, F. Pension fund investment in social infrastructure (February).
- Ducoulombier, F., Lixia, L., and S. Stoyanov. What asset-liability management strategy for sovereign wealth funds? (February).
- Amenc, N., Cocquemas, F., and S. Sender. Shedding light on non-financial risks – a European survey (January).

EDHEC-Risk Institute Publications (2012-2015)

- Amenc, N., F. Cocquemas, R. Deguest, P. Foulquier, Martellini, L., and S. Sender. Ground Rules for the EDHEC-Risk Solvency II Benchmarks. (January).
- Amenc, N., F. Cocquemas, R. Deguest, P. Foulquier, Martellini, L., and S. Sender. Introducing the EDHEC-Risk Solvency Benchmarks – Maximising the Benefits of Equity Investments for Insurance Companies facing Solvency II Constraints - Synthesis -. (January).
- Amenc, N., F. Cocquemas, R. Deguest, P. Foulquier, Martellini, L., and S. Sender. Introducing the EDHEC-Risk Solvency Benchmarks – Maximising the Benefits of Equity Investments for Insurance Companies facing Solvency II Constraints (January).
- Schoeffler.P. Les estimateurs de marché optimaux de la performance de l'immobilier de bureaux en France (January).

EDHEC-Risk Institute Position Papers (2012-2015)

2014

- Blanc-Brude, F. Benchmarking Long-Term Investment in Infrastructure: Objectives, Roadmap and Recent Progress (June).

2012

- Till, H. Who sank the boat? (June).
- Uppal, R. Financial Regulation (April).
- Amenc, N., F. Ducoulombier, F. Goltz, and L. Tang. What are the risks of European ETFs? (January).

Notes

For more information, please contact:
Carolyn Essid on +33 493 187 824
or by e-mail to: carolyn.essid@edhec-risk.com

EDHEC-Risk Institute
393 promenade des Anglais
BP 3116 - 06202 Nice Cedex 3
France
Tel: +33 (0)4 93 18 78 24

EDHEC Risk Institute—Europe
10 Fleet Place, Ludgate
London EC4M 7RB
United Kingdom
Tel: +44 207 871 6740

EDHEC Risk Institute—Asia
1 George Street
#07-02
Singapore 049145
Tel: +65 6438 0030

EDHEC Risk Institute—North America
One Boston Place, 201 Washington Street
Suite 2608/2640 — Boston, MA 02108
United States of America
Tel: +1 857 239 8891

EDHEC Risk Institute—France
16-18 rue du 4 septembre
75002 Paris
France
Tel: +33 (0)1 53 32 76 30

www.edhec-risk.com